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Pathfinder for OLYMPIAD MATHEMATICS

Vikash Tiwari
V. Seshan

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MATHEMATICS

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Preface

"For another hundred years, School will teach children 'to do' rather than 'to think'" observed Bertrand Russell. This statement is still seen to be true without being even remotely contradicted.

NCF 2005 (National Curriculum Framework) provides a vision for perspective planning ofschool education in scholastic and non-scholastic domains. It also emphasizes on 'mathematisation' of the child's thought and processes by recognizing mathematics as an integral part of development of the human potential. The higher aim of teaching mathematics is to enhance the ability to visualize, logically understand, build arguments, prove statements and in a sense, handle abstraction. For motivated and talented students, there is a need to widen the horizon as these students love challenges and always look beyond the curriculum at school.

Hence, we created this book to cater to the needs of these students. With numerous problems designed to develop thinking and reasoning, the book contains statements, definitions, postulates, formulae, theorems, axioms, and propositions, which normally do not appear in school textbooks. These are spelt out and interpreted to improve the student's conceptual knowledge.

The book also presents 'non-routine problems' and detailed, step-by-step solutions to these problems to enable the reader to acquire a better understanding of the concepts as well as to develop analytical and reasoning (logical) abilities. Thus, readers get the 'feel' of problem-solving as an activity which, in turn, reveals the innate pleasure of successfully solving a challenging problem. This 'pleasure' is permanent and helps to build-in them a positive attitude towards the subject. Developing ability for critical analysis and problem solving is an essential requirement if one wants to become successful in life.

No one has yet discovered a way of learning mathematics better than, by solving problems in the subject. This book helps students to face competitive examinations such as the Olympiads (RMO, INMO, IMO), KVPY and IIT-JEE confidently without being befuddled by the intricacies of the subject. It has been designed to enable students and all lovers of mathematics to master the subject at their own pace.

We have made efforts to provide solutions along with the problems in an error-free and unambiguous manner as far as possible. However, if any error is detected by the reader, it may please be brought to our notice, so that we may make necessary corrections in the future editions of the book. We look forward to your suggestions and shall be grateful for them.

Lastly, we share the observation made by Pundit Jawaharlal Nehru: "Giving opportunity to potential creativity is a matter of life and death for an enlightened society because the contributions of a few creative individuals are the mankind's ultimate capital asset."

We wish best of luck at all times to all those using this book.

Vikash Tiwari

V. Seshan

Acknowledgements

First and foremost, we thank the Pearson group for motivating us and rendering all possible assistance in bringing out this book in its present form. We are grateful to the Pearson group for having consented to publish the book on our behalf.

We would also like to thank Ajai Lakheena, who has been instrumental in giving this book its present shape. He has made invaluable contributions to "Geometry" chapter of this book.This section would not have been as effective without his efforts.

We also express our gratitude to Bhupinder Singh Tomar and Abir Bhowmick, who have helped us with their discerning inputs and suggestions for making this book error-free. We are indebted to R.K. Thakur for his inputs and constant encouragement to write this book.

This book is dedicated to my wife Priyanka for her kindness, devotion and endless-support in managing household chores and to my two adorable daughters Tanya and Manya who sacrificed their vacation umpteen times for my (our) sake.

Vikash Tiwari

About the Authors

Vikash Tiwari has been teaching students for Mathematical Olympiads (Pre RMO, RMO, INMO and IMOTC) and other examinations like KVPY and JEE *for the last 20 years*. He is a renowned figure in the field of Mathematics across the geography of the country. His students have always founds his methods of teaching insightful and his approach to problem solving very intriguing. He has guided several of the medal winning students that have done India proud at the International Mathematical Olympiad over the years. He use to conduct Olympiad training camps for several organizations such as Kendriya Vidyalaya Sangathan, Delhi Public School (DPS) etc. He has immersed himself into the service of mankind via the medium of Mathematics for the past couple of decades and has come with this first book of his which is basically an inventory of all the concepts required to ace the Mathematics Olympiad at various levels.

V. Seshan is the key resource person shortlisted by CBSE to provide Olympiad training across India. He is a popular teacher and retd. Principal & Director of Bhartiya Vidya Bhavan, Baroda Centre. He is well known for his unique ability in teaching Mathematics with conceptual clarity. With teaching experience spanning over 40 years, he has been instrumental in setting-up the Olympiad Centre in Tata Institute of Fundamental Research, Mumbai. He has also been awarded with various medals, honors and recognitions by prestigious universities and institutions from across the globe. These bear testament to his immense contribution to the field of Mathematics. Many of his students have taken part in both National & International Mathematics Olympiad and have also won gold and silver medals to their credits. A few of his recognition are listed below.

- Fulbright Teacher Awardee (USA-1970)
- Presidential Awardee (1987)
- Advisor to National Science Olympiad Foundation (Since 1989)
- Rotary (Int) Awardee (1992)
- PEE VEE National Awardee (2000)
- Ramanujan Awardee (2008)

Chapter 1

Niccolò Fontana Tartaglia

(1499/1500–13 Dec 1557) Tartaglia was an Italian mathematician. The name "Tartaglia" is actually a nickname meaning "stammerer", a reference to his injury-induced speech impediment. He was largely self-taught, and was the first person to translate Euclid's *Elements* into a modern European language. He is best remembered for his contributions to algebra, namely his discovery of a formula for the solutions to a cubic equation. Such a formula was also found by Gerolamo Cardano at roughly the same time, and the modern formula is known as the Cardano-Tartaglia formula. Cardano also found a solution to the general quartic equation.

Évariste Galois

(25 Oct 1811–31 May 1832) Galois was a very gifted young French mathematician, and his story is one of the most tragic in the history of mathematics. He was killed at the age of 20 in a duel that is still veiled in mystery. Before that, he made huge contributions to abstract algebra. He helped to found group theory as we know it today, and he was the first to use the term "group". Perhaps most importantly, he proved that it is impossible to solve a fifth-degree polynomial (or a polynomial of any higher degree) using radicals by studying permutation groups associated to polynomials. This area of algebra is still important today, and it is known as *Galois theory* in his honor.

Niels Henrik Abel

(5 Aug 1802–6 Apr 1829) Abel was a Norwegian mathematician who, like Galois, did seminal work in algebra before dying at a very young age. Strangely enough, he proved similar results regarding the insolvability of the quintic independently from Galois. In honor of his work in group theory, abelian groups are named after him. The Abel Prize in mathematics, sometimes thought of as the "Nobel Prize in Mathematics," is also named for him.

Joseph-Louis Lagrange

(25 Jan 1736–10 Apr 1813) Despite his French-sounding name, Lagrange was an Italian mathematician. Like many of the great mathematicians of his time, he made contributions to many different areas of mathematics. In particular, he did some early work in abstract algebra.

Polynomials

1.1 Polynomial Functions

Any function, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, is a polynomial function in 'x' where $a_i (i = 0, 1, 2, 3, \ldots, n)$ is a constant which belongs to the set of real numbers and sometimes to the set of complex numbers, and the indices, $n, n-1, \ldots, 1$ are natural numbers. If $a_n \neq 0$, then we can say that $f(x)$ is a polynomial of degree n. a_n is called leading coefficient of the polynomial. If $a_n = 1$, then polynomial is called monic polynomial. Here, if $n = 0$, then $f(x) = a_0$ is a constant polynomial. Its degree is 0, if $a_0 \neq 0$. If $a_0 = 0$, the polynomial is called zero polynomial. Its degree is defined as $-\infty$ to preserve the first two properties listed below. Some people prefer not to defined degree of zero polynomial. The domain and range of the function are the set of real numbers and complex numbers, respectively. Sometimes, we take the domain also to be complex numbers. If z is a complex number and $f(z) = 0$, then z is called 'a zero of the polynomial'.

If $f(x)$ is a polynomial of degree n and $g(x)$ is a polynomial of degree m then

1. $f(x) \pm g(x)$ is polynomial of degree $\leq \max\{n, m\}$
2. $f(x) \cdot g(x)$ is polynomial of degree $m + n$
3. $f(g(x))$ is polynomial of degree $m \cdot n$, where $g(x)$ is a non-constant polynomial.

Illustrations

1. $x^4 - x^3 + x^2 - 2x + 1$ *is a polynomial of degree* 4 and 1 *is a zero of the polynomial as*
$$1^4 - 1^3 + 1^2 - 2 \times 1 + 1 = 0.$$
2. $x^3 - ix^2 + ix + 1 = 0$ *is a polynomial of degree* 3 and i *is a zero of his polynomial as* $i^3 - i \cdot i^2 + i \cdot i + 1 = -i + i - 1 + 1 = 0$.
3. $x^2 - (\sqrt{3} - \sqrt{2})x - \sqrt{6}$ is *a polynomial of degree* 2 *and* $\sqrt{3}$ *is a zero of this polynomial as* $(\sqrt{3})^2 - (\sqrt{3} - \sqrt{2})\sqrt{3} - \sqrt{6} = 3 - 3 + \sqrt{6} - \sqrt{6} = 0$.

Note: The above-mentioned definition and examples refer to polynomial functions in one variable. Similarly, polynomials in 2, 3, …, n variables can be defined. The domain for polynomial in n variables being the set of (ordered) n tuples of complex numbers and the range is the set of complex numbers.

Illustration *$f(x, y, z) = x^2 - xy + z + 5$ is a polynomial in x, y, z of degree 2 as both x^2 and xy have degree 2 each.*

Note: In a polynomial in n variables, say, $x_1, x_2, \ldots, x_n$, a general term is $x_1^{k_1} \cdot x_2^{k_2} \cdots x_n^{k_n}$. Degree of the term is $k_1 + k_2 + \cdots + k_n$ where $k_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, n$. The degree of a polynomial in n variables is the maximum of the degrees of its terms.

1.2 Division in Polynomials

If $P(x)$ and $\phi(x)$ ($\phi(x) \not\equiv 0$) are any two polynomials, then we can find unique polynomials $Q(x)$ and $R(x)$, such that $P(x) = \phi(x) \times Q(x) + R(x)$ where the degree of $R(x) <$ degree of $\phi(x)$ or $R(x) \equiv 0$. $Q(x)$ is called the quotient and $R(x)$, the remainder.

In particular, if $P(x)$ is a polynomial with complex coefficients, and a is a complex number, then there exists a polynomial $Q(x)$ of degree 1 less than $P(x)$ and a complex number R, such that $P(x) = (x - a)Q(x) + R$.

Illustration $x^5 = (x - a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4) + a^5$.

Here, $\quad P(x) = x^5,\ Q(x) = x^4 + ax^3 + a^2x^2 + a^3x + a^4$,

and $\quad R = a^5$.

Example 1 *What is the remainder when $x + x^9 + x^{25} + x^{49} + x^{81}$ is divided by $x^3 - x$.*

Solution: We have,

$$x + x^9 + x^{25} + x^{49} + x^{81} = x(1 + x^8 + x^{24} + x^{48} + x^{80})$$
$$= x[(x^{80} - 1) + (x^{48} - 1) + (x^{24} - 1) + (x^8 - 1) + 5]$$
$$= x(x^{80} - 1) + x\,(x^{48} - 1) + x(x^{24} - 1) + x(x^8 - 1) + 5x$$

Now, $x^3 - x = x(x^2 - 1)$ and all, but the last term $5x$ is divisible by $x(x^2 - 1)$. Thus, the remainder is $5x$.

Example 2 *Prove that the polynomial $x^{9999} + x^{8888} + x^{7777} + \cdots + x^{1111} + 1$ is divisible by $x^9 + x^8 + x^7 + \cdots + x + 1$.*

Solution: Let,

$$P = x^{9999} + x^{8888} + x^{7777} + \cdots + x^{1111} + 1$$
$$Q = x^9 + x^8 + x^7 + \cdots + x + 1$$
$$P - Q = x^9(x^{9990} - 1) + x^8(x^{8880} - 1) + x^7(x^{7770} - 1) + \cdots + x(x^{1110} - 1)$$
$$= x^9[(x^{10})^{999} - 1] + x^8[(x^{10})^{888} - 1] + x^7[(x^{10})^{777} - 1] + \cdots + x[(x^{10})^{111} - 1] \quad (1)$$

But, $(x^{10})^n - 1$ is divisible by $x^{10} - 1$ for all $n \geq 1$.

$\therefore$ RHS of Eq. (1) divisible by $x^{10} - 1$.

$\therefore$ $P - Q$ is divisible by $x^{10} - 1$

As $x^9 + x^8 + \cdots + x + 1 \,|\, (x^{10} - 1)$

$\Rightarrow$ $x^9 + x^8 + x^7 + \cdots + x + 1 \,|\, (P - Q)$

$\Rightarrow$ $x^9 + x^8 + x^7 + \cdots + x + 1 \,|\, P$

1.3 Remainder Theorem and Factor Theorem

1.3.1 Remainder Theorem

If a polynomial $f(x)$ is divided by $(x - a)$, then the remainder is equal to $f(a)$.

Proof:

$$f(x) = (x - a)Q(x) + R$$

and so, $$f(\mathrm{a}) = (a - a)Q(a) + R = R.$$

If $R = 0$, then $f(x) = (x - a)Q(x)$ and hence, $(x - a)$ is a factor of $f(x)$.

Further, $f(a) = 0$, and thus, a is a zero of the polynomial $f(x)$. This leads to the factor theorem.

1.3.2 Factor Theorem

$(x - a)$ is a factor of polynomial $f(x)$, if and only if, $f(a) = 0$.

Example 3 *If $f(x)$ is a polynomial with integral coefficients and, suppose that $f(1)$ and $f(2)$ both are odd, then prove that there exists no integer n for which $f(n) = 0$.*

Solution: Let us assume the contrary. So, $f(n) = 0$ for some integer n.

Then, $(x - n)$ divides $f(x)$.

Therefore, $f(x) = (x - n)g(x)$

where $g(x)$ is again a polynomial with integral coefficients.

Now, $f(1) = (1 - n)\, g(1)$ and $f(2) = (2 - n)\, g(2)$ are odd numbers but one of $(1 - n)$ and $(2 - n)$ should be even as they are consecutive integers.

Thus one of $f(1)$ and $f(2)$ should be even, which is a contradiction. Hence, the result.

Aliter: See the Example (41) on page 6.24 in Number Theory chapter.

Example 4 *If f is a polynomial with integer coefficients such that there exists four distinct integer a_1, a_2, a_3 and a_4 with $f(a_1) = f(a_2) = f(a_3) = f(a_4) = 1991$, show that there exists no integer b, such that $f(b) = 1993$.*

Solution: Suppose, there exists an integer b, such that $f(b) = 1993$, let $g(x) = f(x) - 1991$.

Now, g is a polynomial with integer coefficients and $g(a_1) = 0$ for $i = 1, 2, 3, 4$.

Thus $(x - a_1)(x - a_2)(x - a_3)$ and $(x - a_4)$ are all factors of $g(x)$.

So, $g(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4) \times h(x)$

where $h(x)$ is polynomial with integer coefficients.

$$\begin{aligned} g(b) &= f(b) - 1991 \\ &= 1993 - 1991 = 2 \text{ (by our choice of } b) \end{aligned}$$

But, $g(b) = (b - a_1)(b - a_2)(b - a_3)(b - a_4)\, h(b) = 2$

Thus, $(b - a_1)(b - a_2)(b - a_3)(b - a_4)$ are all divisors of 2 and are distinct.

$\therefore$ $(b - a_1)(b - a_2)(b - a_3)(b - a_4)$ are $1, -1, 2, -2$ in some order, and $h(b)$ is an integer.

$\therefore$ $g(b) = 4 \cdot h(b) \neq 2$.

Hence, such b does not exist.

1.4 Fundamental Theorem of Algebra

Every polynomial function of degree ≥ 1 has at least one zero in the complex numbers. In other words, if we have

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $n \geq 1$, then there exists atleast one $h \in \mathbb{C}$, such that

$$a_n h^n + a_{n-1} h^{n-1} + \cdots + a_1 h + a_0 = 0.$$

From this, it is easy to deduce that a polynomial function of degree 'n' has exactly n zeroes.
i.e., $f(x) = a(x - r_1)(x - r_2)\ldots(x - r_n)$

Notes:

1. Some of the zeroes of a polynomial may repeat.
2. If a root α is repeated m times, then m is called multiplicity of the root 'α' or α is called m fold root.
3. The real numbers of the form $\sqrt{3}$, $\sqrt{5}$, $\sqrt{12}$, $\sqrt{27}$, …, $\sqrt{5}+\sqrt{3}$, etc. are called, 'quadratic surds'. In general, $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{a}+\sqrt{b}$, etc. are quadratic surds, if a, b are not perfect squares. In a polynomial with integral coefficients (or rational coefficients), if one of the zeroes is a quadratic surd, then it has the conjugate of the quadratic surd also as a zero.

Illustration *$f(x) = x^2 + 2x + 1 = (x + 1)^2$ and the zeroes of $f(x)$ are -1 and -1. Here, it can be said that $f(x)$ has a zero -1 with multiplicity two.*

Similarly, $f(x) = (x + 2)^3(x - 1)$ has zeroes $-2, -2, -2, 1$, i.e., the zeroes of $f(x)$ are -2 with multiplicity 3 and 1.

Example 5 *Find the polynomial function of lowest degree with integral coefficients with $\sqrt{5}$ as one of its zeroes.*

Solution: Since the order of the surd $\sqrt{5}$ is 2, you may expect that the polynomial of the lowest degree to be a polynomial of degree 2.

Let, $P(x) = ax^2 + bx + c;\ a, b, c \in \mathbb{Q}$

$$P(\sqrt{5}) = 5a + \sqrt{5}b + c = 0 \quad \Rightarrow \quad (5a + c) + \sqrt{5}b = 0$$

But, $\sqrt{5}$ is irrational.

So,

$$5a + c = 0 \quad \text{and} \quad b = 0$$

$$\Rightarrow \quad c = -5a \quad \text{and} \quad b = 0.$$

So, the required polynomial function is $P(x) = ax^2 - 5a,\ a \in \mathbb{Z}\setminus\{0\}$

You can find the other zero of this polynomial to be $-\sqrt{5}$.

Aliter: You know that any polynomial function having, say, n zeroes $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be written as $P(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$ and clearly, this function is of nth degree. Here, the coefficients may be rational, real or complex depending upon the zeroes $\alpha_1, \alpha_2, \ldots, \alpha_n$.

If the zero of a polynomial is $\sqrt{5}$, then $P_0(x) = (x - \sqrt{5})$ or $a(x - \sqrt{5})$.

But, we want a polynomial with rational coefficients.

So, here we multiply $(x - \sqrt{5})$ by the conjugate of $x - \sqrt{5}$, *i.e.*, $x + \sqrt{5}$. Thus, we get the polynomial $P(x) = (x - \sqrt{5})(x + \sqrt{5})$, where the other zero of $P(x)$ is $-\sqrt{5}$.

Now, $P_1(x) = x^2 - 5$, with coefficient of $x^2 = 1$, $x = 0$ and constant term -5, and all these coefficients are rational numbers.

Now, we can write the required polynomial as $P(x) = ax^2 - 5a$ where a is a non-zero integer.

Example 6 *Obtain a polynomial of lowest degree with integral coefficient, whose one of the zeroes is* $\sqrt{5}+\sqrt{2}$.

Solution: Let, $P(x) = x - (\sqrt{5}+\sqrt{2}) = [(x-\sqrt{5}) - \sqrt{2}]$.

Now, following the method used in the previous example, using the conjugate, we get:

$$\begin{aligned} P_1(x) &= [(x-\sqrt{5})-\sqrt{2}][(x-\sqrt{5})+\sqrt{2}] \\ &= (x^2 - 2\sqrt{5}x + 5) - 2 \\ &= (x^2 + 3 - 2\sqrt{5}x) \end{aligned}$$

$$\begin{aligned} P_2(x) &= [(x^2+3)-2\sqrt{5}x][(x^2+3)+2\sqrt{5}x] \\ &= (x^2+3)^2 - 20x^2 \\ &= x^4 + 6x^2 + 9 - 20x^2 \\ &= x^4 - 14x^2 + 9 \end{aligned}$$

$$P(x) = ax^4 - 14ax^2 + 9a, \quad \text{where } a \in \mathbb{Z}, a \neq 0.$$

The other zeroes of this polynomial are $\sqrt{5}-\sqrt{2}, -\sqrt{5}+\sqrt{2}, -\sqrt{5}-\sqrt{2}$.

1.4.1 Identity Theorem

A polynomials $f(x)$ of degree n is identically zero if it vanishes for atleast $n + 1$ distinct values of 'x'.

Proof: Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be n distinct values of x at which $f(x)$ becomes zero.
Then we have

$$f(x) = a(x - x_1)(x - x_2)\ldots(x - x_n)$$

Let α_{n+1} be the $n+1^{\text{th}}$ value of x at which $f(x)$ vanishes. Then

$$f(\alpha_{n+1}) = a(\alpha_{n+1} - \alpha_1)(\alpha_{n+1} - \alpha_2)\ldots(\alpha_{n+1} - \alpha_n) = 0$$

As α_{n+1} is different from $\alpha_1, \alpha_2 \ldots \alpha_n$ none of the number $\alpha_{n+1} - \alpha_i$ vanishes for $i = 1, 2, 3, \ldots n$. Hence $a = 0 \Rightarrow f(x) \equiv 0$.
Using above result we can say that,
If two polynomials $f(x)$ and $g(x)$ of degree m, n respectively with $m \leq n$ have equal values at $n + 1$ distinct values of x, then they must be equal.

Proof: Let $P(x) = f(x) - g(x)$, now degree of $P(x)$ is at most 'n' and it vanishes for at least $n + 1$ distinct values of $x \Rightarrow P(x) \equiv 0 \Rightarrow f(x) \equiv g(x)$.

Corollary: The only periodic polynomial function is constant function.
i.e., if $f(x)$ is polynomials with $f(x + T) = f(x)\ \forall\ x \in \mathbb{R}$ for some constant T then $f(x) =$ constant $= c$ (say)

Proof: Let $f(0) = c$
$\Rightarrow f(0) = f(T) = f(2T) = \ldots = c$
$\Rightarrow$ Polynomial $f(x)$ and constant polynomial $g(x) = c$ take same values at an infinite number of points. Hence they must be identical.

Example 7 Let $P(x)$ be a polynomial such that $x \cdot P(x-1) = (x-4)\,P(x)\ \forall\, x \in \mathbb{R}$. Find all such $P(x)$.

Solution: Put $x = 0$, $0 = -4\,P(0)$
$\Rightarrow P(0) = 0$
Put $x = 1$, $1 \cdot P(0) = -3\,P(1)$
$\Rightarrow P(1) = 0$
Put $x = 2$, $2 \cdot P(1) = -2\,P(2)$
$\Rightarrow P(2) = 0$
Put $x = 3$, $3 \cdot P(2) = -P(3)$
$\Rightarrow P(3) = 0$
Let us assume $P(x) = x(x-1)(x-2)(x-3)\,Q(x)$, where $Q(x)$ is some polynomial.
Now using given relation we have

$$x(x-1)(x-2)(x-3)(x-4)Q(x-1) = x(x-1)(x-2)(x-3)(x-4)Q(x)$$
$$\Rightarrow \quad Q(x-1) = Q(x) \quad \forall x \in \mathbb{R} - \{0,1,2,3,4\}$$
$$\Rightarrow \quad Q(x-1) = Q(x) \quad \forall x \in \mathbb{R} \quad \text{(From identity theorem)}$$
$$\Rightarrow \quad Q(x) \text{ is periodic}$$
$$\Rightarrow \quad Q(x) = c$$
$$\Rightarrow \quad P(x) = cx(x-1)(x-2)(x-3)$$

Example 8 Let $P(x)$ be a monic cubic equation such that $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, then find $P(4)$.

Solution: as $P(x)$ is a monic, coefficient of highest degree will be '1'.
Let $Q(x) = P(x) - x$, where $Q(x)$ is also monic cubic polynomial.

$$Q(1) = P(1) - 1 = 0;\ Q(2) = P(2) - 2 = 0;\ Q(3) = P(3) - 3 = 0$$
$$\Rightarrow \quad Q(x) = (x-1)(x-2)(x-3)$$
$$\Rightarrow \quad P(x) = Q(x) + x = (x-1)(x-2)(x-3) + x$$
$$\Rightarrow \quad P(x) = (4-1)(4-2)(4-3) + 4 = 10$$

Build-up Your Understanding 1

1. Find a fourth degree equation with rational coefficients, one of whose roots is, $\sqrt{3} + \sqrt{7}$.
2. Find a polynomial equation of the lowest degree with rational coefficients whose one root is $\sqrt[3]{2} + 3\sqrt[3]{4}$.
3. Form the equation of the lowest degree with rational coefficients which has $2 + \sqrt{3}$ and $3 + \sqrt{2}$ as two of its roots.
4. Show that $(x-1)^2$ is a factor of $x^n - nx + n - 1$.
5. If a, b, c, d, e are all zeroes of the polynomial $(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1)$, find the value of $(1 + a)(1 + b)(1 + c)(1 + d)(1 + e)$.
6. If $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be the roots of the equation $x^n - 1 = 0$, $n \in \mathbb{N}$, $n \geq 2$ show that $n = (1-\alpha_1)(1-\alpha_2)(1-\alpha_3) \ldots (1-\alpha_{n-1})$.
7. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, show that $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2) = (1-q+s)^2 + (p-r)^2$.

8. If $f(x) = x^4 + ax^3 + bx^2 + cx + d$ is a polynomial such that $f(1) = 10$, $f(2) = 20$, $f(3) = 30$, find the value of $\frac{f(12)+f(-8)}{10}$. **[CMO, 1984]**

9. The polynomial $x^{2k} + 1 + (x + 1)^{2k}$ is not divisible by $x^2 + x + 1$. Find the value of $k \in \mathbb{N}$.

10. Find all polynomials $P(x)$ with real coefficients such that

$$(x - 8)P(2x) = 8(x - 1)P(x).$$

11. Let $(x - 1)^3$ divides $(p(x) + 1)$ and $(x + 1)^3$ divides $(p(x)–1)$. Find the polynomial $p(x)$ of degree 5.

1.5 Polynomial Equations

Let, $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$; $a_n \neq 0$, $n \geq 1$ be a polynomial function.

Then, $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ is called a polynomial equation in x of degree n. Thus,

1. Every polynomial equation of degree n has n roots counting repetition.
2. If $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ (1)

$a_n \neq 0$ and a_i, $(i = 0, 1, 2, 3, \ldots, n)$ are all real numbers and if, $\alpha + i\beta$ is a zero of (1), then $\alpha - i\beta$ is also a root. For real polynomial, complex roots occur in conjugate pairs.

However, if the coefficients of Eq. (1) are complex numbers, it is not necessary that the roots occur in conjugate pairs.

Example 9 *Form a polynomial equation of the lowest degree with* $3 + 2i$ *and* $2 + 3i$ *as two of its roots, with rational coefficients.*

Solution: Since, $3 + 2i$ and $2 + 3i$ are roots of polynomial equation with rational coefficients, $3 - 2i$ and $2 - 3i$ are also the roots of the polynomial equation. Thus, we have identified four roots so that there are 2 pairs of roots and their conjugates. So, the lowest degree of the polynomial equation should be 4. The polynomial equation should be

$$P(x) = a\,[x-(3-2i)][x-(3+2i)][x-(2+3i)]\,[x-(2-3i)] = 0$$
$$\Rightarrow \quad a\,[(x-3)^2+4][(x-2)^2+9] = 0$$
$$\Rightarrow \quad a\,((x-3)^2(x-2)^2 + 9(x-3)^2 + 4(x-2)^2 + 36) = 0$$
$$\Rightarrow \quad a\,((x^2-5x+6)^2 + 9(x^2-6x+9) + 4(x^2-4x+4) + 36) = 0$$
$$\Rightarrow \quad a\,(x^4 - 10x^3 + 50x^2 - 130x + 169) = 0, \quad a \in \mathbb{Q}\backslash\{0\}$$

1.5.1 Rational Root Theorem

An important theorem regarding the rational roots of polynomial equations:

If the rational number $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$, $gcd(p, q) = 1$, *i.e.*, p and q are relatively prime, is a root of the equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $a_0, a_1, a_2, \ldots, a_n$ are integers and $a_n \neq 0$, then p is a divisor of a_0 and q that of a_n.

Proof: Since $\frac{p}{q}$ is a root, we have

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\frac{p}{q} + a_0 = 0$$

$$\Rightarrow \quad a_n p^n + a_{n-1}qp^{n-1} + \cdots + a_1 q^{n-1} p + a_0 q^n = 0 \tag{1}$$

$$\Rightarrow \quad a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1 q^{n-2} p + a_0 q^{n-1} = -\frac{a_n p^n}{q} \tag{2}$$

Since the coefficients $a_{n-1}, a_{n-2}, \ldots, a_0$ and p, q are all integers, hence the left-hand side is an integer, so the right- hand side is also an integer. But, p and q are relatively prime to each other, therefore q should divide a_n.

Again,

$$a_n p^n + a_{n-1}qp^{n-1} + \cdots + a_1 q^{n-1} p = a_0 q^n$$

$$\Rightarrow \quad a_n p^{n-1} + a_{n-1}qp^{n-2} + \cdots + a_1 q^{n-1} = \frac{a_0 q^n}{p} \tag{3}$$

$$\Rightarrow \quad p | a_0$$

As a consequence of the above theorem, we have the following corollary.

1.5.2 Corollary (Integer Root Theorem)

Every rational root of $x^n + a_{n-1}x^{n-1} + \cdots + a_0; 0 \le i \le n-1$ is an integer, where $a_i (i = 0, 1, 2, \ldots, n-1)$ is an integer, and each of these roots is a divisor of a_0.

Example 10 *Find the roots of the equation* $x^4 + x^3 - 19x^2 - 49x - 30$, *given that the roots are all rational numbers.*

Solution: Since all the roots are rational by the above corollary, they are the divisors of -30.

The divisors of -30 are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$.

By applying the remainder theorem, we find that $-1, -2, -3$, and 5 are the roots.

Hence, the roots are $-1, -2, -3$ and $+5$.

Example 11 *Find the rational roots of* $2x^3 - 3x^2 - 11x + 6 = 0$.

Solution: Let the roots be of the form $\frac{p}{q}$, where $(p, q) = 1$ and $q > 0$.

Then, since $q/2$, q must be 1 or 2

and $p|6 \quad \Rightarrow \quad p = \pm 1, \pm 2, \pm 3, \pm 6$

By applying the remainder theorem,

$$f\left(\frac{1}{2}\right) = f\left(\frac{-2}{1}\right) = f\left(\frac{3}{1}\right) = 0.$$

(Corresponding to $q = 2$ and $p = 1$; $q = 1, p = -2$; $q = 1, p = 3$, respectively.)

So, the three roots of the equation are $\frac{1}{2}$, -2, and 3.

Example 12 *Solve:* $x^3 - 3x^2 + 5x - 15 = 0.$

Solution: $x^3 - 3x^2 + 5x - 15 = 0 \quad \Rightarrow \quad (x^2 + 5)(x - 3) = 0$

$$\Rightarrow \quad x = \pm\sqrt{5}\,i, 3.$$

So the solution are 3, $\sqrt{5}\,i, -\sqrt{5}\,i$.

Example 13 *Show that* $f(x) = x^{1000} - x^{500} + x^{100} + x + 1 = 0$ *has no rational roots.*

Solution: If there exists a rational root, let it be $\frac{p}{q}$ where $(p, q) = 1$, $q \neq 0$. Then, q should divide the coefficient of the leading term and p should divide the constant term.

Thus, $q \mid 1 \quad \Rightarrow \quad q = \pm 1$,
And $p \mid 1 \quad \Rightarrow \quad p = \pm 1$

Thus, $\frac{p}{q} = \pm 1$

If the root $\frac{p}{q} = 1$,

Then, $f(1) = 1 - 1 + 1 + 1 + 1 = 3 \neq 0$,
so, 1 is not a root.

If $\frac{p}{q} = -1$,

Then, $f(-1) = 1 - 1 + 1 - 1 + 1 = 1 \neq 0$
And hence, (-1) is not a root.
Thus, there exists no rational roots for the given polynomial.

1.6 Vieta's Relations

Francois Viète

1540–23 Feb 1603
Nationality: French

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the polynomial equation

$$a_n x_n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 = 0 \quad (a_n \neq 0),$$

then,
$$\sum_{1\le i\le n} \alpha_i = -\frac{a_{n-1}}{a_n}; \sum_{1\le i<j\le n} \alpha_i \cdot \alpha_j = \frac{a_{n-2}}{a_n}$$

$$\sum_{1\le i<j<k\le n} \alpha_i \alpha_j \alpha_k = -\frac{a_{n-3}}{a_n}, \cdots; \alpha_1\alpha_2\alpha_3 \cdots \alpha_n = (-1)^n \frac{a_0}{a_n}$$

If we represent the sum $\sum\alpha_i$, $\sum\alpha_i\alpha_j$, …, $\sum\alpha_i\alpha_j \ldots \alpha_n$, respectively, as $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$, (Read it 'sigma 1', 'sigma 2', etc.)

then,
$$\sigma_1 = -\frac{a_{n-1}}{a_n}, \sigma_2 = \frac{a_{n-2}}{a_n}, \ldots$$

$$\sigma_r = (-1)^r \frac{a_{n-r}}{a_n}, \ldots, \sigma_n = (-1)^n \frac{a_0}{a_n}$$

These relations are known as Vieta's relations.
Let us consider the following quadratic, cubic and biquadratic equations and see how we can relate $\sigma_1, \sigma_2, \sigma_3, \ldots,$ with the coefficients.

1. $ax^2 + bx + c = 0$, where α and β are its roots. Thus,

$$\sigma_1 = \alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \sigma_2 = \alpha\beta = \frac{c}{a}$$

2. $ax^3 + bx^2 + cx + d = 0$, where α, β and γ are its roots. Thus,

$$\sigma_1 = \alpha + \beta + \gamma = -\frac{b}{a}$$

$$\sigma_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\sigma_3 = \alpha\beta\gamma = \frac{-d}{a}$$

Here, expressing $\sigma_2 = \alpha(\beta + \gamma) + \beta\gamma = \frac{c}{a}$ will be helpful when we apply this property in computations.

3. $ax^4 + bx^3 + cx^2 + dx + e = 0$, where α, β, γ, δ are its roots. Thus,

$$\sigma_1 = \alpha + \beta + \gamma + \delta = \frac{-b}{a}$$

$$\sigma_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\sigma_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a},$$

$$\sigma_4 = \alpha\beta\gamma\delta = \frac{e}{a}$$

Here, again, σ_2 can be written as $(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta$ and σ_3 can be written as $\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta)$.

Example 14 *If* $x^2 + ax + b + 1 = 0$, *where* $a, b \in \mathbb{Z}$ and $b \neq -1$, *has a root in integers then prove* that $a^2 + b^2$ *is a composite.*

Solution: Let, α and β be the two roots of the equation where, $\alpha \in \mathbb{Z}$. Then,

$$\alpha + \beta = -a \tag{1}$$

$$\alpha \cdot \beta = b + 1 \tag{2}$$

$\therefore \quad \beta = -a - \alpha$ is an integer. Also, since $b + 1 \neq 0$, $\beta \neq 0$.

From Eqs. (1) and (2), we get

$$\begin{aligned} a^2 + b^2 &= (\alpha + \beta)^2 + (\alpha\beta - 1)^2 \\ &= \alpha^2 + \beta^2 + \alpha^2\beta^2 + 1 \\ &= (1 + \alpha^2)(1 + \beta^2) \end{aligned}$$

Now, as $\alpha \in \mathbb{Z}$ and β is a non-zero integer, $1 + \alpha^2 > 1$ and $1 + \beta^2 > 1$.

Hence, $a^2 + b^2$ is composite number.

Example 15 *For what value of p will the sum of the squares of the roots of* $x^2 - px = 1 - p$ *be minimum?*

Solution: If r_1 and r_2 are the roots of $x^2 - px + (p - 1) = 0$,
then $r_l + r_2 = p$, $r_1 r_2 = p - 1$

$$r^2{}_1 + r_2{}^2 = (r_1 + r_2)^2 - 2r_1r_2 = p^2 - 2p + 2 = (p-1)^2 + 1$$

and $r^2{}_1 + r_2{}^2$ is minimum when $(p-1)^2$ is minimum, then $p = 1$.

Thus, for $p = 1$, the sum of the squares of the roots is minimum.

Example 16 *Let u, v be two real numbers none equql to −1, such that u, v and uv are the roots of a cubic polynomial with rational coefficients. Prove or disprove that uv is rational.*

Solution: Let, $x^3 + ax^2 + bx + c = 0$ be the cubic polynomial of which u, v, and uv are the roots and a, b, and c are all rationals.

$$u + v + uv = -a$$

$$\Rightarrow \quad u + v = -a - uv, \qquad (1)$$

$$uv + uv^2 + u^2v = b \qquad (2)$$

and $$u^2v^2 = -\text{c} \qquad (3)$$

From (2) $$b = uv + uv^2 + u^2v = uv(1 + v + u)$$

$$= uv(1 - a - uv) \quad \text{(From (1))}$$

$$= (1 - a)\, uv - u^2v^2$$

$$= (1 - a)uv + c$$

$$\Rightarrow \quad (1 - a)\, uv = b - c$$

As $a \neq 1$, $uv = \dfrac{(b-c)}{1-a}$ and since, a, b, c are rational, uv is rational.

Note that $a = 1 \Rightarrow 1 + u + v + uv = 0 \Rightarrow (1 + u)(1 + v) = 0 \Rightarrow u = -1$ or $v = -1$, which is not the case.

Example 17 *Solve the cubic equation* $9x^3 - 27x^2 + 26x - 8 = 0$, *given that one of the root of this equation is double the other.*

Solution: Let the roots be α, 2α and β.

Now, $$3\alpha + \beta = -\frac{-27}{9} = 3$$

$$\Rightarrow \quad \beta = 3(1 - \alpha) \qquad (1)$$

$$2\alpha^2 + 3\alpha\beta = \frac{26}{9} \qquad (2)$$

$$2\alpha^2\beta = \frac{8}{9} \qquad (3)$$

From Eqs. (1) and (2), we get

$$2\alpha^2 + 3\alpha \times 3(1-\alpha) = \frac{26}{9}$$

$$\Rightarrow \quad 63\alpha^2 - 81\alpha + 26 = 0$$

$$\Rightarrow \quad (21\alpha - 13)(3\alpha - 2) = 0$$

So, $\alpha = \dfrac{13}{21}$ or $\dfrac{2}{3}$

If $\alpha = \dfrac{13}{21}$

$$\therefore\ \beta = 3\left(1-\frac{13}{21}\right)=\frac{24}{21}=\frac{8}{7}$$

This leads to $2\alpha^2\beta = 2\times\frac{169}{441}\times\frac{8}{7}\neq\frac{8}{9}$ (a contradiction)

So, taking $\alpha=\frac{2}{3},\ \ \beta=3\left(1-\frac{2}{3}\right)=3\times\frac{1}{3}=1$

Hence, $\alpha+2\alpha+\beta=\frac{2}{3}+\frac{4}{3}+1=3,$

$$2\alpha^2+3\alpha\beta=2\times\frac{4}{9}+\frac{3\times 2}{3}\times 1=\frac{26}{9},$$

and $$2\alpha^2\beta=2\times\frac{4}{9}\times 1=\frac{8}{9}$$

Thus, the roots are $\frac{2}{3}$, $\frac{4}{3}$, and 1.

Example 18 *Solve the equation* $6x^3 - 11x^2 + 6x - 1 - 0$, *given that the roots are in harmonic progression.*

Solution: Let the roots be α, β and γ.

Since they are in HP,

$$\therefore\qquad \beta=\frac{2\alpha\gamma}{\alpha+\gamma} \tag{1}$$

Now,
$$\sigma_1=\alpha+\beta+\gamma=\frac{11}{6} \tag{2}$$

$$\sigma_2=\beta(\alpha+\gamma)+\alpha\gamma=1 \tag{3}$$

$$\sigma_3=\alpha\beta\gamma=\frac{1}{6} \tag{4}$$

Using Eqs. (1) and (3), we get

$$\frac{2\alpha\gamma}{(\alpha+\gamma)}\times(\alpha+\gamma)+\alpha\gamma=1$$

$$\Rightarrow\quad 3\alpha\gamma=1$$

$$\Rightarrow\quad \alpha\gamma=\frac{1}{3} \tag{5}$$

From Eqs. (4) and (5), we get

$$\beta=\frac{1}{6}\div\frac{1}{3}=\frac{1}{2} \tag{6}$$

From Eqs. (2) and (6), we get

$$\alpha+\gamma=\frac{11}{6}-\frac{1}{2}=\frac{8}{6}=\frac{4}{3}$$

$$\therefore\qquad \alpha=\frac{4}{3}-\gamma$$

$$\therefore \qquad \alpha \times \gamma = \frac{1}{3} \quad \Rightarrow \quad \left(\frac{4}{3} - \gamma\right)\gamma = \frac{1}{3}$$

$$\gamma^2 - \frac{4}{3}\gamma + \frac{1}{3} = 0$$

$$3\gamma^2 - 4\gamma + 1 = 0$$

$$(3\gamma - 1)(\gamma - 1) = 0$$

$$\gamma = \frac{1}{3} \quad \text{or} \quad \gamma = 1$$

Hence, $\alpha = 1$ or $\alpha = \frac{1}{3}$.

Thus, the roots are $1, \frac{1}{2}, \frac{1}{3}$ or $\frac{1}{3}, \frac{1}{2}, 1$.

Example 19 *If the product of two roots of the equation* $4x^4 - 24x^3 + 31x^2 + 6x - 8 = 0$ *is* 1, *find all the roots.*

Solution: Suppose, the roots are $\alpha, \beta, \gamma, \delta$ and $\alpha\beta = 1$.

Now,
$$\sigma_1 = (\alpha + \beta) + (\gamma + \delta) = -\frac{-24}{4} = 6 \qquad (1)$$

$$\sigma_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{31}{4}$$

$$\Rightarrow \quad (\alpha + \beta)(\gamma + \delta) + \gamma\delta = \frac{31}{4} - 1 = \frac{27}{4} \qquad (2)$$

$$\sigma_3 = \gamma\delta(\alpha + \beta) + \alpha\beta(\gamma + \beta) = \frac{-3}{2}$$

$$\Rightarrow \quad \gamma\delta(\alpha + \beta) + (\gamma + \delta) = \frac{-3}{2} \qquad (3)$$

$$\sigma_4 = \alpha\beta\gamma\delta = -2$$

$$\Rightarrow \quad \gamma\delta = -2 \qquad (4)$$

From Eqs. (2) and (4), we get

$$(\alpha + \beta)(\gamma + \delta) = \frac{35}{4} \qquad (5)$$

From Eqs. (3) and (4), we get

$$-2(\alpha + \beta) + (\gamma + \delta) = \frac{-3}{2} \qquad (6)$$

From Eqs. (1) and (6), we get

$$3(\alpha + \beta) = \frac{15}{2}$$

or
$$\alpha + \beta = \frac{5}{2} \qquad (7)$$

and
$$\alpha\beta = 1$$

$$\Rightarrow \quad \beta = \frac{1}{\alpha}$$

Putting the value of β in Eq. (7), we get

$$\alpha + \frac{1}{\alpha} = \frac{5}{2}$$
$$\Rightarrow \quad 2\alpha^2 - 5\alpha + 2 = 0$$
$$\Rightarrow \quad (2\alpha - 1)(\alpha - 2) = 0$$
$$\Rightarrow \quad \alpha = \frac{1}{2} \quad \text{or} \quad \alpha = 2$$

Hence, $\beta = 2$ or $\beta = \frac{1}{2}$.

Taking $\alpha = \frac{1}{2}$ and $\beta = 2$, and substituting in Eq. (5), we get $\gamma + \delta = \frac{7}{2}$. We know that $\gamma\delta = -2$.

Again, solving for γ and δ, we get

$$\gamma = \frac{-1}{2} \text{ and } \delta = 4 \text{ or } \delta = \frac{-1}{2} \text{ and } \gamma = 4$$

Thus, the four roots are $\frac{1}{2}, \frac{-1}{2}, 2$, and 4.

Example 20 *One root of the equation* $x^4 - 5x^3 + ax^2 + bx + c = 0$ is $3 + \sqrt{2}$. *If all the roots of the equation are real, find extremum values of a, b, c; given that a, b and c are rational.*

Solution: Since the coefficients are rational, where $3 + \sqrt{2}$ is a root, so $3 - \sqrt{2}$ is also a root.

Thus, if the other two roots are α and β, we have

$$\sigma_1 = \alpha + \beta + 3 + \sqrt{2} + 3 - \sqrt{2} = -(-5) = 5$$
$$\Rightarrow \quad \alpha + \beta = -1$$
$$\sigma_2 = (\alpha + \beta)(3 + \sqrt{2} + 3 - \sqrt{2}) + \alpha\beta + (3 + \sqrt{2})(3 - \sqrt{2}) = a$$

or $$6(\alpha + \beta) + \alpha\beta + 7 = a$$

or $$\alpha\beta = a - 1$$

$$\sigma_3 = \alpha\beta(3 + \sqrt{2} + 3 - \sqrt{2}) + (3 + \sqrt{2})(3 - \sqrt{2})(\alpha + \beta)$$
$$= -b$$
$$= 6\alpha\beta + 7\,(-1) = -b$$

or $$\alpha\beta = \frac{7 - b}{6}$$

$$\sigma_4 = 7\alpha\beta = c$$
$$\Rightarrow \quad \alpha\beta = \frac{c}{7}$$

Since, we are interested in finding a, b and c, we take $\alpha + \beta = -1$, $\alpha\beta = k$. α and β are the roots of $x^2 + x + k = 0$.

Since the roots of the given equation are real and hence, the roots of above equation are real, if

$$D \geq 0 \Rightarrow 1-4k \geq 0$$

$$\text{or,} \quad k \leq \frac{1}{4}$$

Now for a, $k = a - 1$

$$\Rightarrow a-1 \leq \frac{1}{4}$$

$$\Rightarrow a \leq \frac{5}{4}$$

So, the greatest value of a is $\frac{5}{4}$.

For b, $k = \frac{7-b}{6}$

$\Rightarrow \frac{7-b}{6} \leq \frac{1}{4}$

$\Rightarrow b \geq 7 - \frac{3}{2}$

$\Rightarrow b \geq \frac{11}{2}$

So, least value of b will be $\frac{11}{2}$ and for c, take $k = \frac{c}{7}$

$\Rightarrow \frac{c}{7} \leq \frac{1}{4}$

$\Rightarrow c \leq \frac{7}{4}$

So, maximum value of c will be $\frac{7}{4}$

For these extremum values of a, b and c, the equation becomes

$$x^4 - 5x^3 + \frac{5}{4}x^2 + \frac{11}{2}x + \frac{7}{4} = 0$$

The four roots of this equation are

$$3+\sqrt{2}, 3-\sqrt{2}, \frac{-1}{2}, \frac{-1}{2} \qquad \text{(verify this)}$$

Build-up Your Understanding 2

1. Find the rational roots of $x^4 - 4x^3 + 6x^2 - 4x + 1 = 0$.
2. Solve the equation $x^4 + 10x^3 + 35x^2 + 50x + 24 = 0$, if sum of two of its roots is equal to sum of the other two roots.
3. Find the rational roots of $6x^4 + x^3 - 3x^2 - 9x - 4 = 0$.
4. Find the rational roots of $6x^4 + 35x^3 + 62x^2 + 35x + 2 = 0$.
5. Given that the sum of two of the roots of $4x^3 + ax^2 - x + b = 0$ is zero, where $a, b \in \mathbb{Q}$. Solve the equation for all values of a and b.
6. Find all a, b, such that the roots of $x^3 + ax^2 + bx - 8 = 0$ are real and in G.P.
7. Show that $2x^6 + 12x^5 + 30x^4 + 60x^3 + 80x^2 + 30x + 45 = 0$ has no real roots.

8. Construct a polynomial equation, of the least degree with rational coefficients, one of whose roots is sin 10°.
9. Construct a polynomial equation of the least degree with rational coefficients, one of whose roots is sin 20°.
10. Construct a polynomial equation of the least degree, with rational coefficients, one of whose roots is (*a*) cos 10° (*b*) cos 20°.
11. Construct a polynomial equation of the least degree with rational coefficient, one of whose roots is (*a*) tan 10° (*b*) tan 20°.
12. Construct a polynomial equation with rational coefficients, two of whose roots are sin 10° and cos 20°.
13. If p, q, r are the real roots of $x^3 - 6x^2 + 3x + 1 = 0$, determine the possible values of $p^2q + q^2r + r^2p$.
14. The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k.

[USA MO, 1984]

1.7 Symmetric Functions

The following expressions are examples of symmetric functions:

(i) $\alpha + \beta + \gamma$
(ii) $\alpha^2 + \beta^2 + \gamma^2$
(iii) $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$
(iv) $(\alpha + \beta)\alpha\beta + (\beta + \gamma)\beta\gamma + (\gamma + \alpha)\gamma\alpha$
(v) $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$

In the above expressions, you can easily verify that if any two of the variables α, β, and γ are interchanged, the expression remains unaltered. Such functions are called *symmetric functions*.

In general, a function $f(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ of n variables is said to be a symmetric function if it remains unaltered by interchanging any two of the n variables.

Thus, $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ of the previous section are symmetric functions of $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

The functions $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ are called *elementary symmetric functions.*

It can be proved that every rational symmetric function of the roots of a polynomial equation can be expressed in terms of the elementary symmetric functions and coefficients of the polynomial.

Example 21 *If $x + y = 1$ and $x^4 + y^4 = c$, find $x^3 + y^3$ and $x^2 + y^2$ in terms of c.*

Solution: We have, $x + y = 1$

$$\Rightarrow \quad x^2 + y^2 = 1 - 2xy$$

and,

$$(x^2 + y^2)^2 = (1 - 2xy)^2$$

$$\Rightarrow \quad x^4 + y^4 = 1 + 4x^2y^2 - 4xy - 2x^2y^2$$

$$= 2x^2y^2 - 4xy + 1 \quad (1)$$

but, $x^4 + y^4 = c$

So Eq. (1) becomes $2x^2y^2 - 4xy + 1 - c = 0$

So, $$xy = \frac{4 \pm \sqrt{16 + 8c - 8}}{4}$$

$$= \frac{4 \pm \sqrt{8 + 8c}}{4} \quad \text{or} \quad 1 \pm \frac{1}{\sqrt{2}}\sqrt{(1+c)}$$

and hence, $$x^2 + y^2 = 1 - 2\left(1 \pm \frac{1}{\sqrt{2}}\sqrt{(1+c)}\right)$$

$$= -1 \pm \sqrt{2(1+c)}$$

For $$x^3 + y^3 = (x+y)^3 - 3xy(x+y)$$

$$= 1 - 3xy \ (\because x + y = 1)$$

$$= 1 - 3 \times \left[\frac{2 \pm \sqrt{2 + 2c}}{2}\right]$$

$$= \frac{2 - 6 \pm 3\sqrt{2 + 2c}}{2}$$

$$= -2 \pm \frac{3}{2}\sqrt{2 + 2c}$$

Example 22 *Find, all real x, y that satisfy* $x^3 + y^3 = 7$ *and* $x^2 + y^2 + x + y + xy = 4$.

Solution: Let, $x + y = \alpha$ and $xy = \beta$ and hence, $x^2 + y^2 = \alpha^2 - 2\beta$.

Now, $$(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$$

$$= \alpha(\alpha^2 - 3\beta) = 7$$

$$= \alpha^3 - 3\alpha\beta = 7 \tag{1}$$

And, $$x^2 + y^2 + x + y + xy = 4$$

$$\Rightarrow \alpha^2 - 2\beta + \alpha + \beta = 4$$

$$\Rightarrow \alpha^2 - \beta + \alpha = 4$$

$$\Rightarrow \beta = \alpha^2 + \alpha - 4 \tag{2}$$

From Eqs. (1) and (2), we have

$$\alpha^3 - 3\alpha(\alpha^2 + \alpha - 4) = 7 \tag{3}$$

$$\Rightarrow f(\alpha) = 2\alpha^3 + 3\alpha^2 - 12\alpha + 7 = 0$$

$$f(1) = 2 + 3 - 12 + 7 = 0$$

and hence, $(\alpha - 1)$ is a factor.

So, $$f(\alpha) = 2\alpha^3 + 3\alpha^2 - 12\alpha + 7 = 0$$

$$\Rightarrow (\alpha - 1)(2\alpha^2 + 5\alpha - 7) = 0$$

$$\Rightarrow (\alpha - 1)(\alpha - 1)(2\alpha + 7) = 0$$

So, $$\alpha = 1 \text{ or } \alpha = \frac{-7}{2}$$

When $\alpha = 1$, then $\beta = -2$ and when, $\alpha = \frac{-7}{2}, \beta = \frac{19}{4}$.

If we take $\alpha = 1$ and $\beta = -2$, then x and y are the roots of

$$t^2 + t - 2 - 0$$

$$\Rightarrow (t + 2)(t - 1) = 0$$

$$\Rightarrow t = -2 \quad \text{and} \quad t = 1$$

i.e., $x = -2$ and $y = 1$ or $x = 1$ and $y = -2$.

If we take $\alpha = \dfrac{-7}{2}$ and $\beta = \dfrac{19}{4}$, then x, y are the roots of $4t^2 + 14t + 19 = 0$, and here the discriminant $14^2 - 4 \times 4 \times 19 < 0$. Hence, there are no real roots.

Thus, the real values of x, y satisfying the given equations are $(2, -1)$ or $(-1, 2)$.

Example 23 *If α, β, γ, are the roots of $x^3 + px + q = 0$, then prove that*

(i) $\dfrac{\alpha^5+\beta^5+\gamma^5}{5} = \dfrac{\alpha^3+\beta^3+\gamma^3}{3} \times \dfrac{\alpha^2+\beta^2+\gamma^2}{2}$

(ii) $\dfrac{\alpha^7+\beta^7+\gamma^7}{7} = \dfrac{\alpha^5+\beta^5+\gamma^5}{5} \times \dfrac{\alpha^2+\beta^2+\gamma^2}{2}$

Solution:

(i) Since, α, β, γ, are the roots of

$$x^3 + px + q = 0. \tag{1}$$

We have,

$$\left.\begin{aligned} \alpha^3 + p\alpha + q = 0 \\ \beta^3 + p\beta + q = 0 \\ \gamma^3 + p\gamma + q = 0 \end{aligned}\right\} \tag{2}$$

From Eq. (2),

$$\Sigma\alpha^3 + p(\Sigma\alpha) + 3q = 0$$

$$\Rightarrow \quad \Sigma\alpha^3 = -3q \quad (\because \Sigma\alpha = 0) \tag{3}$$

$$\begin{aligned} \Sigma\alpha^2 &= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta \\ &= 0^2 - 2 \times p \quad (\because \ \Sigma\alpha\beta = p) \\ &= -2p \end{aligned} \tag{4}$$

Multiplying Eq. (1) by x^2, we get

$$x^5 + px^3 + qx^2 = 0 \tag{5}$$

and α, β, γ are three roots of Eq. (5).
So,

$$\left.\begin{aligned} \alpha^5 + p\alpha^3 + q\alpha^2 = 0 \\ \beta^5 + p\beta^3 + q\beta^2 = 0 \\ \gamma^5 + p\gamma^3 + q\gamma^2 = 0 \end{aligned}\right\} \tag{6}$$

From Eq. (6),

$$\begin{aligned} \Sigma\alpha^5 + p\Sigma\alpha^3 + q\Sigma\alpha^2 &= 0 \\ \Sigma\alpha^5 &= -(p\Sigma\alpha^3 + q\Sigma\alpha^2) \\ &= -[p\,(-3q) + q\,(-2p)] \\ &= 3pq + 2pq = 5pq \end{aligned} \tag{7}$$

or

$$\begin{aligned} \frac{1}{5}\Sigma\alpha^5 &= pq \\ &= \left(-\frac{1}{2} \times \Sigma\alpha^2\right)\left(-\frac{1}{3}\Sigma\alpha^3\right) \end{aligned}$$

$$= \left[\frac{1}{3}\Sigma\alpha^3\right]\left[\frac{1}{2}\Sigma\alpha^2\right]$$

$$\frac{\alpha^5+\beta^5+\gamma^5}{5} = \left(\frac{\alpha^3+\beta^3+\gamma^3}{3}\right)\times\left(\frac{\alpha^2+\beta^2+\gamma^2}{2}\right) \quad (8)$$

(ii) Multiplying Eq. (1) by x, we get

$$x^4 + px^2 + qx = 0 \quad (9)$$

hence, $$\Sigma\alpha^4 + p\Sigma\alpha^2 + q\Sigma\alpha = 0$$

$$\Rightarrow \quad \Sigma\alpha^4 = -p\Sigma\alpha^2 \quad (\because \quad \Sigma\alpha = 0)$$

Again, multiplying Eq. (1) by x^4, we get

$$x^7 + px^5 + qx^4 = 0 \quad (10)$$

hence, $$\Sigma\alpha^7 = -p\Sigma\alpha^5 - q\Sigma\alpha^4 = 0$$

or $$\Sigma\alpha^7 = -p\Sigma\alpha^5 + q\Sigma\alpha^4$$

$$= -p \times 5pq - q\Sigma\alpha^4$$

$$= -p \times 5pq - q\,(-p\Sigma\alpha^2)$$

$$= -5p^2q - 2p^2q$$

$$= -7p^2q$$

or $$\frac{1}{7}\Sigma\alpha^7 = -p^2q = pq \times (-p)$$

$$= \left(\frac{1}{5}\Sigma\alpha^5\right)\times\left(\frac{1}{2}\Sigma\alpha^2\right)$$

or $$\left(\frac{\alpha^7+\beta^7+\gamma^7}{7}\right) = \left(\frac{\alpha^5+\beta^5+\gamma^5}{5}\right)\times\left(\frac{\alpha^2+\beta^2+\gamma^2}{2}\right)$$

Example 24 *If* $\alpha+\beta+\gamma = 0$, *show that*

$$3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5) = 5(a^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4)$$

Solution: Since $\alpha+\beta+\gamma = 0$; α, β and γ can be the roots of the equation

$$x^3 + px + q = 0 \quad (1)$$

$$\alpha+\beta+\gamma = 0 \quad (2)$$

$$(\alpha^2+\beta^2+\gamma^2) = (\alpha+\beta+r)^2 - 2\Sigma\alpha\beta$$

$$= 0 - 2p = -2p \quad (3)$$

and $$\Sigma\alpha^3 = 3\alpha\beta\gamma \quad (\text{as, } \alpha+\beta+\gamma = 0)$$

$$= -3q$$

Multiplying Eq. (1) by x, we get

$$x^4 + px^2 + qx = 0 \quad (4)$$

Again, α, β and γ are three of the roots of this polynomial. By substituting α, β, γ in Eq. (4), and adding, we get

$$\sum\alpha^4 + p\sum\alpha^2 + q\sum\alpha = 0$$

$$\Rightarrow \quad \Sigma\alpha^4 = -p\Sigma\alpha^2 = -p \times -2p = 2p^2$$

Similarly $\quad \Sigma\alpha^5 = -p\Sigma\alpha^3 - q\Sigma\alpha^2$

$$= 3pq + 2pq = 5pq$$

$$\therefore \quad 3\,(\alpha^2 + \beta^2 + \gamma^2)(\alpha^5 + \beta^5 + \gamma^5) = 3 \times (-2p) \times (5pq)$$
$$= 5(-3q) \times (2p^2)$$
$$= 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^4 + \beta^4 + \gamma^4)$$

Example 25 *Show that there do not exist any distinct natural numbers a, b, c, d, such that*

$$a^3 + b^3 = c^3 + d^3 \text{ and } a + b = c + d.$$

Solution: Suppose that $a^3 + b^3 = c^3 + d^3$ and $a + b = c + d$

Let, $\quad a + b = c + d = m$ (say)

$$\therefore \quad (a + b)^3 = (c + d)^3$$

$$\Rightarrow \quad 3ab(a + b) = 3cd(c + d)$$

$$\Rightarrow \quad ab = cd = n \text{ (say)}$$

If a and b are the roots of a quadratic equation, then the equation is $x^2 - mx + n = 0$
But, $\quad a + b = m$ and $ab = n$

So, a and b are the roots of this equation. For similar reasons, c and d are also the roots of the same equation.

But, a quadratic equation can have at most two distinct roots.

Hence, either $a = c$ or $a = d$, so that b may be one of c or d.

Example 26 *Determine all the roots of the system of simultaneous equations* $x + y + z = 3$, $x^2 + y^2 + z^2 = 3$ *and* $x^3 + y^3 + z^3 = 3$.

Solution: Let, x, y, z be the roots of the cubic equation

$$t^3 - at^2 + bt - c = 0 \quad (1)$$
$$\sigma_1 = x + y + z = a \quad (2)$$
$$\sigma_2 = xy + yz + zx = b \quad (3)$$
$$\Rightarrow \quad 2xy + 2yz + 2zx = 2b \quad (4)$$

From Eq. (2), we get $a = 3$.

From Eqs. (2) and (3), we get

$$2b = 2xy + 2yz + 2zx = (x + y + z)^2 - (x^2 + y^2 + z^2)$$
$$= 9 - 3 = 6$$
$$\Rightarrow \quad b = 3$$

Since, x, y and z are the roots of Eq. (1), substituting and adding, we get

$$(x^3 + y^3 + z^3) - a(x^2 + y^2 + z^2) + b(x + y + z) - 3c = 0$$
$$\Rightarrow \quad 3 - 3a + 3b - 3c = 0$$
$$\Rightarrow \quad 3 - 9 + 9 - 3c = 0$$

or $c = 1$

Thus Eq. (1) becomes

$$t^3 - 3t^2 + 3t - 1 = 0$$
$$\Rightarrow \quad (t-1)^3 = 0$$

Thus, the roots are 1, 1, 1.

Hence, $x = y = z = 1$ is the only solution for the given equations.

Example 27 *Given real numbers* x, y, z, such that $x + y + z = 3$, $x^2 + y^2 + z^2 = 5$, $x^3 + y^3 + z^3 = 7$, find $x^4 + y^4 + z^4$.

Solution: We know $x^2 + y^2 + z^2 = 5$.

$$\therefore \quad 5 = x^2 + y^2 + z^2 = (x + y + z)^2 - 2xy - 2yz - 2xz$$
$$= 9 - 2(xy + yz + xz)$$
$$\Rightarrow \quad xy + yz + zx = 2.$$

We know that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)\,[x^2 + y^2 + z^2 - (xy + yz + xz)]$$
$$\Rightarrow \quad 7 - 3xyz = 3[5 - 2] = 9$$

Or,
$$xyz = \frac{-2}{3}.$$

$$x^4 + y^4 + z^4 = (x^2 + y^2 + z^2)^2 - 2[(xy)^2 + (yz)^2 + (zx)^2]$$
$$= 25 - 2[(xy + yz + zx)^2 - 2(xy^2z + yz^2x + zx^2y)]$$
$$= 25 - 2\,[2^2 - 2xyz(x + y + z)]$$
$$= 25 - 2\left[4 + \frac{4}{3} \times 3\right]$$
$$= 25 - 16 = 9$$

Build-up Your Understanding 3

1. If α and β are the roots of the equation $x^2 - (a + d)x + ad - bc = 0$, show that α^3 and β^3 are the roots of
   $y^2 - (a^3 + d^3 + 3abc + 3bcd)y + (ad - bc)^3 = 0$.
2. If $a^3 + b^3 + c^3 = (a + b + c)^3$, prove that $a^5 + b^5 + c^5 = (a + b + c)^5$. Generalize your result.
3. If p, q and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, find the value of $p^3 + q^3 + r^3$.
4. Find the sum of the 5th powers of the roots of the equation $x^3 + 3x + 9 = 0$.
5. Find the sum of the fifth powers of the roots of the equation $x^3 - 7x^2 + 4x - 3 = 0$.
6. α, β, γ are the roots of the equation $x^3 - 9x + 9 = 0$. Find the value of
   $$\alpha^{-3} + \beta^{-3} + \gamma^{-3} \text{ and } \alpha^{-5} + \beta^{-5} + \gamma^{-5}.$$
7. Form the cubic equation whose roots are α, β, γ such that
   (i) $\alpha + \beta + \gamma = 9$
   (ii) $\alpha^2 + \beta^2 + \gamma^2 = 29$
   (iii) $\alpha^3 + \beta^3 + \gamma^3 = 99$
   Hence, find the value of $(\alpha^4 + \beta^4 + \gamma^4)$.

8. If $\alpha+\beta+\gamma=4$, $\alpha^2+\beta^2+\gamma^2=7$, $\alpha^3+\beta^3+\gamma^3=28$, find $\alpha^4+\beta^4+\gamma^4$ and $\alpha^5+\beta^5+\gamma^5$.
9. Solve: $x^3+y^3+z^3=a^3$, $x^2+y^2+z^2=a^2$, $x+y+z=a$ in terms of a.
10. If α, β, γ be the roots of $2x^3+x^2+x+1=0$, show that

$$\left(\frac{1}{\beta^3}+\frac{1}{\gamma^3}-\frac{1}{\alpha^3}\right)\left(\frac{1}{\gamma^3}+\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)\left(\frac{1}{\alpha^3}+\frac{1}{\beta^3}-\frac{1}{\gamma^3}\right)=16.$$

11. Find $x, y, \in \mathbb{C}$ such that $x^5+y^5=275$, $x+y=5$.
12. Find real x such that $\sqrt[4]{97-x}+\sqrt[4]{x}=5$.

1.8 Common Roots of Polynomial Equations

A number α is a common root of the polynomial equations $f(x)=0$ and $g(x)=0$, if and only if, it is a root of the HCF of the polynomials $f(x)$ and $g(x)$.

HCF of two polynomials, $f(x)$ and $g(x)$, is a polynomial $h(x)$ of the greatest possible degree which divides both $f(x)$ and $g(x)$, exactly.

Note: The HCF of two polynomials is not unique, because $a \cdot h(x)$ is also a HCF, where $a \neq 0$ is a constant (either real or complex). The HCF of two polynomials can be found by the Euclidean algorithm.

Example 28 *Find the common roots of the polynomials x^3+x^2-2x-2 and x^3-x^2-2x+2.*

Solution: Find the HCF by using the Euclidean algorithm,

$$\begin{array}{r|l|l}
x^3+x^2-2x-2 & x^3-x^2-2x+2 & 1 \\
 & x^3+x^2-2x-2 & \\
 & (-)\ (-)\ (+)\ (+) & \\
\end{array}$$

$$\begin{array}{r|l|l}
-2x^2+4 & x^3+x^2-2x-2 & \frac{-1}{2}x \\
 & x^3 \qquad -2x & \\
 & (-) \qquad (+) & \\
\end{array}$$

$$\begin{array}{r|l|l}
x^2-2 & -2x^2+4 & -2 \\
 & -2x^2+4 & \\
 & (+) \quad (-) & \\
\end{array}$$

$$0$$

Thus, the HCF is x^2-2 and hence, the common roots of the given equations are the roots of $x^2-2=0$, *i.e.*, $\pm\sqrt{2}$.

Example 29 *Find the common roots of $x^4+5x^3-22x^2-50x+132=0$ and $x^4+x^3-20x^2+16x+24=0$, and solve the equations.*

Solution: You can see that $4(x^2-5x+6)$ is HCF of the two equations and hence, the common roots are the roots of $x^2-5x+6=0$, *i.e.*, $x=3$ or $x=2$.

Now, $$x^4 + 5x^3 - 22x^2 - 50x + 132 = 0 \quad (1)$$

and $$x^4 + x^3 - 20x^2 + 16x + 24 = 0 \quad (2)$$

have 2 and 3 as their common roots.

If the other roots of Eq. (1) are α and β, then

$$\alpha + \beta + 5 = -5,$$

$\Rightarrow \quad \alpha + \beta = -10$ from Eq. (1)

$$6\alpha\beta = 132$$

$\Rightarrow \quad \alpha\beta = 22$

So, α and β are also the roots of the quadratic equation $x^2 + 10x + 22 = 0$.

$$\therefore \quad x = \frac{-10 \pm \sqrt{100-88}}{2} = \frac{-10 \pm 2\cdot\sqrt{3}}{2} = -5 \pm \sqrt{3}$$

So, the roots of Eq. (1) are 2, 3, $(-5+\sqrt{3})$, $(-5-\sqrt{3})$.

For Eq. (2), if α_1 and β_1 be the roots of Eq. (2), then we have

$$\alpha_1 + \beta_1 + 5 = -1$$

$$\alpha_1 + \beta_1 = -6$$

$$6\alpha_1\beta_1 = 24 \quad \text{or} \quad \alpha_1\beta_1 = 4$$

So, α_1 and β_1 are the roots of

$$x^2 + 6x + 4 = 0$$

$$x = \frac{-6 \pm \sqrt{36-16}}{2} = -3 \pm \sqrt{5}$$

So, the roots of Eq. (2) are 2, 3, $-3 + \sqrt{5}$, $-3 - \sqrt{5}$.

Example 30 *Show that the set of polynomials*

$$P = \{p_k(x) \colon p_k(x) = x^{5k+4} + x^3 + x^2 + x + 1, \quad k \in \mathbb{N}\}$$

has a common non-trivial polynomial divisor.

Solution: If $k = 1$

$$\begin{aligned} p_1(x) &= x^9 + x^3 + x^2 + x + 1 \\ &= x^9 - x^4 + x^4 + x^3 + x^2 + x + 1 \\ &= x^4(x^5 - 1) + (x^4 + x^3 + x^2 + x + 1) \\ &= x^4(x-1)(x^4 + x^3 + x^2 + x + 1) + (x^4 + x^3 + x^2 + x + 1) \\ &= (x^4 + x^3 + x^2 + x + 1)[x^4(x-1) + 1] \end{aligned}$$

Thus, $x^4 + x^3 + x^2 + x + 1$ is a non-trivial polynomial divisor of $p_1(x)$.

$$\begin{aligned} p_k(x) &= x^{(5k+4)} - x^4 + (x^4 + x^3 + x^2 + x + 1) \\ &= x^4[x^{5k} - 1] + (x^4 + x^3 + x^2 + x + 1) \end{aligned}$$

$(x^5 - 1)$ divides $(x^5)^k - 1$, $x^4 + x^3 + x^2 + x + 1$ divides $x^5 - 1$ and hence, $x^{5k} - 1$.

Therefore, $x^4 + x^3 + x^2 + x + 1$ divides $P_k(x)$ for all k.

Build-up Your Understanding 4

1. Find the common roots of the equations
$$x^3 - 3x^2 - 4x + 12 = 0 \text{ and } x^3 + 9x^2 + 26x + 24 = 0.$$
2. Find the common roots of the equations
$$x^4 - 5x^3 + 2x^2 + 20x - 24 = 0 \quad \text{and}$$
$$x^4 + 7x^3 + 8x^2 - 28x - 48 = 0.$$
3. If d, e, f are in GP and the two quadratic equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then prove that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in HP.
4. If n is an even and α, β, are the roots of the equation $x^2 + px + q = 0$ and also of the equation $x^{2n} + p^n x^n + q^n = 0$ and $f(x) = \frac{(1+x)^n}{1+x^n}$ where $\alpha^n + \beta^n \neq 0$, $p \neq 0$, find the value of $f\left(\frac{\alpha}{\beta}\right)$.

1.9 Irreducibility of Polynomials

An irreducible polynomial is, a non-constant polynomial that cannot be factored into the product of two non-constant polynomials. The property of irreducibility depends on the set (usually we take $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$) to which the coefficients are considered to belong.

A polynomial that is not irreducible over a set is said to be reducible over the set.

Observe the following illustrations to understand reducible and irreducible polynomials over the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

1. $p_1(x) = 6x^2 - 19x + 15 = (2x-3)(3x-5)$
2. $p_2(x) = x^2 - \frac{16}{25} = \left(x - \frac{4}{5}\right)\left(x + \frac{4}{5}\right)$,
3. $p_3(x) = x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$,
4. $p_4(x) = x^2 + 4 = (x + 2i)(x - 2i)$

Over the integers, only first polynomial is reducible the last two are irreducible. The second is not a polynomial over the integers).

Over the rational numbers, the first two polynomials are reducible, but the other two are irreducible

Over the real numbers, the first three polynomials are reducible, but last one is reducible.

Over the complex numbers, all four polynomials are reducible.

Example 31 *Factorize $x^4 + 4$ as a product of irreducible polynomials over each of the following sets:*

(i) $\mathbb{Q}$

(ii) $\mathbb{R}$

(iii) $\mathbb{C}$

Solution:

(i) Over $\mathbb{Q}$:

$$x^4+4=x^4+4x^2+4-4x^2=(x^2+2)^2-(2x)^2=(x^2+2x+2)(x^2-2x+2)$$

(ii) Over $\mathbb{R}$:
It is same as in $\mathbb{Q}$,
i.e., $x^4+4=(x^2+2x+2)(x^2-2x+2)$

(iii) Over $\mathbb{C}$:

We need further factorization of x^2+2x+2 and x^2-2x+2, for this let us solve

$$x^2+2x+2=0 \Rightarrow x=\frac{-2\pm\sqrt{2^2-4\times1\times2}}{2} \Rightarrow x=-1\pm i$$

$$\text{And } x^2-2x+2=0 \Rightarrow x=\frac{2\pm\sqrt{2^2-4\times1\times2}}{2} \Rightarrow x=1\pm i$$

Hence, $x^2+4=(x-(-1+i))(x-(-1-i))(x-(1+i))(x-(1-i))$.

Example 32 *Check whether following polynomials are reducible or irreducible over* $\mathbb{Z}$.

(i) x^4+x^3-x-1

(ii) x^3+x^2+x+3

Solution:

(i) $x^4+x^3-x-1=x^3(x+1)-(x+1)=(x+1)(x^3-1)=(x+1)(x-1)(x^2+x+1)$
Hence it is reducible over $\mathbb{Z}$.

(ii) As it is a cubic, if this is reducible then it would have to have a linear factor $x-\alpha$, hence a root ($\alpha\in\mathbb{Z}$). But by integer root theorem α would have to be an integer divisor of constant 3, hence would have to be 1, –1, 3 or –3. By direct checking we see that none of these is a root, and hence the polynomial is irreducible.

Example 33 *Show that* x^4+x^3-x+1 *is irreducible over* $\mathbb{Z}$.

Solution: As in previous example here also if there were a linear factor then there would be an integer root which, since it would have to divide the constant term, could be only ±1, but clearly neither of these is a root; hence no linear factor.

To determine whether it factorizes as the product of two quadratics, let us try:

$$x^4+x^3-x+1=(x^2+ax+b)(x^2+cx+d)$$

Now by equating coefficients, we get $a+c=1$, $b+ac+d=0$, $ad+bc=-1$, $bd=1$. Bearing in mind that a, b, c, d all are integers, we have either

$$b=d=1 \text{ or } b=d=-1.$$

In the first case the other equations become $a+c=0$, $ac=-2$, $a+c=-1$ which is impossible.

And in the second case we obtain $a+c=1$, $ac=2$ which has no integer solution. Thus there is no factorization, and the polynomial is irreducible.

Example 34 *Prove that if the integer 'a' is not divisible by 5, then* $f(x)=x^5-x+a$ *cannot be factored as the product of two non-constant polynomials with integer coefficients.*

Solution: Suppose f can be factored, then

$$f(x) = (x-n)g(x) \text{ or } f(x) = (x^2 - bx + c)\ g(x)$$

In the former case, $f(n) = n^5 - n + a = 0$. Now $n^5 \equiv n \pmod 5$ by Fermat's little theorem $\Rightarrow 5 \mid (b - b^5) = a$, a contradiction.

In the later case, dividing $f(x) = x^5 - x + a$ by $x^2 - bx + c$, we get the remainder $(b^4 + 3b^2c + c^2 - 1)x + (b^3c + 2bc^2 + a)$. Since $x^2 - bx + c$ is a factor of $f(x)$, both coefficients of remainder equal to 0.

That is,

$$b^4 + 3b^2c + c^2 - 1 = 0 \tag{1}$$

and

$$b^3c + 2bc^2 + a = 0 \tag{2}$$

Now $b(1) - 3(2)$ gives

$b(b^4 + 3b^2c + c^2 - 1) - 3(b^3c + 2bc^2 + a) = b^5 - b - 5bc^2 - 3a = 0$

$\Rightarrow 3a = b^5 - b - 5bc^2$ is divisible by 5.

$\Rightarrow 5 \mid a$ which is a contradiction.

Example 35 *Let* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *be a polynomial with integer coefficients, such that* $|a_0|$ *is prime and* $|a_0| > |a_1| + |a_2| + \cdots + |a_n|$. *Prove that* $f(x)$ *is irreducible over* $\mathbb{Z}$.

Solution: Let α be any complex zero of f.

Case 1: Consider $|\alpha| \le 1$, then $|a_0| = |a_1\alpha + \cdots + a_n\alpha^n| \le |a_1| + \cdots + |a_n|$, which is a contradiction.

Case 2: Therefore, all the zeros of f satisfies $|\alpha| > 1$.

Now, let us assume that $f(x) = g(x)h(x)$, where g and h are non-constant integer polynomials. Then $a_0 = f(0) = g(0)h(0)$. Since $|a_0|$ is a prime, one of $|g(0)|, |h(0)|$ equals 1. Say $|g(0)| = 1$, and let b be the leading coefficient of g.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the roots of g, then $|\alpha_1\alpha_2\ldots\alpha_k| = \dfrac{1}{|b|} \le 1$. (As $b \in \mathbb{Z}\setminus\{0\} \Rightarrow |b| \ge 1$)

But, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are also zeros of f, and from case 1 we have magnitude of each α_i greater than 1.

$\Rightarrow |\alpha_1\alpha_2\ldots\alpha_k| > 1$. Which is a contradiction.

Hence, f is irreducible.

Note: If a polynomial has integer coefficients, then the concepts of (ir) reducibility over the integers and over the rationals are equivalent. This is true because of a Lemma by Gauss.

Johann Carl Friedrich Gauss

30 Apr 1777–23 Feb 1855
Nationality: German

1.9.1 Gauss Lemma

If a polynomial with integer coefficients is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$.

The following theorem is very useful for deciding irreducibility of some integer polynomials over $\mathbb{Z}$.

1.9.2 Eisenstein's Irreducibility Criterion Theorem

Ferdinand Gotthold Max Eisenstein

16 Apr 1823–11 Oct 1852
Nationality: German

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients and there exist a prime p such that,

1. $p \mid a_i$ for $0 \le i \le n-1$,
2. $p \nmid a_n$
3. $p^2 \nmid a_0$.

Then $f(x)$ is irreducible over the integers.

Proof: If possible let us assume $f(x) = g(x)\cdot h(x)$, such that

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + b_1 x + b_0,$$

$$h(x) = c_k x^k + c_{k-1}x^{k-1} + c_1 x + c_0,$$

where $b_i,\ c_i \in \mathbb{Z}\ \forall i = 0, 1, 2, \ldots;\ \ b_m \ne 0,\ c_k \ne 0;\ \ 1 \le m,\ k \le n-1.$

Comparing leading coefficient on both side we get $a_n = b_m\, c_k$.

As $p \nmid a_n \Rightarrow p \nmid b_m c_k \Rightarrow p \nmid b_m$ and $p \nmid c_k$.

Comparing constant term on both side we get $a_0 = b_0\, c_0$. As $p \mid a_0$ and $p^2 \nmid a_0$ $\Rightarrow p \mid b_0 c_0$ but p cannot divide both b_0 and c_0. Without loss of generality, suppose that $p \mid b_0$ and $p \nmid c_0$. Suppose i be the smallest index such that b_i is not divisible by p. There is such an index i since $p \nmid b_m$ where $1 \le i \le m$. Depending upon i viz a viz k we have following two cases:

Case 1: for $i \le k$, $a_i = b_i c_0 + b_i c_1 + \cdots + b_0 c_i$

Case 2: for $i > k$, $a_i = b_i c_0 + b_i c_1 + \cdots + b_{i-m} c_m$

We have $p \mid a_i$ and by supposition p divides each one of $b_0, b_1, \ldots, b_{i-1} \Rightarrow p \mid b_i c_0$.

But $p \nmid c_0 \Rightarrow p \mid b_i$, which is a contradiction. Therefore $f(x)$ is irreducible.

Example 36 *Prove that* $16x^3 - 35x^2 + 105x + 175$ *is irreducible over* $\mathbb{Z}$.

Solution: This is irreducible by Eisenstein's Criterion with the prime p being taken to be 7: for 7 does not divide the leading coefficient but it divides all the others, and its square, 49, does not divide 175. Note that using the prime 5 is not valid since 5^2 does divide the constant coefficient 175.

Example 37 *Prove that* $x^3 - 3x^2 + 3x + 22$ *is irreducible over* $\mathbb{Z}$.

Solution: Let $f(x) = x^3 - 3x^2 + 3x + 22$. Eisenstein Criterion does not apply since there is no suitable prime. Substituting $x - 1$ for x gives the polynomial $x^3 - 6x^2 + 6x + 21$ to which Eisenstein does apply, with $p = 3$. Writing $f(x)$ for the original polynomial, we deduce that $f(x - 1)$ is irreducible. But a factorization of $f(x)$ would give a factorization of $f(x - 1)$, hence $f(x)$ is irreducible over $\mathbb{Z}$.

Example 38 *Let p be a prime number. Show that* $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ *is irreducible.*

Solution: The given polynomial is called pth **Cyclotomic polynomial**

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1 = \frac{x^p - 1}{x - 1}$$

Consider $\Phi_p(x+1) = \dfrac{(x+1)^p - 1}{(x+1)-1} = \dfrac{x^p + \binom{p}{1}x^{p-1} + \binom{p}{2}x^{p-2} + \cdots + \binom{p}{p-2}x^2 + \binom{p}{p-1}x}{x}$

$$= x^{p-1} + \binom{p}{1}x^{p-2} + \binom{p}{2}x^{p-3} + \cdots + \binom{p}{p-2}x + \binom{p}{p-1}$$

As $p \mid \binom{p}{i} \ \forall\ i = 1, 2, 3, \ldots, p-1$, so all the lower coefficients are divisible by p, and the constant coefficient is exactly p, so is not divisible by p^2. Thus, Eisenstein's criterion applies, and $\Phi_p(x+1)$ is irreducible. Certainly if $\Phi_p(x) = g(x)h(x)$ then $\Phi_p(x+1) = g(x+1)h(x+1)$ gives a factorization of $\Phi_p(x+1)$. Thus, Φ_p has no proper factorization.

1.9.3 Extended Eisenstein's Irreducibility Criterion Theorem

Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with integer coefficient. If there exist a prime number p and an integer $k \in \{0, 1, \ldots, n-1\}$ such that $p \mid a_0, a_1, \ldots, a_k$; $p \nmid a_{k+1}$ and $p^2 \nmid a_0$, then $f(x)$ has an irreducible factor of a degree at least $k+1$.

In particular, if p can be taken so that $k = n-1$, then $f(x)$ is irreducible.

Proof: Like in the proof of Eisenstein's irreducibility criterion, suppose that $f(x) = g(x) \cdot h(x)$ such that

$$g(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0,$$
$$h(x) = c_r x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0,$$

where $b_i, c_i \in \mathbb{Z}\ \forall i = 0, 1, 2, \ldots;\ b_k \neq 0, c_r \neq 0;\ 1 \leq m, r \leq n-1$.

Since $a_0 = b_0 c_0$ is divisible by p and not by p^2, exactly one of b_0, c_0 is a multiple of p. without loss of generality assume that $p \mid b_0$ and $p \nmid c_0$.

Now, $p \mid a_1 = b_0 c_1 + b_1 c_0 \Rightarrow p \mid b_1 c_o \Rightarrow p \mid b_1$.

Similarly, $p \mid a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0 \Rightarrow p \mid b_2 c_o \Rightarrow p \mid b_2$ and so on.

We conclude that all coefficients $b_0, b_1, \ldots, b_k$ are divisible by p,

Now, $a_{k+1} = b_k c_1 + b_{k-1}c_2 + b_{k-2}c_2 + \cdots \Rightarrow p \mid a_{k+1}$ but $p \nmid a_{k+1}$. It follows that degree of $g \geq k+1$.

Example 39 Let, $f(x) = x^n + 5x^{n-1} + 3$, $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as a product of two polynomials, each of which has all its coefficient integers and degree at least 1. **[IMO, 1993]**

Solution: Rewrite the given polynomial as

$$f(x) = x^n + 5x^{n-1} + 0 \cdot x^{n-2} + 0 \cdot x^{n-3} + \cdots + 0 \cdot x + 3.$$

Now take prime $p = 3$, obviously $3 \mid a_i \forall i = 0, 1, 2, \ldots, n-2;\ 3^2 \nmid a_0 = 3,\ 3 \nmid a_{n-1} = 5$.

Hence by the extended Eisenstein criterion, f has an irreducible factor of degree at least $n - 1$.

If possible, let us take one factor of degree $n - 1$ then other must be linear and monic (as f is monic) this implies f has integral roots. By integer root theorem this root must be an integer divisor of constant 3, hence would have to be 1, –1, 3 or –3. By direct checking we see that none of these is a root, and hence the polynomial is irreducible.

Build-up Your Understanding 5

1. Prove that for any prime p, polynomial, $x^n - p$ is irreducible over $\mathbb{Z}$.
2. Prove that $x^7 + 48x - 24$ is irreducible over $\mathbb{Z}$.
3. Prove that $x^4 + 2x^2 + 2x + 2$ is not the product of two polynomials $x^2 + ax + b$ and $x^2 + cx + d$ where a, b, c, d are integers.
4. Prove that $x^5 - 36x^4 + 6x^3 + 30x^2 + 24$ is irreducible over $\mathbb{Z}$.
5. Prove that $x^3 + 3x^2 + 5x + 5$ is irreducible over $\mathbb{Z}$.
6. Prove that $x^6 + 5x^2 + 8$ is reducible over $\mathbb{Z}$.
7. Prove that if $x^p + px + p - 1$ is reducible for some prime of then p must be '2'.
8. Let $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is polynomial over $\mathbb{Z}$ and irreducible over it. Prove that $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ is also irreducible over $\mathbb{Z}$ and use this to show that $21x^5 - 49x^3 + 14x^2 - 4$ is irreducible over $\mathbb{Z}$.
9. If $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ are distinct, then prove that $(x-a_1)(x-a_2)\ldots(x-a_n)-1$ is irreducible over $\mathbb{Z}$.
10. Prove that $1 + x^p + x^{2p} + \cdots + x^{(p-1)p}$ is irreducible over $\mathbb{Z}$.

Solved Problems

Note: In solving some problems, you may have to use simple trigonometric identities. These formulae would be given wherever they are used in solving problems, and also given in appendix.

Problem 1 *Solve for x: $2p(p-2)x = (p-2)$.*

Solution:
$$2p(p-2)x = (p-2) \quad (1)$$
$$\Rightarrow \quad x = \frac{(p-2)}{2p(p-2)} \quad (2)$$

If $p = 0$ or 2, the above Eq. (2) is undefined.

However, if $p = 0$, then Eq. (1) becomes $0 = -2$, which is inconsistent. Hence, no value of x will satisfy Eq. (1), and there is no solution for $p = 0$.

If $p = 2$, then by Eq. (1), $0 = 0$.

Thus, every value from the domain of x will satisfy Eq. (1) and hence, there exists an infinite number of solution for Eq. (1), when $p = 2$.

If $p \neq 2$, $p \neq 0$; then Eq. (2) is well-defined and the solution is $x = \dfrac{1}{2p}$.

Aliter: $2p(p-2)x = p - 2$

$$\Rightarrow \quad 2p(p-2)x - (p-2) = 0$$
$$\Rightarrow \quad (p-2)(2px-1) = 0$$
$$\Rightarrow \quad p = 2 \text{ or } 2px = 1$$
$$\Rightarrow \quad p = 2 \text{ or } x = \frac{1}{2p}$$

Thus, $p = 2$ guarantees infinitely many values for x, where $p = 2$ is itself sufficient to get $(p - 2)(2px - 1) = 0$ and if, $p \neq 2$, $x = \dfrac{1}{2p}$ must be true and hence, $p = 0$ does not satisfy.

Problem 2 *If x_1 and x_2 are non-zero roots of the equation $ax^2 + bx + c = 0$, and $-ax^2 + bx + c = 0$, respectively, prove* that $\dfrac{a}{2}x^2 + bx + c = 0$ has a root between x_1 and x_2, where $a \neq 0$.

Solution: x_1 and x_2 are roots of

$$ax^2 + bx + c = 0 \tag{1}$$

and

$$-ax^2 + bx + c = 0 \tag{2}$$

We have

$$a{x_1}^2 + bx_1 + c = 0$$

and

$$-ax_2^2 + bx_2 + c = 0$$

Let,

$$f(x) = \frac{a}{2}x^2 + bx + c.$$

This,

$$f(x_1) = \frac{a}{2}{x_1}^2 + bx_1 + c \tag{3}$$

$$f(x_2) = \frac{a}{2}{x_2}^2 + bx_2 + c \tag{4}$$

Adding $\dfrac{1}{2}ax_1^2$ in Eq. (3), we get

$$f(x_1) + \frac{1}{2}ax_1^2 = ax_1^2 + bx_1 + c = 0$$

$$\Rightarrow \quad f(x_1) = -\frac{1}{2}ax_1^2 \tag{5}$$

Subtracting $\dfrac{3}{2}ax_2^2$ from Eq. (4), we get

$$f(x_2) - \frac{3}{2}ax_2^2 = -ax_2^2 + bx_2 + c = 0$$

$$\Rightarrow \quad f(x_2) = \frac{3}{2}ax_2^2.$$

Thus, $f(x_1)$ and $f(x_2)$ have opposite signs and hence, $f(x)$ must have a root between x_1 and x_2.

Problem 3 Let, *$P(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Show that there is an integer M, such that $P(n) \cdot P(n + 1) = P(M)$ for any integer n.*

Solution: Clearly, $P(n) \times P(n + 1)$ is of 4th degree in 'n' as $P(n)$ and $P(n + 1)$ are of second degree each in n, and so $P(n) \times P(n + 1)$ will be a polynomial of 4th degree in n with leading coefficient 1.

So, if there exists an M, so that $P(M) = P(n) \times P(n + 1)$, then M must be in the form of a quadratic in n, with leading coefficient 1.

Let $M = n^2 + cn + d$, where c and d are integers.

Now,

$$P(M) = P(n^2 + cn + d)$$
$$= (n^2 + cn + d)^2 + a(n^2 + cn + d) + b$$
$$= n^4 + 2cn^3 + (c^2 + 2d + a)n^2 + (2cd + ac)n + d^2 + ad + b$$

and

$$P(n) \times P(n + 1) = (n^2 + an + b)[(n + 1)^2 + a(n + 1) + b]$$
$$= n^4 + 2\,(a + 1)n^3 + [(a + 1)^2 + (a + 2b)]n^2 + (a + 1)(a + 2b)n + b(a + b + 1)$$

Now, comparing the coefficients of n^3 and the constant terms of $P(M)$ and $P(n) \times P(n+1)$, we get

$$2c = 2(a + 1)$$
$$\Rightarrow \quad c = (a + 1)$$

and
$$d^2 + ad + b = ab + b^2 + b$$
$$\Rightarrow \quad d^2 - b^2 + ad - ab = (d - b)(d + a + b) = 0$$
$$\Rightarrow \quad d = b \text{ or } d = -(a + b)$$

Using these values of $d = b$ and $c = a + 1$, the coefficient of n^2 and n in $P(M)$ are

$$c^2 + 2d + a = (a + 1)^2 + 2b + a$$

and
$$2cd + ac = 2(a + 1)\,b + a(a + 1)$$
$$= (a + 1)(2b + a), \text{ respectively.}$$

But, these are the coefficients of n^2 and n in $P(n) \times P(n + 1)$. Thus, with these values for c and d, $P(M) = P(n) \times P(n + 1)$. So, the M of the desired property is $n^2 + (a + 1)n + b$.

Thus, we can verify that $d = -(a + b)$, $c = (a + 1)$, if $P(M)$ and $P(n) \times P(n + 1)$ are identical and hence, show that there exists exactly one M for every n which is a function of n,

i.e.,
$$M = f(n) = n^2 + (a + 1)n + b$$

Aliter: Let $P(x) = x^2 + ax + b \equiv (x - \alpha)\,(x - \beta)$, where $\alpha + \beta = -a$, $\alpha \cdot \beta = b$.

Now, $P(n)\,P(n + 1) = (n - \alpha)(n - \beta) \cdot (n + 1 - \alpha)(n + 1 - \beta)$

$$= (n - \alpha)(n + 1 - \beta)\,(n - \beta)(n + 1 - \alpha)$$
$$= (n^2 - (\alpha + \beta - 1)n + \alpha\beta - \alpha)\,(n^2 + (\alpha + \beta - 1)n + \alpha\beta - \beta)$$
$$= (n^2 + (a + 1)n + b - \alpha)\,(n^2 + (a + 1)\,n + b - \beta)$$
$$= (M - \alpha)\,(M - \beta)$$
$$= P(M) \quad \text{where } M = n^2 + (a + 1)\,n + b.$$

Problem 4 *Prove that, if the coefficients of the quadratic equation $ax^2 + bx + c = 0$ are odd integers, and then the roots of the equation cannot be rational numbers.*

Solution: Let there be a rational root $\dfrac{p}{q}$, where $(p, q) = 1$. Then,

$$\frac{ap^2}{q^2}+\frac{bp}{q}+c=0$$

$$\Rightarrow \quad ap^2+bpq+cq^2=0$$

Now, p, q both may be odd or one of p, q be even.

If both p and q are odd, then $ap^2 + bpq + cq^2$ is an odd number and cannot be equal to zero.

Again, if one of p and q is even, then two of the terms of the left-hand side of the equation are even, and the third term is odd and again, its sum is odd and cannot be equal to zero.

Hence, the above equation cannot have rational roots.

Problem 5 *If* $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, *then prove that*

$\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}$ *for all odd n.*

Solution: We have, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$.

None of a, b, c and $a + b + c$ are zero.

Now,
$$\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b+c}-\frac{1}{c}$$

$$\Rightarrow \quad \frac{a+b}{ab}=\frac{-(a+b)}{(a+b+c)c}$$

$$\Rightarrow \quad c\,(a+b)(a+b+c)+ab(a+b)=0$$

$$\Rightarrow \quad (a+b)\,(b+c)\,(c+a)=0$$

$$\Rightarrow \quad a=-b \text{ or } b=-c \text{ or } c=-a$$

If $a = -b$, then $a^n = -b^n$ for n odd $\Rightarrow \frac{1}{a^n}=-\frac{1}{b^n}$

So,
$$\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{c^n}=\frac{1}{0+c^n}=\frac{1}{a^n+b^n+c^n}$$

The equality can be proved similarly in the other two cases also.

Problem 6 *Show that*

$$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-a)(b-c)}+\frac{c^3}{(c-a)(c-b)}=a+b+c$$

Solution: We have,
$$\frac{a^3}{(a-b)(a-c)}=\frac{-a^3}{(a-b)(c-a)}$$

$$\frac{b^3}{(b-a)(b-c)}=\frac{-b^3}{(a-b)(b-c)}$$

and
$$\frac{c^3}{(c-a)(c-b)}=\frac{-c^3}{(b-c)(c-a)}$$

$$\sum\frac{a^3}{(a-b)(a-c)}=-\left[\frac{(b-c)a^3+(c-a)b^3+(a-b)c^3}{(a-b)(b-c)(c-a)}\right]$$

Numerator of RHS is a cyclic symmetric expression in a, b, c in 4th degree and writing $b = c$, we get $0 + (c - a)b^3 + (a - b)c^3 = 0$.

So $(b - c)$, and hence $(c - a)$ and $(a - b)$ are factors. Since it is a fourth degree symmetric expression, $(a + b + c)$ is also a factor.

Thus, we have $k(a + b + c)(a - b)(b - c)(c - a) = (b - c)a^3 + (c - a)b^3 + (a - b)c^3$

If $a = 1$, $b = -1$, $c = 2$, we get on

$$\text{LHS} = k \times 2(2)(-3) \times 1 = -12k$$

and $$\text{RHS} = -3 + (-1) + 16 = 12 \quad \Rightarrow \quad k = -1$$

$\therefore$ The expression $= \dfrac{(a+b+c)(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)} = (a+b+c)$.

Problem 7 *Let, $a_1, a_2, \ldots, a_n$ be non negative real numbers not all zero. Prove that*

$$x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n = 0$$

has exactly one positive real root.

Solution:

$$x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n = 0$$

$$\Rightarrow \quad -x^n\left[-1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}\right] = 0$$

Let, $$f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}$$

$f(x)$ is a decreasing function as x increases in $(0, \infty)$, $f(x)$ decreases in $(\infty, 0)$. Hence, there exists a unique positive real number R, such that

$$f(R) = \frac{a}{R} + \frac{a_2}{R^2} + \cdots + \frac{a_n}{R^n} = 1$$

$$\therefore \quad x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n = -x^n\left(-1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}\right)$$

and for $x = R$, we get

$$-R^n\left(-1 + \frac{a_1}{R} + \frac{a_2}{R^2} + \cdots + \frac{a_n}{R^n}\right) = -R^n[-1+1] = 0$$

Therefore, R is a root of the given equation.

Problem 8 *Let $P(x)$ be a real polynomial function, and $P(x) = ax^3 + bx^2 + cx + d$. Prove, if $|P(x)| \le 1$ for all x, such that $|x| \le 1$, then $|a| + |b| + |c| + |d| \le 7$.*

[IMO, 1996 Short List]

Solution: Considering the polynomials $\pm P(\pm x)$ we may assume without loss of generality that $a, b \ge 0$.

Case 1: If $c, d \ge 0$, then

$$|a| + |b| + |c| + |d| = a + b + c + d = p(1) \le 1 < 7$$

Case 2: If $d \le 0$ and $c \ge 0$, then $|a| + |b| + |c| + |d|$

$$= a + b + c - d = (a + b + c + d) - 2d$$
$$= P(1) - 2P(0) \le 1 + 2 = 3 < 7$$

Case 3: If $d \geq 0$, $c < 0$

$$|a| + |b| + |c| + |d| = a + b - c + d$$

$$= \frac{4}{3}P(1) - \frac{1}{3}P(-1) - \frac{8}{3}P\left(\frac{1}{2}\right) + \frac{8}{3}P\left(\frac{-1}{2}\right)$$

$$\leq \frac{4}{3} + \frac{1}{3} + \frac{8}{3} + \frac{8}{3} = \frac{21}{3} = 7$$

Case 4: If $d < 0$, $c < 0$

$$|a| + |b| + |c| + |d| = a + b - c - d$$

$$= \frac{5}{3}P(1) - 4P\left(\frac{1}{2}\right) + \frac{4}{3}P\left(\frac{-1}{2}\right)$$

$$\leq \frac{5}{3} + 4 + \frac{4}{3} = \frac{21}{3} = 7.$$

Problem 9 *A person who left home between* 4 *p.m. and* 5 *p.m. returned between* 5 *p.m. and* 6 *p.m. and found that the hands of his watch has exactly changed places. When did he go out?*

Solution: The dial of a clock is divided into 60 equal divisions. In one hour, the minute hand makes one complete revolution, *i.e.*, it moves through 60 divisions and the hour hand moves through 5 divisions.

Suppose, when the man went out, the hour hand was x divisions ahead of the point labeled 12 on the dial, where $20 < x < 25$ (as he went out between 4 p.m. and 5 p.m.). Also suppose, when the man returned, the hour hand was y divisions ahead of zero mark and $25 < y < 30$.

Since the minute hand and hour hand exactly interchanged places during the interval that the man was out, the minute hand was at y when he went out and at x when he returned.

Since the minute hand moves 12 times as fast as the hour hand, we have

$$y = 12(x - 20)$$

and

$$x = 12(y - 25)$$

$$\Rightarrow \quad y = 12[12(y - 25) - 20]$$

$$= 144y - 3600 - 240$$

or

$$143y = 3840$$

$$\Rightarrow \quad y = \frac{3840}{143} = 26\frac{122}{143}$$

The minute hand was at y when he went out. So, he went out at $26\frac{122}{143}$ minutes past 4 p.m.

Problem 10 If $\alpha^{13} = 1$ *and* $\alpha \neq 1$, *find the quadratic equation whose roots are* $\alpha + \alpha^3 + \alpha^4 + \alpha^{-4} + \alpha^{-3} + \alpha^{-1}$ *and* $\alpha^2 + \alpha^5 + \alpha^6 + \alpha^{-6} + \alpha^{-5} + \alpha^{-2}$.

Solution: Let

$$A = \alpha + \alpha^3 + \alpha^4 + \alpha^{-4} + \alpha^{-3} + \alpha^{-1}$$

$$= \alpha + \alpha^3 + \alpha^4 + \alpha^9 + \alpha^{10} + \alpha^{12} \quad (\because \ \alpha^{13} = 1)$$

and

$$B = \alpha^2 + \alpha^5 + \alpha^6 + \alpha^{-6} + \alpha^{-5} + \alpha^{-2}$$

$$= \alpha^2 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^{11}$$

$$\begin{aligned} A+B &= \alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^7+\alpha^8+\alpha^9+\alpha^{10}+\alpha^{11}+\alpha^{12} \\ &= (1+\alpha+\alpha^2+\alpha^3+\cdots+\alpha^{12})-1 \\ &= \frac{(\alpha^{13}-1)}{(\alpha-1)}-1=-1 \end{aligned}$$

$$\begin{aligned} (A\times B) &= (\alpha+\alpha^3+\alpha^4+\alpha^9+\alpha^{10}+\alpha^{12})\times(\alpha^2+\alpha^5+\alpha^6+\alpha^8+\alpha^{11}) \\ &= 3(\alpha+\alpha^2+\alpha^3+\cdots+\alpha^{12}) \\ &= 3\,(-1)=-3. \end{aligned}$$

Therefore, the required equation is $x^2+x-3=0$.

Problem 11 *Determine all pairs of positive integers* (m, n), *such that* $(1+x^n+x^{2n}+\cdots+x^{mn})$ *is divisible by* $(1+x+x^2+\cdots+x^m)$.

Solution:

$$1+x^n+x^{2n}+\cdots+x^{mn}=\frac{x^{(m+1)n}-1}{x^n-1} \quad \text{(verify)}$$

and
$$1+x+x^2+\cdots+x^m=\frac{x^{m+1}-1}{x-1}$$

We must find m and n, so that $\dfrac{1+x^n+x^{2n}+\cdots+x^{mn}}{1+x+x^2+\cdots+x^m}$ is a polynomial in 'x', *i.e.*,

$$\frac{x^{(m+1)n}-1}{x^n-1}\div\frac{x^{m+1}-1}{x-1}=\frac{\left(x^{(m+1)}-1\right)(x-1)}{(x^n-1)(x^{m+1}-1)}$$

must be a polynomial.

Now, if k and l are relatively prime, then (x^k-1) and (x^l-1) have just one common factor which is $x-1$. For $x^k-1=0$, say $1, w_1, w_2, \ldots, w_{k-1}$, are all distinct roots. Similarly, those of $x^l-1=1,\ w_1',\ w_2',\ \ldots,\ w_{l-1}'$ are distinct roots.

By Demoivre's theorem, the roots of $x^k-1=0$ are $\cos\dfrac{2n\pi}{k}+i\sin\dfrac{2n\pi}{k}$ for $n=0, 1, 2, \ldots, k-1$ and those of $x^l-1=0$ are $\cos\dfrac{2n\pi}{l}+i\sin\dfrac{2n\pi}{l}$ for $n=0, 1, 2, \ldots, l-1$. If l and K are co-prime integer other than zero, $\cos\dfrac{2n\pi}{l}+i\sin\dfrac{2n\pi}{l}$ and $\cos\dfrac{2n\pi}{k}+i\sin\dfrac{2n\pi}{k}$, will be different.

Since, all the factors of $x^{n(m+1)}-1$ are distinct, $x^{m+1}-1$, x^n-1 cannot have any common factors other than $(x-1)$. Thus, $(m+1)$ and 'n' must be relatively prime.

Again, $x^{n(m+1)}-1=(x^n)^{m+1}-1=(x^{m+1})^n-1$.

So, $x^{n(m+1)}-1$ is divisible by (x^n-1) and, also by $(x^{m+1})-1$

Thus, $\dfrac{[x^{(m+1)n}-1](x-1)}{(x^n-1)(x^{m+1}-1)}$ is a polynomial which shows that the condition $(m+1)$ and n must be relatively prime is also sufficient.

Problem 12 *Show that* $(a-b)^2+(a-c)^2=(b-c)^2$ *is not solvable when a, b and c are all distinct.*

Solution: We have, $(a-b)^2+(a-c)^2=(b-c)^2$

$$\Rightarrow \quad 2a^2-2ab-2ac+2bc=0$$
$$\Rightarrow \quad a^2-a(b+c)+bc=0$$
$$\Rightarrow \quad (a-b)(a-c)=0$$
$$\Rightarrow \quad a=b \text{ or } a=c$$

Thus, the equation has no solution, if a, b and c are all distinct.

Aliter: Let $a-b=x$ and $a-c=y$

$$\Rightarrow \quad b-c=y-x$$

Hence, given equation becomes

$$x^2+y^2=(y-x)^2 \quad \Rightarrow \quad 2xy=0$$
$$\Rightarrow \quad x=0 \quad \text{or} \quad y=0$$
$$\Rightarrow \quad a=b \quad \text{or} \quad a=c$$

Problem 13 *If $P(x)$ is a polynomial of degree n such that $P(x)=2^x$ for $x=1, 2, 3, \ldots, n+1$, find $P(x+2)$.*

Solution: $2^m=(1+1)^m=\binom{m}{0}+\binom{m}{1}+\binom{m}{2}+\cdots+\binom{m}{m}$ for $m=1, 2, \ldots, n+1$.

Now, consider the polynomial

$$f(x)=2\left[\binom{x-1}{0}+\binom{x-1}{1}+\binom{x-1}{2}+\cdots+\binom{x-1}{n}\right]$$

where $\binom{x-1}{r}=\dfrac{(x-1)(x-2)\ldots(x-r)}{r!}$

Clearly, $f(x)$ is of degree n.

Now, $f(r)=2\left[\binom{r-1}{0}+\binom{r-1}{1}+\cdots+\binom{r-1}{r-1}+\binom{r-1}{r}+\cdots+\binom{r-1}{n}\right]$

where $1 \le r \le n+1$

But, $\binom{r-1}{k}=0$ for all $k>r-1$ where k and r are integers

So $f(r)=2\cdot 2^{r-1}=2^r$ for all $r=1, 2, \ldots, n+1$

$\therefore$ Thus, $f(x)$ is the required polynomial

$$\therefore \quad f(n+2)=2\left[\binom{n+1}{0}+\binom{n+1}{1}+\cdots+\binom{n+1}{n}\right]$$
$$=2[2^{n+1}-1]=2^{n+2}-2$$

Similarly, $\qquad p(x+2)=2^{x+2}-2.$

Problem 14 *If a, b, c, d are all real and $a^2+b^2+c^2+d^2=ab+bc+cd+da$, then show that $a=b=c=d$.*

Solution: We have, $2(a^2 + b^2 + c^2 + d^2) - 2(ab + bc + cd + da) = 0$

$$\Rightarrow \quad (a^2 - 2ab + b^2) + (b^2 + c^2 - 2bc) + (c^2 + d^2 - 2cd) + (d^2 + a^2 - 2da) = 0$$

$$\Rightarrow \quad (a - b)^2 + (b - c)^2 + (c - d)^2 + (d - a)^2 = 0$$

$$\Rightarrow \quad a = b,\ b = c,\ c = d,\ d = a$$

$$\Rightarrow \quad a = b = c = d.$$

Problem 15 *Determine $x, y, z \in \mathbb{R}$, such that*
$2x^2 + y^2 + 2z^2 - 8x + 2y - 2xy + 2xz - 16z + 35 = 0.$

Solution: $2x^2 + y^2 + 2z^2 - 8x + 2y - 2xy + 2xz - 16z + 35 = 0$

$$\Rightarrow \quad (x - y)^2 + (x + z)^2 + z^2 - 16z - 8x + 2y + 35 = 0$$

$$\Rightarrow \quad (x - y - 1)^2 + (x + z - 3)^2 + z^2 - 10z + 25 = 0$$

$$\Rightarrow \quad (x - y - 1)^2 + (x + z - 3)^2 + (z - 5)^2 = 0$$

Thus, $x - y = 1$, $x + z = 3$, $z = 5$ and hence, $x = -2$, $y = -3$.
Thus, the solution is $x = -2$, $y = -3$ and $z = 5$.

Problem 16 *Find all real numbers satisfying $x^8 + y^8 = 8xy - 6$.*

Solution: We know $x^8 + y^8 + 6 = 8xy.$
$\Rightarrow$ x and y must be of same sign, otherwise LHS > 0 and RHS < 0
Moreover (x, y) is a solution $\Leftrightarrow$ $(-x, -y)$ also WLOG $x, y > 0$

$$\text{Now } x^8 + y^8 + 1 + 1 + 1 + 1 + 1 + 1 = 8xy$$

By AM–GM inequality.

$$x^8 + y^8 + 1 + 1 + 1 + 1 + 1 + 1$$

$$\geq 8 \times \sqrt[8]{x^8 \times y^8 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1}$$

$$\geq 8 \times \sqrt[8]{x^8 \times y^8} = 8\,|x\,y|$$

But, by hypothesis, equality holds. Hence, all the 8 terms are equal. Therefore,

$$x^8 = y^8 = 1.$$

Hence, $(x, y) \equiv (1, 1), (-1, -1)$ is the solution set.

Problem 17 *Solve the systems of equations for real x and y.*

$$5x\left(1 + \frac{1}{x^2 + y^2}\right) = 12,\ 5y\left(1 - \frac{1}{x^2 + y^2}\right) = 4.$$

Solution: Given that

$$5x\left(1 + \frac{1}{x^2 + y^2}\right) = 12$$

$$\therefore \quad 25x^2 = \frac{144}{\left(1 + \frac{1}{x^2 + y^2}\right)^2} \tag{1}$$

And similarly, we can find by the second equation

$$25y^2 = \frac{16}{\left(1-\frac{1}{x^2+y^2}\right)^2} \tag{2}$$

By adding Eqs. (1) and (2), we get

$$25(x^2+y^2) = \frac{144}{\left(1+\frac{1}{x^2+y^2}\right)^2} + \frac{16}{\left(1-\frac{1}{x^2+y^2}\right)^2} \tag{3}$$

Let, $\quad \dfrac{1}{x^2+y^2} = t$ so that $x^2+y^2 = \dfrac{1}{t}$.

Now $\quad \dfrac{25}{t} = \dfrac{144}{(1+t)^2} + \dfrac{16}{(1-t)^2}$

$$\Rightarrow \quad 144t(1-t)^2 + 16t(1+t)^2 = 25(1-t^2)^2$$

$$\Rightarrow \quad 32t(5t^2-8t+5) = 25(t^4-2t^2+1)$$

Dividing both sides by t^2, we get

$$32\left[5\left(t+\frac{1}{t}\right)-8\right] = 25\left[\left(t+\frac{1}{t}\right)^2-4\right]$$

Putting $t+\dfrac{1}{t}=\alpha$ in the above equation, we get

$$25\alpha^2 - 160\alpha + 156 = 0 \quad \Rightarrow \quad \alpha = \frac{6}{5}, \frac{26}{5}$$

$$\Rightarrow \quad t+\frac{1}{t} = \alpha = \frac{6}{5} \quad \text{or} \quad \frac{26}{5}$$

$$\Rightarrow \quad 5t^2-6t+5=0 \quad \text{or} \quad 5t^2-26t+5=0$$

Since the discriminant of $5t^2-6t+5=0$ is $36-100<0$, there is no real root.

$5t^2-26t+5=0$, the roots are 5 and $\dfrac{1}{5}$.

Thus, $x^2+y^2 = \dfrac{1}{5}$ or $x^2+y^2=5$

If $x^2+y^2=5$, then $5x\left(1+\dfrac{1}{5}\right)=12$ and $5y\left(1-\dfrac{1}{5}\right)=4$

Thus, by solving, we get

$$x=2 \quad \text{and} \quad y=1$$

If $x^2+y^2=\dfrac{1}{5}$ then $5x(1+5)=12$ and $5y(1-5)=4$

Thus, by solving, we get

$$x=\frac{2}{5} \quad \text{and} \quad y=\frac{-1}{5}.$$

The two solution are $x = 2, y = 1$ and $x = \frac{2}{5},\ \ y = \frac{-1}{5}$.

Aliter: Let $z = x + iy \quad \Rightarrow \quad x^2 + y^2 = |z|^2 = z \cdot \bar{z}$

Now Eq. (1) + i Eq. (2) $\quad \Rightarrow \quad 5\left(x + iy + \frac{x - iy}{x^2 + y^2}\right) = 12 + 4i$

$$\Rightarrow \quad 5\left(z + \frac{\bar{z}}{z \cdot \bar{z}}\right) = 12 + 4i \quad \Rightarrow \quad 5z^2 - (12 + 4i)z + 5 = 0$$

$$\Rightarrow \quad z = \frac{12 + 4i \pm \sqrt{(12 + 4i)^2 - 100}}{2(5)} = \frac{12 + 4i \pm \sqrt{28 + 2 \times 48i}}{10}$$

$$= \frac{12 + 4i \pm \sqrt{64 - 36 + 2 \times 8 \times 6i}}{10} = \frac{12 + 4i \pm (8 + 6i)}{10}$$

$$= 2 + i, \frac{2}{5} - \frac{1}{5}i$$

$$\Rightarrow \quad (x, y) \equiv (2,1), \left(\frac{2}{5}, \frac{-1}{5}\right)$$

Problem 18 Solve the system

$$(x + y)(x + y + z) = 18$$
$$(y + z)(x + y + z) = 30$$
$$(z + x)(x + y + z) = 2L \qquad \text{in terms of } L.$$

Where $x, y, z, L \in \mathbb{R}^+$

Solution: Adding the three equations, we get

$$2(x + y + z)^2 = 48 + 2L$$

or $$x + y + z = \sqrt{24 + L}.$$

Dividing the three equation by $x + y + z = \sqrt{24 + L}$, we get

$$x + y = \frac{18}{\sqrt{24 + L}}, y + z = \frac{30}{\sqrt{24 + L}}, z + x = \frac{24}{\sqrt{24 + L}}.$$

Also, by solving, we get

$$x = \frac{(\sqrt{24 + L})^2 - 30}{\sqrt{24 + L}} = \frac{L - 6}{\sqrt{24 + L}},$$

$$y = \frac{(24 + L) - 2L}{\sqrt{24 + L}} = \frac{24 - L}{\sqrt{24 + L}},$$

and $$z = \frac{24 + L - 18}{\sqrt{24 + L}} = \frac{L + 6}{\sqrt{24 + L}} \qquad \text{where } 6 < L < 24$$

Problem 19 *Solve*:

$$x + y - z = 4 \qquad (1)$$
$$x^2 - y^2 + z^2 = -4 \qquad (2)$$
$$xyz = 6 \qquad (3)$$

Where $x, y, z \in \mathbb{R}$

Solution: From Eq. (1), $(x - z) = (4 - y)$

$$\Rightarrow \quad x^2 - 2xz + z^2 = 16 - 8y + y^2$$
$$\Rightarrow \quad (x^2 + z^2 - y^2) - 2xz + 8y - 16 = 0$$
$$\Rightarrow \quad xz = 2(2y - 5) \quad (\because\ x^2 + z^2 - y^2 = -4) \tag{4}$$

From Eqs. (3) and (4), we get

$$y \times 2(2y - 5) = 6$$
$$\Rightarrow \quad 2y^2 - 5y - 3 = 0$$
$$\Rightarrow \quad (2y + 1)(y - 3) = 0$$
$$\Rightarrow \quad y = -\frac{1}{2} \text{ or } y = 3.$$

Putting the value of $y = -\frac{1}{2}$ in Eqs. (1) and (3), we get

$x - z = 4\frac{1}{2}$ and $xz = -12$

$$(x + z)^2 = (x - z)^2 + 4xz = \left(4\frac{1}{2}\right)^2 - 48 < 0.$$

So, $y = 3$ is the only valid solution for y.

$$x - z = 1,\ xz = 2 \tag{5}$$
$$\Rightarrow \quad (x + z)^2 = (x - z)^2 + 4xz = 9$$
$$\Rightarrow \quad x + z = \pm 3 \tag{6}$$

Solving Eqs. (5) and (6), we get

$$x = 2 \quad \text{and} \quad z = 1 \quad \text{or} \quad x = -1 \quad \text{and} \quad z = -2.$$

So, the solution is $x = 2$, $y = 3$ and $z = 1$
or, $x = -1, y = 3, z = -2$.

Problem 20 *Solve:*

$$3x(x + y - 2) = 2y$$
$$y(x + y - 1) = 9x$$

Solution:

$$3x(x + y - 2) = 2y \tag{1}$$
$$y(x + y - 1) = 9x \tag{2}$$

Multiplying Eqs. (1) and (2), we get

$$3xy(x + y - 2)(x + y - 1) = 18xy$$
$$\Rightarrow \quad 3xy[(x + y - 2)(x + y - 1) - 6] = 0$$
$$\Rightarrow \quad 3xy[(x + y)^2 - 3(x + y) - 4] = 0$$
$$\Rightarrow \quad 3xy(x + y - 4)(x + y + 1) = 0 \tag{3}$$

So, $x = 0$ or $y = 0$ or $x + y = 4$ or $x + y = -1$. Putting $x + y = 4$ in Eq. (1), we get

$$6x = 2y$$
$$\Rightarrow \quad y = 3x$$
$$\Rightarrow \quad x = 1, y = 3$$

Putting $x + y = -1$ in. Eq. (1), we get

$$y = \frac{-9x}{2}$$

$$\Rightarrow \frac{-7}{2}x = -1 \qquad \text{(As } x + y = -1\text{)}$$

$$\Rightarrow x = \frac{2}{7},\ y = \frac{-9}{7}$$

Also, $x = 0 \quad \Leftrightarrow \quad y = 0$

Thus, the solutions are (0, 0), (1, 3), $\left(\frac{2}{7}, \frac{-9}{7}\right)$.

Problem 21 *Solve:*

$$xy + x + y = 23$$
$$yz + y + z = 31$$
$$zx + z + x = 47.$$

Solution: We know

$$xy + x + y = 23 \qquad (1)$$
$$yz + y + z = 31 \qquad (2)$$
$$zx + z + x = 47 \qquad (3)$$

Adding 1 in both sides of Eq. (1), we get

$$xy + x + y + 1 = 24$$
$$\Rightarrow \quad (x + 1)(y + 1) = 24 \qquad (4)$$

Similarly, we get

$$(y + 1)(z + 1) = 32 \qquad (5)$$

and

$$(z + 1)(x + 1) = 48 \qquad (6)$$

By multiplying Eqs. (4), (5) and (6), we get

$$(x + 1)^2(y + 1)^2(z + 1)^2 = 24 \times 32 \times 48$$
$$\Rightarrow \quad (x + 1)(y + 1)(z + 1) = \pm(24 \times 8)$$

Since none of $(x + 1)$, $(y + 1)$ and $(z + 1)$ is zero, we get

$$z + 1 = \pm 8$$
$$x + 1 = \pm 6$$
$$y + 1 = \pm 4$$

Thus, we have two solutions $x = 5, y = 3, z = 7$ and $x = -7, y = -5, z = -9$.

Problem 22 *Find all the solutions of the system of equations* $y = 4x^3 - 3x$, $z = 4y^3 - 3y$ *and* $x = 4z^3 - 3z$.

Solution: If $x > 1$, then $y = x^3 + 3x(x^2 - 1) > x^3 > x > 1$,
$z = 4y^3 - 3y = y^3 + 3y(y^2 - 1) > y^3 > y > 1$
and $x = 4z^3 - 3z = z^3 + 3z(z^2 - 1) > z^3 > z > 1$.

Thus, $z > y > x > z$, which is impossible, $\Rightarrow x \le 1$ and, again, $x < -1$, and lead to $x > y > z > x$, so $x \ge -1$.

So, $|x| \le 1$, $|y| \le 1$, $|z| \le 1$.

And hence, we can write $x = \cos\theta$, where $0 \le \theta \le \pi$.

Now, $y = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$, $z = 4y^3 - 3y = 4\cos^3 3\theta - 3\cos 3\theta = \cos 3\times 3\theta = \cos 9\theta$ and $x = 4z^3 - 3z = 4\cos^3 9\theta - 3\cos 9\theta = \cos 3\times 9\theta = \cos 27\theta$

Since trigonometric functions are periodic, it is possible.

Thus,

$$\begin{aligned} & \cos\theta = \cos 27\theta \\ \Rightarrow\ & \cos\theta - \cos 27\theta = 0 \\ \Rightarrow\ & 2\sin 14\theta \sin 13\theta = 0 \\ \Rightarrow\ & \sin 14\theta = 0 \text{ or } \sin 13\theta = 0 \end{aligned}$$

$$\text{so}\quad \theta = \frac{k\pi}{13} \quad \text{where } k = 0, 1, 2, \ldots, 12, 13$$

$$\text{or}\quad \theta = \frac{k\pi}{14} \quad \text{where } k = 1, 2, \ldots, 13$$

and the solution is $(x, y, z) = (\cos\theta, \cos 3\theta, \cos 9\theta)$ where θ takes all the above values.

Problem 23 *Let, $x = p$, $y = q$, $z = r$ and $w = s$ be the unique solutions of the system of linear equations $x + a_i y + a_i^2 z + a_i^3 w = a_i^4$, $i = 1, 2, 3, 4$. Express the solution of the following system in terms of p, q, r and s.*

$$x + a_i^2 y + a_i^4 z + a_i^6 w = a_i^8,\ i = 1, 2, 3, 4$$

Assume the uniqueness of the solution.

Solution: Consider: the quadratic equation

$$p + qt + rt^2 + st^3 = t^4$$

or

$$t^4 - st^3 - rt^2 - qt - p = 0.$$

Now, by our assumption of the problem, a_1 a_2, a_3 and a_4 are the solution of this equation and hence,

$$\begin{aligned} \sigma_1 &= a_1 + a_2 + a_3 + a_4 = s \\ \sigma_2 &= (a_1 + a_2)(a_3 + a_4) + a_1 a_2 + a_3 a_4 = -r \\ \sigma_3 &= a_1 a_2 (a_3 + a_4) + a_3 a_4 (a_1 + a_2) = q \\ \sigma_4 &= a_1 a_2 a_3 a_4 = -p \end{aligned}$$

The second system of equation is

$$(t^2)^4 - w(t^2)^3 - z(t^2)^2 - y(t^2) - x = 0$$

Putting $t^2 = u$, we have

$$u^4 - wu^3 - zu^2 - yu - x = 0$$

and the roots can be seen to be a_1^2, a_2^2, a_3^2 and a_4^2

and

$$\sigma_1 = a_1^2 + a_2^2 + a_3^2 + a_4^2 = w$$

$$\Rightarrow\quad w = \left(\sum a_i\right)^2 - 2\sum_{i<j} a_i a_j = s^2 + 2r$$

$$\sigma_2 = \sum_{i<j} a_i^2 a_j^2 = -z$$

or

$$z = -\sum_{i<j} a_i^2 a_j^2 = -\left(\sum_{i<j} a_i a_j\right)^2 + 2\left(\sum a_i\right)\sum_{i<j<k} a_i a_j a_k - 2a_1 a_2 a_3 a_4$$

[As $(a_1^2 a_2^2 + a_1^2 a_3^2 + a_1^2 a_4^2 + a_2^2 a_3^2 + a_2^2 a_4^2 + a_3^2 a_4^2) = (a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_2a_4)^2 - 2(a_1 + a_2 + a_3 + a_4)(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) + 2a_1a_2a_3a_4$]

and hence, $z = -r^2 + 2qs + 2p$,

$$\sigma_3 = a_1^2a_2^2a_3^2 + a_1^2a_2^2a_4^2 + a_1^2a_3^2a_4^2 + a_2^2a_3^2a_4^2 = y$$

$$y = (a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)^2$$
$$-2(a_1a_2a_3a_4)(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)$$
$$= q^2 - 2pr$$

Finally, $\sigma_4 = a_1^2a_2^2a_3^2a_4^2 = -x$

or $x = -(a_1^2a_2^2a_3^2a_4^2) = -(a_1a_2a_3a_4)^2 = -p^2$

$\therefore$ $x = -p^2, y = q^2 - 2pr, z = -r^2 + 2qs + 2p$

and $w = s^2 + 2r$ is the solution.

Problem 24 *Find out all values of a and b, for which*

$$xyz + z = a \tag{1}$$
$$xyz^2 + z = b \tag{2}$$

and
$$x^2 + y^2 + z^2 = 4 \tag{3}$$

has only one solution.

Solution: You may observe that both (x, y, z) and $(-x, -y, z)$ satisfy the system. Since, by the condition of the problem, there must be just one solution, we get $x = y = 0$ and so, $z^2 = 4 \Rightarrow z = \pm 2$ by Eq. (3).

But, by Eqs. (1) and (2), $z = a$ or $z = b$. Since, there should be only one solution, either, $a = b = 2$ or $a = b = -2$.

If $a = b = 2$, we have

$$xyz + z = 2 \tag{4}$$
$$xyz^2 + z = 2 \tag{5}$$
$$x^2 + y^2 + z^2 = 4 \tag{6}$$

Eq. (5) – Eq. (4) gives

$xyz(z - 1) = 0$ either x, y or $z = 0$ or $z = 1$. If z = 0, from Eq. (4) 0 = 2, contradiction

If $z = 1$, then x, y are not zero $\Rightarrow$ More than one solution of the system

Hence, $a = b = 2$ does not satisfy the condition.

If $a = b = -2$, we have

$$xyz + z = -2 \tag{7}$$
$$xyz^2 + z = -2 \tag{8}$$
$$x^2 + y^2 + z^2 = 4 \tag{9}$$

Eq. (8) – Eq. (7) $\Rightarrow$ $xyz(z - 1) = 0$ $\Rightarrow$ any of x, y, and $z = 0$ or $z = 1$.

For z = 0, Eq. (7) becomes 0 = –2, contradiction.

If $z = 1$, then $xy + 1 = -2 \Rightarrow xy = -3$ and $x^2 + y^2 = 3$

$(x + y)^2 = x^2 + y^2 + 2xy = 3 - 6 = -3$ cannot be true for any real x, y and hence, $z \neq 1$.

If one of x, y is zero, say $x = 0$, then

$$z = -2$$
$$x^2 + y^2 + z^2 = 4$$
$$\Rightarrow \quad 0 + y^2 + 4 = 4$$
$$\Rightarrow \quad y = 0$$

Thus, for $a = b = -2$, the given system has a unique solution, namely, (0, 0, –2).

Problem 25 *Given, a, b, and c are positive real numbers, such that*

$$a^2+ab+\frac{b^2}{3}=25,\ \frac{b^2}{3}+c^2=9,\ c^2+ca+a^2=16.$$

Find out the value of $ab+2bc+3ca$.

Solution: Let, $A=a^2+ab+\frac{b^2}{3}=25,\ \ B=\frac{b^2}{3}+c^2=9$ and $C=c^2+ca+a^2=16.$

Hence, $25=A=9+16=B+C$

$$\Rightarrow\quad a^2+ab+\frac{b^2}{3}=\frac{b^2}{3}+c^2+c^2+ca+a^2$$

$$\Rightarrow\quad 2c^2+ac-ab=0$$

$$\Rightarrow\quad ab=c(2c+a)$$

$$\Rightarrow\quad a+2c=\frac{ab}{c} \tag{1}$$

Again, $A-B+C=25-9+16=32$

$$\Rightarrow\quad 2a^2+ab+ca=a(2a+b+c)=32$$

$$\Rightarrow\quad 2a+b+c=\frac{32}{a} \tag{2}$$

If $\quad S=ab+2bc+3ca$

then, $\quad S=b(a+2c)+3ca$

$$=\frac{b\times ab}{c}+3ca \qquad \text{[from Eq. (1)]}$$

$$=\frac{3a}{c}\left(\frac{b^2}{3}+c^2\right)$$

$$=\frac{3a}{c}\times 9=\frac{27a}{c} \tag{3}$$

But, S can also be written using Eq. (1), we get

$$S=ab+2bc+3ca$$
$$=2c^2+ac+2bc+3ca=2c^2+2bc+4ac$$
$$=2c(c+b+2a)$$

$$\therefore\quad 2a+b+c=\frac{S}{2c} \tag{4}$$

From Eqs. (2), (3), and (4), we have

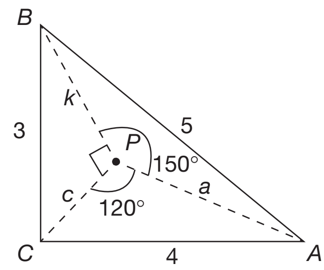
$$\frac{32}{a}=\frac{27a}{c}\times\frac{1}{2c}=\frac{27a}{2c^2}$$

$$\Rightarrow\quad \frac{a^2}{c^2}=\frac{64}{27}$$

$$\therefore\quad \frac{a}{c}=\frac{8}{3\sqrt{3}}\left(\text{as } \frac{a}{c}>0\right)$$

But, by Eq. (3),

$$S=27\times\frac{a}{c}=24\sqrt{3}$$



Aliter: Let $b = \sqrt{3}\,k$ then system of equations becomes

$$a^2 + \sqrt{3}\,ak + k^2 = 25 \text{ or } a^2 + k^2 - 2ak\cos 150° = 5^2$$
$$k^2 + c^2 = 9 \qquad \text{or } k^2 + c^2 - 2kc\cos 90° = 3^2$$
$$c^2 + ac + a^2 = 16 \quad \Rightarrow \quad a^2 + c^2 - 2ac\cos 120° = 4^2$$

Now consider a ΔABC of sides 3, 4, 5 and a point P in it such that $AP = a$, $BP = k$, $CP = c$

Now consider $ab + 2bc + 3ca = a\sqrt{3}\,k + 2\sqrt{3}\,kc + 3ca$ (1)

$$\text{Area of } \Delta ABC = \frac{1}{2}kc + \frac{1}{2}ac\sin 120° + \frac{1}{2}ak\sin 150° = \frac{1}{2}\times 3\times 4$$

$$\Rightarrow \frac{1}{2}kc + \frac{\sqrt{3}ac}{4} + \frac{1}{4}ak = 6$$
$$\Rightarrow 2\sqrt{3}kc + 3ca + ak\sqrt{3} = 24\sqrt{3}$$
$$\Rightarrow ab + 2bc + 3ca = 24\sqrt{3} \quad (\text{as } \sqrt{3}k = b)$$

Problem 26 *Solve:* $\log_3(\log_2 x) + \log_{1/3}\cdot(\log_{1/2} y) = 1$

$$xy^2 = 4$$

Solution: We have, $\log_3(\log_2 x) + (\log_{1/3}\cdot\log_{1/2} y) = 1$

$$\Rightarrow \quad \log_3(\log_2 x) - \log_3(\log_{1/2} y) = 1$$
$$\Rightarrow \quad \log_3\left(\frac{\log_2 x}{\log_{1/2} y}\right) = 1 \quad \Rightarrow \quad \frac{\log_2 x}{\log_{1/2} y} = 3^1$$
$$\Rightarrow \quad \log_2 x = 3\log_{1/2} y$$
$$\Rightarrow \quad \log_2 x = -3\log_2 y = -\log_2 y^3$$
$$\Rightarrow \quad \log_2 xy^3 = 0$$
$$\Rightarrow \quad xy^3 = 1$$

But, we have $xy^2 = 4$. So, by using the above equation, we get $y = \dfrac{1}{4}$ and $x = 64$. Which satisfy the parent equations.

Problem 27 *Solve:*

$$\log_2 x + \log_4 y + \log_4 z = 2$$
$$\log_3 y + \log_9 z + \log_9 x = 2$$
$$\log_4 z + \log_{16} x + \log_{16} y = 2$$

Solution:

We know that, $\log_a x = \log_{(a^n)}(x^n)$

So, $\log_2 x = \log_{2^2} x^2 = \log_4 x^2$,

$$\log_3 y = \log_{3^2} y^2 = \log_9 y^2$$
$$\log_4 z = \log_{4^2} z^2 = \log_{16} z^2$$

So, $\log_2 x + \log_3 y + \log_4 z = 2$

$$\Rightarrow \quad \log_4 x^2yz = 2$$

$$\Rightarrow \quad x^2yz = 4^2 = 16 \qquad (1)$$

Similarly,
$$y^2xz = 81 \qquad (2)$$

and
$$z^2xy = 256 \qquad (3)$$

Hence,
$$x^2yz \times y^2xz \times z^2xy = 16 \times 81 \times 256$$

$$\Rightarrow \quad (xyz)^4 = 2^4 \times 3^4 \times 4^4$$

$$xyz = 24 \quad \text{as } x, y, z > 0$$

Dividing Eqs. (1), (2), and (3) by $xyz = 24$, we get

$$x = \frac{16}{24}, y = \frac{81}{24} \quad \text{and} \quad z = \frac{256}{24}$$

$$\Rightarrow \quad x = \frac{2}{3}, y = \frac{27}{8}, z = \frac{32}{3}.$$

Problem 28 *Find all real numbers x and y satisfying* $\log_3 x + \log_2 y = 2;\ 3^x - 2^y = 23.$

Solution: By observation one solution is $x = 3, y = 2$

As $\log_3 3 + \log_2 2 = 2$ and $3^3 - 2^2 = 23$

If $x < 3$, then $\log_3 x < 1$. Since, $\log_3 x + \log_2 y = 2$, $\log_2 y > 1$ and $y > 2$.

Hence,

$$3^x < 3^3 = 27 \quad \text{and} \quad 2^y > 2^2 = 4$$

$$\Rightarrow \quad 3^x - 2^y < 27 - 4 = 23$$

So, x cannot be less than 3.

If $x > 3$, then $\log_3 x > 1$ and $\log_2 y < 1$ and so $y < 2$, $3^x > 3^3 = 27$ and $2^y < 2^2 = 4$.

So $3^x - 2^y > 27 - 4 = 23$

So, x cannot be greater than 3.

Hence, $x = 3$

$\Rightarrow \quad y = 2$

Here, the only solution for the given equation is $x = 3$ and $y = 2$.

Check Your Understanding

1. Find the value of $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} + \dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

2. Find the value of $\dfrac{444445 \times 888885 \times 444442 + 444438}{444444^2}$ using algebra.

3. Solve: $\sqrt{x^2 - 4x + 3} \geq 2 - x$.

4. Let α, β, γ be the roots of $x^3 - x^2 - 1 = 0$. Then find the value of $\dfrac{1+\alpha}{1-\alpha} + \dfrac{1+\beta}{1-\beta} + \dfrac{1+\gamma}{1-\gamma}$.

5. Show that $(x - 1)^2$ is a factor of $x^{m+1} - x^m - x + 1$.

6. Find all real solution x of the equation $x^{10} - x^8 + 8x^6 - 24x^4 + 32x^2 - 48 = 0$.

7. Solve $2x^{99}+3x^{98}+2x^{97}+3x^{96}+\cdots+2x+3=0$ in $\mathbb{R}$.

8. Prove that $1+x^{111}+x^{222}+x^{333}+x^{444}$ divides $1+x^{111}+x^{222}+x^{333}+\cdots+x^{999}$.

9. If x, y, z are rational and strictly positive and if $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$ show that $\sqrt{x^2+y^2+z^2}$ is rational.

10. If $a^2x^3+b^2y^3+c^2z^3=p^5$, $ax^2=by^2=cz^2$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{p}$, find $\sqrt{a}+\sqrt{b}+\sqrt{c}$ only in terms of p.

11. If $ax^3=by^3=cz^3$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$; prove that $\sqrt[3]{ax^2+by^2+cz^2}=\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}$.

12. Prove that, if (x, y, z) is a solution of the system of equations, $x+y+z=a$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{a}$. Then, at least one of the numbers x, y, z is 'a'.

13. If one root of the equation $2x^2-6x+k=0$ is $\frac{1}{2}(a+5i)$ where $i^2=-1$; $k, a \in \mathbb{R}$, find the values of 'a' and 'k'.

14. If $x^3+px^2+q=0$, where $q \neq 0$ has a root of multiplicity 2, prove that $4p^3+27q=0$.

15. If $f(x)$ is a quadratic polynomial with $f(0)=6$, $f(1)=1$ and $f(2)=0$, find $f(3)$.

16. Show that, if a, b, c are real numbers and $ac=2(b+d)$, then, at least one of the equations $x^2+ax+b=0$ and $x^2+cx+d=0$ has real roots.

17. Given any four positive, distinct, real numbers, show that one can choose three numbers A, B, C among them, such that all the quadratic equations have only real roots or all of them have only imaginary roots. $Bx^2+x+C=0$; $Cx^2+x+A=0$; $Ax^2+x+B=0$.

18. Show that the equation $x^4-x^3-6x^2-2x+9=0$ cannot have negative roots.

19. If $a, b, c, d \in \mathbb{R}$ such that $a<b<c<d$, then show that, the roots of the equation $(x-a)(x-c)+2(x-b)(x-d)=0$ are real and distinct.

20. Find the maximum number of positive and negative real roots of the equation $x^4+x^3+x^2-x-1=0$.

21. If $P(x)=ax^2+bx+c$ and $Q(x)=-ax^2+bx+c$, where $ac \neq 0$, show that the equation $P(x)\cdot Q(x)=0$ has at least two real roots.

22. Let $f(x)$ be the cubic polynomial x^3+x+1; suppose $g(x)$ is a cubic polynomial, such that $g(0)=-1$ and the roots of $g(x)=0$ are squares of the roots of $f(x)=0$. Determine $g(9)$.

23. If $p, q, r, s \in \mathbb{R}$, show that the equation

    $(x^2+px+3q)(x^2+rx+q)(-x^2+sx+2q)=0$ has at least two real roots.

24. If t_n denotes the nth term of an AP and $t_p=\frac{1}{q}, t_q=\frac{1}{p}$, then show that t_{pq} is a root of the equation $(p+2q-3r)x^2+(q+2r-3p)x+(r+2p-3q)=0$.

25. If p and q are odd integers, show that the equation $x^2+2px+2q=0$ has no rational roots.

26. Show that there cannot exist an integer n, such that n^3-n+3 divides n^3+n^2+n+2.

27. If $s_n = 1 + q + q^2 + \cdots + q^n$ and $S_n = 1 + \frac{1+q}{2} + \left(\frac{1+q}{2}\right)^2 + \cdots + \left(\frac{1+q}{2}\right)^n$

prove that $\binom{n+1}{1} + \binom{n+1}{2} s_1 + \binom{n+1}{3} s_2 + \cdots + \binom{n+1}{n+1} s_n = 2^n S_n$.

28. Solve for x, y, z, the equations

$$a = \frac{xy}{x+y}, b = \frac{yz}{y+z}, \text{ and } c = \frac{xz}{x+z} (a, b, c \neq 0)$$

29. Solve and find the non-trivial solutions

$$x^2 + xy + xz = 0$$
$$y^2 + yz + yx = 0$$
$$z^2 + zx + zy = 0.$$

30. Solve:

$$x^2 + xy + y^2 = 7$$
$$y^2 + yz + z^2 = 19$$
$$z^2 + zx + x^2 = 3.$$

31. Determine all solutions of the equation in $\mathbb{R}$,
$(x^2 + 3x - 4)^3 + (2x^2 - 5x + 3)^3 = (3x^2 - 2x - 1)^3$

32. Show that there is no positive integer, satisfying the condition that $(n^4 + 2n^3 + 2n^2 + 2n + 1)$ is a perfect square.

33. Find the possible solutions of the system of equations:
$a^x = (x + y + z)^y$; $a^y = (x + y + z)^z$; $a^z = (x + y + z)^x$.

34. If a and b are given integers, prove that the systems of equations, $x + y + 2z + 2t = a$ and $2x - 2y + z - t = b$ has a solution in integers x, y, z, t.

35. Show that $2x^3 - 4x^2 + x - 5$ cannot be factored into polynomials with integer coefficients.

36. The product of two of the four roots of the equation
$x^4 + 7x^3 - 240x^2 + kx + 2000 = 0$ is -200, determine k.

37. The product of two of the four roots of $x^4 - 20x^3 + kx^2 + 590x - 1992 = 0$ is 24, find k.

38. Let a, b, c, d be any four real numbers not all equal to zero. Prove that the roots of the polynomial $f(x) = x^6 + ax^3 + bx^2 + cx + d$ can not all be real.

39. If a, b, c and p, q, r are real numbers, such that for every real number x, $ax^2 + 2bx + c \geq 0$ and $px^2 + 2qx + r \geq 0$, then prove that $apx^2 + bqx + cr \geq 0$ for all real number x.

40. Find a necessary and sufficient condition on the natural number n, for the equation $x^n + (2 + x)^n + (2 - x)^n = 0$ to have an integral root.

41. Given that α, β, and γ are the angles of a right angled triangle. Prove that
$\sin\alpha \sin\beta \sin(\alpha - \beta) + \sin\beta \sin\gamma \sin(\beta - \gamma) + \sin\gamma \sin\alpha \sin(\gamma - \alpha) + \sin(\alpha - \beta)$
$\sin(\beta - \gamma) \sin(\gamma - \alpha) = 0$.

42. For a given pair of values x and y satisfy $x = \sin\alpha$, $y = \sin\beta$, there can be four different values of $z = \sin(\alpha + \beta)$:

(i) Set up a relation between x, y, and z not involving trigonometric functions or radicals.

(ii) Find those pairs of values (x, y) for which $z = \sin(\alpha + \beta)$ takes on fewer than four distinct values.

43. Suppose, a, b, and c are three real numbers, such that the quadratic equation
$$x^2 - (a+b+c)x + (ab+bc+ca) = 0$$
has roots of the form $a \pm i\beta$, where $\alpha > 0$ and $\beta \neq 0$ are real numbers [here, $i = \sqrt{-1}$]. Show that

(i) the numbers a, b, and c are all positive.

(ii) the numbers $\sqrt{a}, \sqrt{b}$, and $\sqrt{c}$, form the sides of a triangle.

44. Find the number of quadratic polynomials $ax^2 + bx + c$, which satisfy the following conditions:

(i) a, b, c, are distinct

(ii) $a, b, c \in \{1, 2, 3, \ldots, 999\}$

(iii) $(x + 1)$ divides $(ax^2 + bx + c)$

45. Show that there are infinitely many pairs (a, b) of relatively prime integers (not necessarily positive) such that both quadratic equations $x^2 + ax + b = 0$ and $x^2 + 2ax + b = 0$ have integer roots. **[INMO, 1995]**

46. If the magnitude of the quadratic function $f(x) = ax^2 + bx + c$ never exceeds 1 for $0 \leq x \leq 1$, prove that the sum of the magnitudes of the coefficients cannot exceed 17.

47. Suppose that $-1 \leq ax^2 + bx + c \leq 1$ for $-1 \leq x \leq 1$, where a, b, c are real numbers, prove that $-4 \leq 2ax + b \leq 4$ for $-1 \leq x \leq 1$.

48. Find the polynomial $p(x) = x^2 + px + q$ for which $\max\limits_{x \in [-1,1]} |P(x)|$ is minimal.

49. Find real numbers a, b, c for which $|ax^2 + bx + c| \leq 1 \ \forall\ |x| < 1$ and $\frac{8}{3}a^2 + 2b^2$ is maximal.

50. Let $a, b, c, \in \mathbb{R}$ and $a < 3$ and all roots of $x^3 + ax^2 + bx + c = 0$ are negative real numbers. Prove that $b + c < 4$.

Challenge Your Understanding

1. $xp(x-1) = (x-30)p(x)\ \forall x \in \mathbb{R}$, find all such polynomial $p(x)$.

2. Find a polynomial $p(x)$ if it exist such that $xp(x-1) = (x+1)p(x)$.

3. Let $f(x)$ be a quadratic function suppose $f(x) = x$ has no real roots. Prove that $f(f(x)) = x$ has also no real roots.

4. If $7 \mid (ax^4 + bx^3 + cx^2 + dx + e)\ \forall x \in \mathbb{Z}$ where $a, b, c, d, e \in \mathbb{Z}$. Prove that $7 \mid a, 7 \mid b, 7 \mid c, 7 \mid d, 7 \mid e$.

5. Prove that $a^2 + ab + b^2 \geq 3(a+b-1)\ \forall a, b \in \mathbb{R}$.

6. Let $p(x) = x^4 + x^3 + x^2 + x + 1$. Find the remainder on dividing $p(x^5)$ by $p(x)$.

7. Find the remainder when x^{2025} is divided by $(x^2+1)(x^2+x+1)$.

8. If $A, B, C, \ldots, a, b, c, \ldots, K$ are all constants, show that all the roots of the equation
$$\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + \cdots + \frac{H^2}{x-h} = x + K \text{ are real.}$$

9. Prove that there does not exist a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, such that $p(0)$, $p(1)$, $p(2), \ldots$ are all prime numbers.

10. Solve the following equations for real 'x' depending upon real parameter 'a':

(a) $x+\sqrt{a+\sqrt{x}}=a$

(b) $x^2-\sqrt{a-x}=a$

(c) $\sqrt{a-\sqrt{a+x}}=x$

11. The polynomial ax^3+bx^2+cx+d has integral coefficients a, b, c, d with ad odd and bc even. Prove that all roots cannot be rational.

12. If roots of $x^4+ax^3+bx^2+ax+1=0$ has real roots then find the minimum value of a^2+b^2.

13. If the coefficient of x^k upon the expansion and collecting of terms in the expression $\underbrace{\left(\ldots\left(\left((x-2)^2-2\right)^2-2\right)^2-\cdots-2\right)^2}_{n\text{ times}}$ is a_k, then find a_0, a_1, a_2, a_3 and a_{2^k}.

14. Prove that the equations

$$x^2-3xy+2y^2+x-y=0 \text{ and } x^2-2xy+y^2-5x+7y=0$$

imply the equation $xy-12x+15y=0$.

15. If a and b are integers and the solutions of the equation

$$y-2x-a=0 \text{ and } y^2-xy+x^2-b=0$$

are rational, then prove that the solutions are integers.

16. Solve the following system of equations for real numbers a, b, c, d, e:

$$3a=(b+c+d)^3,\ 3b=(c+d+e)^3,\ 3c=(d+e+a)^3,$$

$$3d=(e+a+b)^3,\ 3e=(a+b+c)^3.$$

[INMO, 1996]

17. Solve for real numbers x and y, simultaneously the equations given by $xy^2=15x^2+17xy+15y^2$ and $x^2y=20x^2+3y^2$.

18. Solve the system of equations in integers: $3x^2-3xy+y^2=7,\ 2x^2-3xy+2y^2=14$.

19. In the sequence $a_1, a_2, a_3, \ldots, a_n$, the sum of any three consecutive terms is 40; if the third term is 10 and the eighth term is 8; find the 2013th term.

20. A sequence has first term 2007, after which every term is the sum of the squares of the digits of the preceding term. Find the sum of this sequence upto 2013 terms.

21. Find a finite sequence of 16 numbers, such that

(a) it reads same from left to right as from right to left

(b) the sum of any 7 consecutive terms is –1

(c) the sum of any 11 consecutive terms is +1.

22. A two-pan balance is inaccurate since its balance arms are of different lengths and its pans are of different weights. Three objects of different weights A, B and C are each weighed separately. When they are placed on the left pan, they are balanced by weights A_1, B_1, and C_1 respectively. When A and B are placed on the right pan, they are balanced by A_2 and B_2, respectively. Determine the true weight of C in terms of A_1, B_1, C_1, A_2 and B_2.

[USA MO, 1980]

23. If a and b are two of the roots of $x^4+x^3-1=0$, prove that ab is a root of $x^6+x^4+x^3-x^2-1=0$.

[USA MO, 1977]

24. If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials, such that $P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)\, S(x)$, prove that $(x - 1)$ is a factor of $P(x)$.

[USA MO, 1976]

The generalization of the above problem is: if $P_0(x), P_1(x), \ldots, P_{(n-3)}(x)$, $n \geq 3$ and $S(x)$ are polynomials, such that

$$P_0(x^n) + xP_1(x^n) + \cdots + x^{n-3} P_{(n-3)}(x^n) = (x^{n-1} + x^{n-2} + \cdots + x + 1)S(x)$$

then $(x - 1)$ is a factor of $P_i, (x)$ for all i.

25. If $x^5 - x^3 + x = a$, prove that $x^6 \geq 2a - 1$. **[INMO, 1994]**

26. The solutions x_1, x_2, and x_3 of the equation $x^3 + ax + a = 0$, where a is real and $a \neq 0$, satisfy $\dfrac{x_1^2}{x_2} + \dfrac{x_2^2}{x_3} + \dfrac{x_3^2}{x_1} = -8$, find x_1, x_2, and x_3. **[AMTI, 1994]**

27. Let $p(x)$ be a polynomial with degree 2008 and leading coefficient 1 such that $p(0) = 2007, p(1) = 2006, p(2) = 2005, \ldots, p(2007) = 0$; determine $p(2008)$.

28. If $P(x)$ denotes a polynomial of degree n, such that $P(k) = \dfrac{1}{k}$ for $k = 1, 2, 3, \ldots, n + 1$, determine $P(n + 2)$.

29. If $P(x)$ denotes a polynomial of degree n, such that $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$, determine $P(n + 1)$. **[USA MO, 1975]**

30. Let a, b and c denote three distinct integers and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$ and $P(c) = a$. **[USA MO, 1974]**

31. Let, a_i, $i = 1, 2, \ldots, n$ be distinct real numbers $b_1\ b_2, \ldots, b_n$ be real numbers, such that the product $\prod_{j=1}^{n}(a_i + b_j)$ is the same for each i. Prove that the product $\prod_{i=1}^{n}(a_i + b_j)$ is also constant for all j.

32. In the polynomial $P(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + 1$, the coefficients $a_1, a_2, \ldots, a_{n-1}$ are non-negative and it has n real roots. Prove that $P(2) \geq 3^n$.

33. Determine all polynomials of degree n with each of its $(n + 1)$ coefficients equal to ± 1, which have only real roots.

34. Let $p(x)$ be polynomial over $\mathbb{Z}$ and at three distinct integers it takes ± 1 value, prove that it has no integral root.

35. Let α, β be the roots of $x^2 - 6x + 1 = 0$. Prove that $\alpha^n + \beta^n \in \mathbb{Z}\ \forall\, n \in \mathbb{N}_0$, also prove that $5 \nmid (\alpha^n + \beta^n)\ \forall n \in \mathbb{N}_0$.

36. Let $P(x)$ be a polynomial with real coefficients such that $P(x) \geq 0$ for every real x. Prove that

$$P(x) = f_1(x)^2 + f_2(x)^2 + \cdots + f_n(x)^2$$

[Putnam, 1999]

37. Is it possible to find three quadratic polynomials $f(x)$, $g(x)$, $h(x)$ such that the equation $f(g(h(x))) = 0$ has eight roots 1, 2, 3, 4, 5, 6, 7, 8?

[Russian MO, 1995]

38. Let $P(z) = az^3 + bz^2 + cz + d$, where a, b, c, d are complex numbers with $|a| = |b| = |c| = |d| = 1$. Show that $|P(z)| \geq \sqrt{6}$ for at least one complex number z satisfying $|z| = 1$.

39. Consider two monic polynomials $f(x)$ and $g(x)$ of degree 4 and 2 respectively over real numbers. Let there be an interval (a, b) of length more than 2 such that both $f(x)$ and $g(x)$ are negative for $x \in (a, b)$ and both are positive for $x < a$ or $x > b$. Prove that there is a real number 'α' such that $f(\alpha) < g(\alpha)$.

40. Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))\ \forall\, j = 2, 3, \ldots$. Show that for any positive integer n, the roots of the equation $P_n(x) = x$ are real and distinct. **[IMO, 1976]**

41. Find all polynomials f satisfying $f(x^2) + f(x) \cdot f(x+1) = 0\ \ \forall x \in \mathbb{R}$.

42. Find all polynomials $P(x)$, for which $P(x) \cdot P(2x^2) = P(2x^3 + x) \forall x \in \mathbb{R}$.

43. Find all polynomials $f(x)$ such that $f(x) \cdot f(x+1) - f(x^2 + x + 1) = 0\ \forall x \in \mathbb{R}$.

44. Find all polynomials $f(x)$ such that $f(x) \cdot f(-x) - f(x^2) = 0\ \forall x \in \mathbb{R}$.

45. Prove that if a polynomial of degree 7 over $\mathbb{Z}$ is equal to $+1$ or -1 for 7 different integers then it is irreducible over $\mathbb{Z}$.

46. Prove that $(x - a_1)^2 (x - a_2)^2 \ldots (x - a_n)^2 + 1$ is irreducible over $\mathbb{Z}$.

47. Prove that $(x + 1^2)(x + 2^2) \ldots (x + n^2) + 1$ is irreducible over $\mathbb{Z}$.

48. Let $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ are distinct, find them for which $(x - a_1)(x - a_2) \ldots (x - a_n) + 1$ can be expressible as product of two polynomials with integral coefficients.

49. Let $p(x)$ be a polynomial over $\mathbb{Z}$ such that $|p(a)| = |p(b)| = 1$ for $a, b \in \mathbb{Z}$, $a < b$; If $p(x) = 0$ has rational root α, then prove that $a - b = 1$ or 2 and $\alpha = \dfrac{a+b}{2}$.

50. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two distinct collections of n positive integers, where each collection may contain repetitions. If the two collections of integers $a_i + a_j (1 \leq i < j \leq n)$ and $b_i + b_j (1 \leq i < j \leq n)$ are the same, then prove that n is a power of 2.

Chapter 2

Godfrey Harold Hardy

7 Feb 1877–1 Dec 1947
Nationality: United Kingdom

Hardy could be named 'the father of the Discipline of Inequalities'. He was the founder of the *Journal of the London Mathematical Society*, a proper publication for many papers on inequalities. In addition, together with Littlewood and Polya, Hardy was the editor of the volume *Inequalities*, a book that was the first monograph on inequalities. The work on the book started in 1929 and it was issued in 1934. The authors confessed that the historical and bibliographical accounts are difficult "in a subject like this, which has applications in every part of mathematics but has never been developed systematically" (Hardy, Littlewood, & Polya, 1934). Their contribution was to track down, document, solve and carefully present a volume comprising of 408 inequalities, and to officially write the first page of the history of inequalities. One of the interesting aspects of the book is the philosophy inequalities, presented in the introduction: generally an inequality that is elementary should be given an elementary proof, the proof should be "inside" the theory it belongs to, and finally the proof should try to settle the cases of equality. This introductory chapter is recommended reading with ideas that are still applicable today.

Inequalities

2.1 Basic Rules

2.1.1 Transitivity

The transitive property of inequality states:

If $a > b$ and $b > c$, then $a > c$.

More generally, if $a_1 > a_2, a_2 > a_3, \ldots, a_{n-1} > a_n$, then $a_1 > a_n$.

2.1.2 Addition and Subtraction

A common constant c may be added to or subtracted from both sides of an inequality:

If $a > b$, then for every c, $a + c > b + c$ and $a - c > b - c$.

2.1.3 Multiplication and Division

For any real numbers, a, b and non-zero c,

If c is positive, then multiplying or dividing by c does not change the inequality:

If $a < b$ and $c > 0$, then $ac < bc$ and $a/c < b/c$.

If c is negative, then multiplying or dividing by c inverts the inequality:

If $a < b$ and $c < 0$, then $ac > bc$ and $a/c > b/c$.

2.1.4 Addition and Multiplication of Two Inequalities

If $a_1 > b_1, a_2 > b_2, \ldots, a_n > b_n$, then $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n$.

If $a_1 > b_1 > 0, a_2 > b_2 > 0, \ldots, a_n > b_n > 0$, then $a_1 a_2 \ldots a_n > b_1 b_2 \ldots b_n$.

2.1.5 Applying a Function to Both Sides of an Inequality

Any monotonically increasing function may be applied to both sides of an inequality (provided they are in the domain of that function) and it will still hold. Applying a monotonically decreasing function to both sides of an inequality means the opposite inequality now holds.

If the inequality is strict ($a < b$, $a > b$) and the function is strictly monotonic, then the inequality remains strict. If only one of these conditions is strict, then the resultant inequality is non-strict.

A few examples of this rule are:

1. Taking reciprocal of both side of an inequality:

   If $0 < a \le b$, then $\frac{1}{a} \ge \frac{1}{b} > 0$.

   If $a \le b < 0$, then $0 > \frac{1}{a} \ge \frac{1}{b}$.

   If $a < 0 < b$, then $\frac{1}{a} < 0 < \frac{1}{b}$.

2. Exponentiating both sides of an inequality by $r > 0$, when $0 < a < b$, then $a^r < b^r$ and $a^{-r} > b^{-r}$.
   Similarly for $r > 0$ and $0 < a < 1 < b$, then $0 < a^r < 1 < a^{-r}$ and $0 < b^{-r} < 1 < b^r$.
3. Taking the natural logarithm to both sides of an inequality when x and y are positive real numbers:
   If $b > 1$ and $x > y > 0$, then $\log_b x > \log_b y$,
   If $0 < b < 1$ and $x > y > 0$, then $\log_b x < \log_b y$

These are true because the logarithm is a strictly increasing (or decreasing) function for base 'b' greater (or less) than 1.

Example 1 *Show that,* $\frac{10^{2013}+1}{10^{2014}+1} > \frac{10^{3013}+1}{10^{3014}+1}$

Solution: Let $a = 10^{2013}$ and $b = 10^{1000}$; then we need to prove that,

$$\left(\frac{a+1}{10a+1}\right) > \left(\frac{ab+1}{10ab+1}\right)$$

This is equivalent to $(a + 1)(10ab + 1) > (10a + 1)(ab + 1)$

This holds only iff $10a^2b + a + 10ab + 1 > 10a^2b + 10a + ab + a$

i.e., $9ab > 9a \Leftrightarrow b > 1$

Since, $b = 10^{1000}$, $b > 1$.

Hence, it is true.

Example 2 *If $a > b > 0$, which of the two numbers* $\frac{1+a+a^2+\cdots+a^{n-1}}{1+a+a^2+\cdots+a^n}$ *and* $\frac{1+b+b^2+\cdots+b^{n-1}}{1+b+b^2+\cdots+b^n}$ *is greater?*

Solution:

Let, $$A = \frac{1+a+a^2+\cdots+a^{n-1}}{1+a+a^2+\cdots+a^n}$$

and $$B = \frac{1+b+b^2+\cdots+b^{n-1}}{1+b+b^2+\cdots+b^n}$$

$$\frac{1}{A}=\frac{1+a+a^2+\cdots+a^n}{1+a+a^2+\cdots+a^{n-1}}=1+\frac{a^n}{1+a+a^2+\cdots+a^{n-1}}=1+\frac{1}{\frac{1+a+a^2+\cdots+a^{n-1}}{a^n}}$$

$$=1+\frac{1}{\frac{1}{a^n}+\frac{1}{a^{n-1}}+\frac{1}{a^{n-2}}+\cdots+\frac{1}{a}}$$

Similarly, $\dfrac{1}{B}=1+\dfrac{1}{\frac{1}{b^n}+\frac{1}{b^{n-1}}+\cdots+\frac{1}{b}}$

As $\quad a>b$

$\Rightarrow \quad a^k>b^k$ for all $k\in\mathbb{N}$

$\Rightarrow \quad \dfrac{1}{a^k}<\dfrac{1}{b^k}$

$\Rightarrow \quad \displaystyle\sum_{k=1}^{n}\frac{1}{a^k}<\sum_{k=1}^{n}\frac{1}{b^k}$

$\Rightarrow \quad \dfrac{1}{\sum_{k=1}^{n}\frac{1}{a^k}}>\dfrac{1}{\sum_{k=1}^{n}\frac{1}{b^k}}$

$\Rightarrow \quad \dfrac{1}{A}>\dfrac{1}{B}$

$\Rightarrow \quad A<B$

Karl Theodor Wilhelm Weierstrass

31 Oct 1815–19 Feb 1897
Nationality: German

2.2 Weirstras's Inequality

For positive numbers $a_1, a_2, \ldots, a_n (n\ge 2)$ we have

$$(1+a_1)(1+a_2)\cdots(1+a_n)>1+a_1+a_2+\cdots+a_n$$

If $a_1, a_2, \ldots, a_n$ are positive numbers less than unity, then

$$(1-a_1)(1-a_2)\cdots(1-a_n)>1-(a_1+a_2+\cdots+a_n).$$

Build-up Your Understanding 1

1. If $a_1, a_2, a_3, \ldots, a_n$ are n positive real numbers, then prove that $(1+a_1)(1+a_2)\cdots(1+a_n)>1+a_1+a_2+\cdots+a_n$ for $n\ge 2$.
2. Let a, b, p, q are positive reals such that $a<b$ and $q<p$. Then prove that $(a^p+b^p)(a^q-b^q)<(a^q+b^q)(a^p-b^p)$.
3. In a right angled triangle ABC, which is right angled at C, prove that $a^n+b^n<c^n$ for all $n>2$.
4. For positive real numbers a, b and c, prove that $a^{b+c}b^{c+a}c^{a+b}\le(a^ab^bc^c)^2$.
5. For positive real numbers a and b, prove that
$$\frac{a+b}{1+a+b}<\frac{a}{1+a}+\frac{b}{1+b}.$$
6. For $n=1,2,3,\ldots$, let $A_n=\dfrac{3}{4}-\left(\dfrac{3}{4}\right)^2+\left(\dfrac{3}{4}\right)^3-\cdots+(-1)^{n-1}\left(\dfrac{3}{4}\right)^n$, and $B_n=1-A_n$.

   Find the smallest natural number n_0 such that $B_n>A_n$ for all $n\ge n_0$.

2.3 Modulus Inequalities

$$|x| = \begin{cases} x & if \ x \geq 0 \\ -x & if \ x < 0 \end{cases}.$$

Also, note that $|x| = \max\{-x, x\}$. Modulus function is also called distance function. It denotes distance of x from origin.

1. $-|a| \leq a \leq |a|$ for each $a \in \mathbb{R}$.
2. If $b \geq 0$, then $|x - a| \leq b$ if and only if $a - b \leq x \leq a + b$.
3. $|a + b| \leq |a| + |b|$. More generally, $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$.
4. $||a| - |b|| \leq |a - b|$

The above inequality (3) explain that in a triangle, sum of lengths of any two sides is greater than the third side. Equality holds when both x and y have same sign or atleast one of them is '0'.
Similarly inequality (4) explain that in a triangle, difference of lengths of any two sides is less than the third side. Equlity holds when both x and y have same sign or atleast one of them '0'.

2.3.1 Triangular Inequalities

Let a, b, c be sides of a triangle, then we have following equivalent results:

1. $a + b > c$, $b + c > a$, $c + a > b$
2. If c is maximum, then $a + b > c$
3. $a > |b - c|$, $b > |c - a|$, $c > |a - b|$
4. $|a - b| < c < a + b$
5. $(a + b - c)(b + c - a)(c + a - b) > 0$
6. $a = y + z$, $b = z + x$, $c = x + y$, where $x, y, z \in \mathbb{R}^+$

Example 3 *Let $A_1A_2A_3$ and $B_1B_2B_3$ be triangles. If $p = A_1A_2 + A_2A_3 + A_3A_1 + B_1B_2 + B_2B_3 + B_3B_1$ and $q = A_1B_1 + A_1B_2 + A_1B_3 + A_2B_1 + A_2B_2 + A_2B_3 + A_3B_1 + A_3B_2 + A_3B_3$, prove that $3p \leq 4q$.*

Solution: Note that, $AB + BC \geq AC$
Now

$$\left.\begin{aligned} A_1B_1 + B_1A_2 &\geq A_1A_2 \\ A_1B_1 + B_1A_3 &\geq A_1A_3 \\ A_1B_2 + B_2A_2 &\geq A_1A_2 \\ A_1B_2 + B_2A_3 &\geq A_1A_3 \\ A_1B_3 + B_3A_2 &\geq A_1A_2 \\ A_1B_3 + B_3A_3 &\geq A_1A_3 \end{aligned}\right\} \text{6 ineqaulities}$$

Similarly write six inequalities starting with each of A_2, A_3, B_1, B_2, B_3 and add all 36 inequalities to get
$8(A_1B_1 + A_1B_2 + A_1B_3 + A_2B_1 + A_2B_2 + A_2B_3 + A_3B_1 + A_3B_2 + A_3B_3) \geq 6(A_1A_2 + A_2A_3 + A_3A_1 + B_1B_2 + B_2B_3 + B_3B_1)$

$$\Rightarrow 8q \geq 6p$$

$$\Rightarrow 4q \geq 3p.$$

Example 4: *Let $n \geq 3$ be a natural number and let P be a polygon with 'n' sides. Let $a_1, a_2, a_3, \ldots, a_n$ be the lengths of the sides of P and let p be its perimeter. Prove that,*

$$\frac{a_1}{p-a_1}+\frac{a_2}{p-a_2}+\frac{a_3}{p-a_3}+\cdots+\frac{a_n}{p-a_n}<2.$$

Solution:

Lemma: Let 'r' and 's' be two positive real numbers, such that $r < s$ or $\frac{r}{s}<1$. Then $\frac{r}{s}<\frac{r+x}{s+x}$ for any positive real x.

Proof: $\frac{r}{s}<\frac{r+x}{s+x} \Leftrightarrow r(s+x)<s\cdot(r+x) \Leftrightarrow rx<sx \Leftrightarrow r<s$

By polygon inequality,

$$a_1 < a_2+a_3+\cdots+a_n$$
$$\Rightarrow 2a_1 < a_1+a_2+\cdots+a_n = p$$

Similarly $\forall i, 2a_i < p \Rightarrow a_i < p-a_i \Rightarrow \frac{a_i}{p-a_i}<1$

$\therefore \frac{a_i}{p-a_i}<\frac{a_i+a_i}{(p-a_i)+a_i}<\frac{2a_i}{p}$ for all $i = 1, 2, 3, \ldots, n$ (By applying Lemma)

Summing up this inequality over i, we get,

$$\frac{a_1}{p-a_1}+\frac{a_2}{p-a_2}+\frac{a_3}{p-a_3}+\cdots+\frac{a_n}{p-a_n}<\frac{2\sum a_i}{p}$$
$$=\frac{2(a_1+a_2+a_3+\cdots+a_n)}{p}=\frac{2p}{p}=2.$$

Example 5 *If a, b, and c are the three sides of a triangle, and $a+b+c=2$, then prove that $a^2+b^2+c^2+2abc<2$.*

Solution: We know that $a+b+c=2$. By squaring, we get

$$4=(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$
$$\Rightarrow \quad a^2+b^2+c^2=2(2-ab-bc-ca)$$

Adding $2abc$ to both sides, we get

$$a^2+b^2+c^2+2abc=2(2-ab-bc-ca+abc)$$

To prove $a^2+b^2+c^2+2abc<2$, it is enough to prove that

$$2(2-ab-bc-ca+abc)<2 \text{ or } 2+abc-ab-bc-ca<1$$

or $\quad ab+bc+ca-abc-1>0$

as $\quad a+b+c=2s=2$

$\Rightarrow \quad s=1$

Now, $1(1-a)(1-b)(1-c)>0$ as the expression on the left is the square of the area of the triangle with sides a, b, c.

But, this implies

$$1^3-(a+b+c)1^2+(ab+bc+ca)1-abc>0$$

or $\quad 1-2+ab+bc+ca-abc>0$

or $\quad ab+bc+ca-abc-1>0$ as desired.

Example 6 *Show that for any ΔABC, the following inequality is true*

$$a^2+b^2+c^2 \quad a^2+b^2+c^2-\sqrt{3}(a^2-c^2)>0$$

where a, b, and c are the sides of the triangle in the usual notation.

Solution: Without loss of generality, we may assume $a \geq b \geq c$, so that $|c^2-a^2| = a^2-c^2$ is the maximum of $|a^2-b^2|$, $|b^2-c^2|$ and $|c^2-a^2|$.

It is enough to prove that $a^2+b^2+c^2-\sqrt{3}(a^2-c^2)>0$

Now,

$$a^2+b^2+c^2-\sqrt{3}(a^2-c^2) > a^2+(a-c)^2+c^2-\sqrt{3}(a^2-c^2)$$

(as $b > a-c$, by triangle inequality)

$$= 2a^2+2c^2-2ac-\sqrt{3}a^2+\sqrt{3}c^2$$

$$= (2-\sqrt{3})a^2+(2+\sqrt{3})c^2-2ac.$$

But, $(\sqrt{3}-1)^2 = 2(2-\sqrt{3})$ and $(\sqrt{3}+1)^2 = 2(2+\sqrt{3})$

So $a^2+b^2+c^2-\sqrt{3}(a^2-c^2) > \dfrac{[(\sqrt{3}-1)a]^2-4ac+[(\sqrt{3}+1)c]^2}{2}$

$$= \frac{1}{2}[(\sqrt{3}-1)a-(\sqrt{3}+1)c]^2 \geq 0.$$

and hence the result.

2.4 Sum of Squares (SOS)

Let x be a real number then we have $x^2 \geq 0$. This seems "trivial" but is the basis for every other inequality!

In general sum of squares of real numbers is non negative.

That is, $\Sigma x^2 \geq 0$.

Example 7 *Prove that* $x^2+y^2+z^2 \geq xy+yz+zx \ \forall\ x, y, z \in \mathbb{R}$.

Solution: Inequality is equivalent to

$\dfrac{1}{2}(x-y)^2+(y-z)^2+(z-x)^2 \geq 0,$ which is true.

Example 8 *If x, y, z are real and unequal numbers, prove that,* $2016x^2+2016y^2+6z^2 > 2(2013xy+3yz+3zx)$

Solution: We have, $(x-y)^2>0; (y-z)^2>0; (z-x)^2>0$

This implies that,

$$x^2+y^2>2xy \tag{1}$$

$$y^2+z^2>2yz \tag{2}$$

$$z^2+x^2>2zx \tag{3}$$

Multiply Inequality (1) by 2013 and Inequalities (2) and (3) by 3, then we have

$$2013x^2 + 2013y^2 > 2(2013xy) \quad (4)$$

$$3y^2 + 3z^2 > 2(3yz) \quad (5)$$

$$3z^2 + 3x^2 > 2(3zx) \quad (6)$$

Adding Inequalities (4), (5) and (6), we get the desired results.

Example 9 *Find all real numbers x and y, so that,*

$$x^2 + 2y^2 + \frac{1}{2} \le x(2y+1)$$

Solution: Multiply the given inequality by 2

Then, $2x^2 + 4y^2 + 1 \le 2x(2y + 1) = 4xy + 2x$

i.e., $(4y^2 + x^2 - 4xy) + (x^2 - 2x + 1) \le 0$

i.e., $(2y - x)^2 + (x - 1)^2 \le 0$

But, by trivial inequality, $a^2 \ge 0 \ \forall$ real 'a'.

Hence, $(2y - x) = (x - 1) = 0 \quad \Rightarrow \quad x = 1$ and $y = \dfrac{1}{2}$.

Example 10 *Three positive real numbers a, b, c are such that,* $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0$. *Can a, b, c be the lengths of the sides of a triangle? Justify your answer.*

Solution: Now,

$$a^2 + 5b^2 + 4c^2 - 4ab - 4bc = (a^2 + 4b^2 - 4ab) + (b^2 + 4c^2 - 4bc)$$
$$= (a - 2b)^2 + (b - 2c)^2$$

$\therefore$ Expression $= (a-2b)^2 + (b-2c)^2 = 0 \quad \Rightarrow \quad a - 2b = 0$ and $b - 2c = 0$ or $a = 2b$ and $b = 2c$

$\therefore$ $a = 4c$; this implies $a : b : c = 4 : 2 : 1$.

Now, $(b + c) : a = 3 : 4 \Rightarrow$ the triangle law is violated.

$\therefore$ a, b, c cannot form a triangle.

Example 11 *For* $x, y \in \mathbb{R}$, *prove that* $3(x + y + 1)^2 + 1 \ge 3xy$.

Solution: $3(x + y + 1)^2 + 1 - 3xy \ge 0$

LHS $= 3x^2 + 3y^2 + 3xy + 6x + 6y + 4$

$$= 3\left(x + \frac{1}{2}y + 1\right)^2 + \left(\frac{3}{2}y + 1\right)^2 \ge 0.$$

Example 12 *For* $x, y, z, \in \mathbb{R}$ *such that* $xy + yz + zx = -1$. *Prove that* $x^2 + 5y^2 + 8z^2 \ge 4$.

Solution: $x^2 + 5y^2 + 8z^2 - 4 = x^2 + 5y^2 + 8z^2 + 4(xy + yz + zx)$

$$= (x + 2y + 2z)^2 + (y - 2z)^2 \ge 0$$

$\Rightarrow \quad x^2 + 5y^2 + 8z^2 \ge 4.$

Example 13 For $x, y, z \in \mathbb{R}^+$, *prove that*

$$\frac{x^2 + yz}{y + z} + \frac{y^2 + zx}{z + x} + \frac{z^2 + xy}{x + y} \ge x + y + z.$$

Solution: Consider $\dfrac{x^2+yz}{y+z}-x+\dfrac{y^2+zx}{z+x}-y+\dfrac{z^2+xy}{x+y}-z$

$$=\frac{x^2-(y+z)x+yz}{y+z}+\frac{y^2-(z+x)y+zx}{z+x}+\frac{z^2-(x+y)z+xy}{x+y}$$

$$=\frac{(x-y)(x-z)}{y+z}+\frac{(y-z)(y-x)}{z+x}+\frac{(z-x)(z-y)}{x+y}$$

$$=\frac{(x^2-y^2)(x^2-z^2)+(y^2-z^2)(y^2-x^2)+(z^2-x^2)(z^2-y^2)}{(x+y)(y+z)(z+x)}$$

$$=\frac{x^4+y^4+z^2-x^2y^2-y^2z^2-z^2x^2}{(x+y)(y+z)(z+x)}$$

$$=\frac{(x^2-y^2)^2+(y^2-z^2)^2+(z^2-x^2)^2}{2(x+y)(y+z)(z+x)}\geq 0.$$

Example 14 *Let $a_3, a_4, \ldots, a_{2005}, a_{2006}$ be real numbers with $a_{2006} \neq 0$. Prove that there are not more than 2004 real numbers x such that,*

$$1+x+x^2+a_3x^3+a_4x^4+\cdots+a_{2005}x^{2005}+a_{2006}x^{2006}=0.$$

Solution: Replace x by $\dfrac{1}{x}$ in equation and multiply by x^{2006}, we get

$$x^{2006}+x^{2005}+x^{2004}+a_3x^{2003}+\cdots+a_{2006}=0$$

Now $$\sum \alpha_i=-1,\quad \sum_{1\leq i<j\leq 2006}\alpha_i\alpha_j=1$$

As $$\sum \alpha_i^2=\left(\sum_{i=1}^{2006}\alpha_i\right)^2-2\sum_{1\leq i<j\leq 2006}\alpha_i\alpha_j$$
$$=(-1)^2-2(1)=-1$$

$\Rightarrow\ \sum \alpha_i^2<0$ which is not possible if all α_i are real.

Hence, at least two non-real roots $\Rightarrow$ at most 2004 real roots.

Example 15 *Let a, b, c, d, e, f be real numbers such that the polynomial*

$$P(x)=x^8-4x^7+7x^6+ax^5+bx^4+cx^3+dx^2+ex+f$$

factorises into eight linear factors $x-x_i$, with $x_i>0$ for $i=1, 2, \ldots, 8$. Determine all possible values of f.

Solution: $$\sum_{i=1}^{8}x_i=4 \tag{1}$$

and $$\sum_{1\leq i<j\leq 8}x_ix_j=7 \tag{2}$$

$$\Rightarrow\ \sum_{i=1}^{8}x_i^2=\left(\sum_{i=1}^{8}x_i\right)^2-2\sum_{1\leq i<j\leq 8}x_ix_j$$
$$=16-14=2$$

Now
$$\sum_{1\le i<j\le 8}(x_i-x_j)^2 = 7\cdot\sum_{k=1}^{8}x_k^2 - 2\sum_{1\le i<j\le 8}x_ix_j$$
$$= 7\times 2 - 2\times 7$$
$$= 0$$

$$\Rightarrow x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8$$

$$\Rightarrow \quad x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = \frac{1}{2} \qquad \text{From Eq. (1)}$$

$$\Rightarrow f = x_1\cdot x_2\cdot x_3\cdot x_4\cdot x_5\cdot x_6\cdot x_7\cdot x_8 = \frac{1}{2^8} = \frac{1}{256}.$$

Example 16 *Let $a, b, c > 0$ satisfy $abc = 1$. Prove that*
$$\frac{1}{\sqrt{b+\frac{1}{a}+\frac{1}{2}}}+\frac{1}{\sqrt{c+\frac{1}{b}+\frac{1}{2}}}+\frac{1}{\sqrt{a+\frac{1}{c}+\frac{1}{2}}} \ge \sqrt{2}.$$

Solution: $a, b, c > 0$ and $abc = 1$

Let
$$a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}; x, y, z > 0$$

Given inequality becomes,
$$\frac{1}{\sqrt{\frac{y}{z}+\frac{y}{x}+\frac{1}{2}}}+\frac{1}{\sqrt{\frac{z}{x}+\frac{z}{y}+\frac{1}{2}}}+\frac{1}{\sqrt{\frac{x}{y}+\frac{x}{z}+\frac{1}{2}}} \ge \sqrt{2}$$

$$\Leftrightarrow \sqrt{\frac{\frac{1}{y}}{\frac{2}{z}+\frac{2}{x}+\frac{1}{y}}}+\sqrt{\frac{\frac{1}{z}}{\frac{2}{x}+\frac{2}{y}+\frac{1}{2}}}+\sqrt{\frac{\frac{1}{x}}{\frac{2}{y}+\frac{2}{z}+\frac{1}{x}}} \ge 1$$

Let, $\frac{1}{x} = p$, $\frac{1}{y} = q$, $\frac{1}{z} = r$ and let us also normalize it with $p + q + r = 1$

Given inequality becomes, $\sqrt{\frac{p}{2-p}}+\sqrt{\frac{q}{2-q}}+\sqrt{\frac{r}{2-r}} \ge 1$

Now

Claim: $\sqrt{\frac{u}{2-u}} \ge u \quad \forall u > 0$

Proof: $\sqrt{\frac{u}{2-u}} \Leftrightarrow \frac{u}{2-u} \ge u^2$

$$\Leftrightarrow 1 \ge u(2-u) \quad (\text{as } u > 0)$$
$$\Leftrightarrow u^2 - 2u + 1 \ge 0$$
$$\Leftrightarrow (u-1)^2 \ge 0 \quad \text{which is true}$$

Hence,

$$\sqrt{\frac{p}{2-p}} \geq p$$

$$\sqrt{\frac{q}{2-q}} \geq q$$

$$\sqrt{\frac{r}{2-r}} \geq r$$

Add all, $\sqrt{\frac{p}{2-p}} + \sqrt{\frac{q}{2-q}} + \sqrt{\frac{r}{2-r}} \geq p+q+r=1.$

2.4.1 Quadratic Inequality

If $x \in \mathbb{R}$, and $Ax^2 + Bx + C = 0$, then $B^2 - 4AC \geq 0$

If $4AC - B^2 \geq 0$ and x is real, then $A(Ax^2 + Bx + C) \geq 0$ for all real x. Converse also true.

Example 17 If $a, b, c \in \mathbb{R}$, such that $a \geq b \geq c$. Prove that

$$a^2 + ac + c^2 \geq 3b(a - b + c).$$

Solution: Rewrite as quadratic in b, as

$$3b^2 - 3(a+c)b + a^2 + ac + c^2 \geq 0 \tag{1}$$

$$D = 9(a+c)^2 - 12(a^2 + ac + c^2)$$

$$= -3(a-c)^2 \leq 0$$

$\Rightarrow$ Inequality (1) is true $\forall\ a, b, c \in \mathbb{R}$.

Build-up Your Understanding 2

1. For every natural number n, prove that $n^n > 1 \cdot 3 \cdot 5 \cdots (2n-1)$.
2. In a triangle ABC, prove that $\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$
3. If a, b, c be the length of the sides of a scalene triangle, prove that $(a+b+c)^3 > 27\,(a+b-c)\,(b+c-a)\,(c+a-b)$.
4. If a, b, c are positive real numbers representing the sides of a scalene triangle, prove that $ab + bc + ca < a^2 + b^2 + c^2 < 2(ab + bc + ca)$ or $1 < \frac{a^2+b^2+c^2}{ab+bc+ca} < 2,$ and hence prove that $3(ab + bc + ca) < (a + b + c)^2 < 4(ab + bc + ca)$ or $3 < \frac{(a+b+c)^2}{ab+bc+ca} < 4.$
5. If a, b, c are distinct real number, prove that $\left(\frac{a}{b-c}\right)^2 + \left(\frac{b}{c-a}\right)^2 + \left(\frac{c}{a-b}\right)^2 \geq 2.$

6. Let $a, b, c \in \mathbb{R}^+$, such that $abc = 1$, prove that $1+\dfrac{3}{a+b+c} \geq \dfrac{6}{ab+bc+ca}$.

7. Let $x, y \in \mathbb{R}^+$, prove that $\dfrac{1}{(1+x)^2}+\dfrac{1}{(1+y)^2} \geq \dfrac{2}{1+xy}$.

8. Let $x, y \in (0, 1)$, prove that $\dfrac{1}{1-x^2}+\dfrac{1}{1-y^2} \geq \dfrac{2}{1-xy}$.

2.5 Arithmetic Mean ≥ Geometric Mean ≥ Harmonic Mean

Given any n positive real numbers $a_1, a_2, ..., a_n$, the positive numbers A, G and H, defined by $A = \dfrac{1}{n}(a_1 + a_2 + \cdots + a_n)$, $G = (a_1, a_2, \ldots, a_n)^{1/n}$ and $\dfrac{1}{H} = \dfrac{1}{n}\left(\dfrac{1}{a_1}+\dfrac{1}{a_2}+\cdots+\dfrac{1}{a_n}\right)$ are called respectively the arithmetic mean (AM), geometric mean (GM) and harmonic mean (HM) of $a_1, a_2, \ldots, a_n$.

Note: A, G and H all are lie between the least and the greatest of $a_1, a_2, \ldots, a_n$. Equality holds in $A \geq G \geq H$, only when all the a_i are equal.

2.5.1 Derived Inequalities from AM ≥ GM ≥ HM

The following inequalities derived from AM ≥ GM ≥ HM, will be very useful for problem solving:

- $x^2 + y^2 + xy \geq \dfrac{3}{4}\ (x + y)^2$ **(Sophie Inequality)**
- $x^2 + y^2 - xy \geq xy$
- $x^3 + y^3 \geq xy(x + y)$
- $\dfrac{ab}{a+b} \leq \dfrac{a+b}{4}$
- $\dfrac{a^2+b^2}{a+b} \geq \dfrac{a+b}{2}; \dfrac{a^2+b^2+c^2}{a+b+c} \geq \dfrac{a+b+c}{3}$, etc.
- $xy \leq \left(\dfrac{x+y}{2}\right)^2$

Example 18 *If a, b, c, d are any four positive real numbers, then prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4.$$

Solution: We use AM–GM inequality for the four numbers $\dfrac{a}{b}, \dfrac{b}{c}, \dfrac{c}{d}$ and $\dfrac{d}{a}$.

$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}{4} \geq \sqrt[4]{\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{a}}$$

or $$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4\times 1 = 4.$$

Example 19 *If a, b, c, and d are four positive real numbers, such that abcd* = 1, *then prove that*

$$(1 + a)(1 + b)(1 + c)(1 + d) \geq 16.$$

Solution: We know that $(1 + a)(1 + b)(1 + c)(1 + d)$

$= 1 + (a + b + c + d) + (ab + ac + ad + bc + bd + cd)$
$\quad + (abc + acd + abd + bcd) + abcd$

$= 1 + abcd + (a + bcd) + (b + acd) + (c + abd) + (d + abc) + (ab + cd) + (ac + bd) +$
$\quad (ad + bc)$

$$= 1 + 1 + \left(a + \frac{1}{a}\right) + \left(b + \frac{1}{b}\right) + \left(c + \frac{1}{c}\right) + \left(d + \frac{1}{d}\right)$$
$$+ \left(ab + \frac{1}{ab}\right) + \left(ac + \frac{1}{ac}\right) + \left(ad + \frac{1}{ad}\right)$$

But, for all real $k > 0$, $k + \frac{1}{k} \geq 2$. Hence

$$(1 + a)(1 + b)(1 + c)(1 + d)$$
$$= 2 + \left(a + \frac{1}{a}\right) + \left(b + \frac{1}{b}\right) + \left(c + \frac{1}{c}\right) + \left(d + \frac{1}{d}\right)$$
$$+ \left(ab + \frac{1}{ab}\right) + \left(ac + \frac{1}{ac}\right) + \left(ad + \frac{1}{ad}\right)$$
$$\geq 2 + 2 \times 7 = 16$$

Aliter: AM $\geq$ GM

$$1 + a \geq 2\sqrt{a}$$
$$1 + b \geq 2\sqrt{b}$$
$$1 + c \geq 2\sqrt{c}$$
$$1 + d \geq 2\sqrt{d}$$
$$\Rightarrow \quad (1 + a)(1 + b)(1 + c)(1 + d) \geq 16\sqrt{abcd} = 16.$$

Example 20 *If* $b_1, b_2, \ldots, b_n$ *is a permutation of the n positive numbers* $a_1, a_2, \ldots, a_n$, then, $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n$.

Solution: Applying the AM–GM inequality on n numbers $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$, we have

$$\frac{1}{n}\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right) \geq \sqrt[n]{\frac{a_1}{b_1} \times \frac{a_2}{b_2} \times \cdots \times \frac{a_n}{b_n}} = \sqrt[n]{1} = 1$$

$$\therefore \quad \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n.$$

Example 21 If $a_1, a_2, \ldots,$ *an are all positive, then*

$$\sqrt{a_1a_2}+\sqrt{a_1a_3}+\cdots+\sqrt{a_1a_n}+\sqrt{a_2a_3}+\sqrt{a_2a_4}+\cdots+\sqrt{a_2a_n}+\cdots+\sqrt{a_{n-1}a_1}+\sqrt{a_{n-1}a_2}$$
$$+\cdots+\sqrt{a_{n-1}a_{n-1}}+\sqrt{a_{n-1}a_n}\le\frac{n-1}{2}(a_1+a_2+\cdots+a_n).$$

Solution: By AM–GM inequality,

$$\sqrt{a_1a_2}\le\frac{a_1+a_2}{2}$$

$$\sqrt{a_1a_3}\le\frac{a_1+a_3}{2}$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$\sqrt{a_1a_n}\le\frac{a_1a_n}{2}$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$\sqrt{a_ia_j}\le\frac{a_i+a_j}{2}\quad(\text{Where } i\ne j,\ i,j=1,2,\ldots,n)$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$\sqrt{a_{n-1}a_n}\le\frac{a_{n-1}+a_n}{2}$$

There are $\dfrac{n(n-1)}{2}$ inequalities. On the right-hand side, each a_i occurs $(n-1)$ times. Adding these inequalities, we get

$$\sqrt{a_1a_2}+\sqrt{a_1a_3}+\cdots+\sqrt{a_ia_j}+\cdots+\sqrt{a_{n-i}a_n}\le(n-1)\frac{(a_1+a_2+\cdots+a_n)}{2}$$
$$=\frac{n-1}{2}(a_1+a_2+\cdots+a_n).$$

Example 22 *If* $a_1+a_2+a_3+\cdots+a_n=1$, $a_i>0$ *for all a, show that*

$$\sum_{i=1}^{n}\frac{1}{a_i}\ge n^2.$$

Solution: $(a-b)^2\ge 0$

$$\Rightarrow\quad a^2+b^2\ge 2ab$$

$$\Rightarrow\quad \frac{a}{b}+\frac{b}{a}\ge 2$$

$$a_1+a_2+a_3+\cdots+a_n=1 \tag{1}$$

Dividing Eq. (1) by $a_1, a_2, a_3, \ldots, a_n$ successively and adding, we get

$$1+\frac{a_2}{a_1}+\frac{a_3}{a_1}+\cdots+\frac{a_n}{a_1}=\frac{1}{a_1};$$

$$\frac{a_1}{a_2}+1+\frac{a_3}{a_2}+\cdots+\frac{a_n}{a_1}=\frac{1}{a_1};$$

$$\frac{a_1}{a_r}+\frac{a_2}{a_r}+\cdots+\frac{a_{r-1}}{a_r}+1+\frac{a_{r+1}}{a_r}+\cdots+\frac{a_n}{a_r}=\frac{1}{a_r};$$

and $\frac{a_1}{a_n}+\frac{a_2}{a_n}+\frac{a_3}{a_n}+\cdots+\frac{a_{n-1}}{a_n}+1=\frac{1}{a_n}$

Adding $\underbrace{1+1+1+\cdots+1}_{n \text{ terms}}+\sum_{\substack{1\le i,\, j\le n\\ i\ne j}}\frac{a_i}{a_j}=\sum_{i=1}^{n}\frac{1}{a_i}$

In $\sum\frac{a_i}{a_j}$, there are $n(n-1)$ fractions $\frac{a_i}{a_j}$ are all distinct. Pairing $\frac{a_i}{a_j}$ and $\frac{a_j}{a_i}$, there are $\frac{n(n-1)}{2}$ pairs of fractions of the form $\frac{a_i}{a_j}+\frac{a_j}{a_i}$.

But, each $\frac{a_i}{a_j}+\frac{a_j}{a_i}\ge 2$

$$\therefore \qquad \sum_{i=1}^{n}\frac{1}{a_i}\ge n+\frac{n(n-1)}{2}\times 2$$

$$\Rightarrow \qquad \sum_{i=1}^{n}\frac{1}{a_i}\ge n+n^2-n=n^2$$

Equality holds when all *ai*, are equal, *i.e.*, each is equal to $\frac{1}{n}$.

Aliter: By AM–HM inequality

$$\frac{\sum a_i}{n}\ge\frac{n}{\sum\frac{1}{a_i}} \quad\Rightarrow\quad \frac{1}{n}\ge\frac{n}{\sum\frac{1}{a_i}} \quad\Rightarrow\quad \sum\frac{1}{a_i}\ge n^2.$$

Example 23 *A and B are the AM and GM between two positive numbers a and b; prove that,* $B<\frac{(a-b)^2}{8(A-B)}<A.$

Solution: Let $A=\frac{a+b}{2}$ and $B=\sqrt{ab}$;

Now, $A>B$ as $\left(\frac{a+b}{2}\right)-\sqrt{ab}=\frac{a+b-2\sqrt{ab}}{2}=\frac{(\sqrt{a}-\sqrt{a})^2}{2}\ge 0$

and as A, B are positive, we have shown $A>B$.

Also, $\dfrac{(a-b)^2}{8(A-B)}$ can be written as $\dfrac{(a-b)^2(A+B)}{8(A^2-B^2)}$

i.e., $\dfrac{(a-b)^2}{8(A-B)} = \dfrac{(a-b)^2}{8}\left[\dfrac{A+B}{A^2-B^2}\right]$

Now $A^2 - B^2 = \left(\dfrac{a^2+b^2+2ab}{4}\right) - ab = \dfrac{(a-b)^2}{4}$

$\therefore \quad \dfrac{(a-b)^2}{8(A-B)} = \dfrac{(a-b)^2}{8} \times \dfrac{(A+B)}{(a-b)^2} \times 4 = \dfrac{A+B}{2}$

As $A > B$,

$\Rightarrow B < \dfrac{A+B}{2} < A$

$\Rightarrow B < \dfrac{(a-b)^2}{8(A-B)} < A.$

Example 24 *Let a, b, c, d be distinct positive numbers in HP. Then prove that*

(i) $a + d > b + c$ (ii) $ad > bc$

Solution:

(i) AM > HM $\Rightarrow \dfrac{a+c}{2} > b \Leftrightarrow a + c > 2b$ (1)

similarly, $b + d > 2c$ (2)

Adding Inequalities (1) and (2), we get

$$a + b + c + d > 2(b + c)$$
$$\Leftrightarrow a + d > b + c.$$

(ii) GM > HM $\Rightarrow \sqrt{ac} > b$ and $\sqrt{bd} > c$

Multiplying, $\Rightarrow \sqrt{abcd} > bc \Rightarrow \sqrt{ad} > \sqrt{bc}$

squaring, $\Rightarrow ad > bc$.

Example 25: *If a, b, c are positive real numbers that satisfy $a^2 + b^2 + c^2 = 1$, find the minimal value of*

$$S = \frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}.$$

Solution: $\dfrac{\dfrac{a^2b^2}{c^2} + \dfrac{b^2c^2}{a^2}}{2} \geq \left(\dfrac{a^2b^2}{c^2} \cdot \dfrac{b^2c^2}{a^2}\right)^{\frac{1}{2}} = b^2$

Or $$\frac{1}{2}\left(\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2}\right) \geq b^2 \quad (1)$$

Similarly $$\frac{1}{2}\left(\frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}\right) \geq c^2 \quad (2)$$

and $$\frac{1}{2}\left(\frac{c^2a^2}{b^2} + \frac{a^2b^2}{c^2}\right) \geq a^2 \quad (3)$$

Adding Inequalities (1), (2) and (3), we get

$$\frac{a^2b^2}{c^2}+\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}\geq a^2+b^2+c^2\geq 1$$

Equality holds when $a^2=b^2=c^2=\frac{1}{3}$.

Example 26 *Given that the equation* $x^4+px^3+qx^2+rx+s=0$ *has four positive roots, prove that*

(i) $pr-16s\geq 0$,
(ii) $q^2-36s\geq 0$.

Solution: Let $\alpha, \beta, \gamma, \delta$ be the four positive roots of the given polynomial. Then,

$$\alpha+\beta+\gamma+\delta=-p \quad (1)$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=q \quad (2)$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-r \quad (3)$$

$$\alpha\beta\gamma\delta=s \quad (4)$$

(i) Using AM–GM inequality in Eqs. (1) and (3), we get

$$\frac{\alpha+\beta+\gamma+\delta}{4}\cdot\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{4}$$

$$\geq\sqrt[4]{\alpha\beta\gamma\delta}\sqrt[4]{\alpha^3\beta^3\gamma^3\delta^3}=\alpha\beta\gamma\delta=s$$

$$\Rightarrow \quad \frac{-p}{4}\cdot\left(\frac{-r}{4}\right)\geq s$$

$$\Rightarrow \quad pr\geq 16s \quad \text{or} \quad pr-16s>0.$$

(ii) Applying AM–GM inequality in Eq. (2), we get

$$\frac{q}{6}\geq\sqrt[6]{\alpha^3\beta^3\gamma^3\delta^3}=\sqrt{s}$$

$$\Rightarrow \quad q^2\geq 36s \text{ or } q^2-36s\geq 0.$$

Example 27 *a, b, c are real numbers, such that* $a+b+c=0$ and $a^2+b^2+c^2=1$. *Prove that,* $a^2b^2c^2\leq\frac{1}{54}$.

Solution: If one of a, b, c is zero, the result is trivial.
Since $a+b+c=0$, without loss of generality assume that $a>0$, $b>0$ and $c<0$ (as $a+b+c=0$, two terms must have the same sign and one term the opposite sign)

$$\therefore \quad c=-(a+b) \quad (1)$$

Now, $1=a^2+b^2+c^2=a^2+b^2+(a+b)^2=2(a^2+ab+b^2)$ (2)

$$\Rightarrow a^2+ab+b^2=\frac{1}{2} \quad (3)$$

By AM–GM inequality, $(a^2+b^2)+ab\geq 3ab$ (4)

$$\therefore \quad 3ab\leq\frac{1}{2} \quad \Rightarrow \quad ab\leq\frac{1}{6} \quad (5)$$

Equality holds only when $a = b = \dfrac{1}{\sqrt{6}}$ (6)

Now, $$c^2 = (a+b)^2 = a^2 + b^2 + 2ab = \frac{1}{2} + ab \le \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$\Rightarrow \quad a^2b^2c^2 = (ab)^2 \cdot c^2 \le \left(\frac{1}{6}\right)^2 \cdot \frac{2}{3} = \frac{1}{54}$$

i.e., $a^2b^2c^2 \le \dfrac{1}{54}$ as desired.

Equality holds, iff $a = b = \dfrac{1}{\sqrt{6}}$ and $c = -\dfrac{2}{\sqrt{6}}$ (as $c = -(a+b)$)

If the sign restriction is removed, we have two of them are $\pm\dfrac{1}{\sqrt{6}}$ and the third as $\mp\dfrac{2}{\sqrt{6}}$.

Example 28 *If a, b, and c are positive real numbers, such that* $a + b + c = 1$, *then prove that* $(1 + a)(1 + b)(1 + d) \ge 8(1 - a)(1 - b)(1 - c)$.

Solution: We know that $a + b + c = 1$

$\Rightarrow \quad 1 + a = 1 + 1 - (b + c) = (1 - b) + (1 - c)$

Since, $a + b + c = 1$ where a, b, and c are positive real numbers, so $1 - b$ and $1 - c$ are positive.

Applying AM–GM inequality, we get

$$1 + a = (1-b) + (1-c) \ge 2\sqrt{(1-b)(1-c)} \tag{1}$$

Similarly $1 + b = (1-a) + (1-c) \ge 2\sqrt{(1-a)(1-c)}$ (2)

and $$1 + c = (1-b)(1-a) \ge 2\sqrt{(1-b)(1-a)} \tag{3}$$

Multiplying Eqs. (1), (2), and (3), we get

$$(1 + a)(1 + b)(1 + c) \ge 8(1 - a)(1 - b)(1 - c).$$

Example 29 *Let a, b, c be real numbers with* $0 < a, b, c < 1$ *and* $a + b + c = 2$. *Prove that*

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \ge 8.$$

Solution: Here, we use AM $\ge$ GM

$$a = \frac{(a+b-c)+(a-b+c)}{2} \ge \sqrt{(a+b-c)(a-b+c)}$$

$$b = \frac{(b+a-c)+(b-a+c)}{2} \ge \sqrt{(b+a-c)(b-a+c)}$$

$$c = \frac{(c+a-b)+(c-a+b)}{2} \ge \sqrt{(c+a-b)(c-a+b)}$$

$$abc \ge \sqrt{\begin{aligned}&(a+b-c)(a-b+c)(b+a-c)(b-a+c)\\ &\qquad\qquad (c+a-b)(c-a+b)\end{aligned}}$$

$$= (a + b - c)(b + c - a)(c + a - b)$$

i.e., $a \cdot b \cdot c \geq (a+b-c)(b+c-a)(c+a-b)$

$$= (2-2c)\,(2-2a)(2-2b) \quad [\text{as } a+b+c=2]$$

$$= 8(1-a)(1-b)(1-c)$$

$$\therefore \qquad \frac{a}{1-a}\cdot\frac{b}{1-b}\cdot\frac{c}{1-c} \geq 8.$$

Example 30 *If* $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ *for* $x, y, z > 0$, *prove that* $(x-1)(y-1)(z-1) \geq 8$.

Solution: $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1;\ x, y, z > 0$

Let, $\qquad x=\frac{1}{a},\ y=\frac{1}{b},\ z=\frac{1}{c} \ \Rightarrow\ a+b+c=1$

Also $\qquad (x-1)(y-1)(z-1) \geq 8 \Leftrightarrow \left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8$

$$\Leftrightarrow (1-a)(1-b)(1-c) \geq 8abc$$

$$\Leftrightarrow (b+c)(c+a)(a+b) \geq 8abc$$

Now $\qquad a+b \geq 2\sqrt{ab}$

Similarly $\qquad b+c \geq 2\sqrt{bc}$

and $\qquad c+a \geq 2\sqrt{ca}$

$$\Rightarrow \quad (a+b)(b+c)(c+a) \geq 8abc.$$

Example 31 *Let* a, b, c *be positive real numbers, such that,* $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \leq 1.$ *Prove that,* $(1+a^2)(1+b^2)(1+c^2) \geq 125.$ *When does equality holds?*

Solution: Now

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \leq 1 \ \Rightarrow\ \frac{1}{1+b}+\frac{1}{1+c} \leq 1-\frac{1}{1+a}=\frac{a}{1+a};$$

$$\therefore\ \frac{a}{1+a} \geq \frac{1}{1+b}+\frac{1}{1+c} \qquad (1)$$

Similarly $\frac{b}{1+b} \geq \frac{1}{1+c}+\frac{1}{1+a}$ $\qquad (2)$

and $\frac{c}{1+c} \geq \frac{1}{1+a}+\frac{1}{1+b}$ $\qquad (3)$

Apply AM–GM for $\frac{1}{1+b}+\frac{1}{1+c}$

$$\therefore\ \frac{1}{1+b}+\frac{1}{1+c} \geq \frac{2}{\sqrt{(1+b)(1+c)}} \ \Rightarrow\ \frac{a}{1+a} \geq \frac{2}{\sqrt{(1+b)(1+c)}} \qquad (4)$$

Similarly $\frac{b}{1+b} \geq \frac{2}{\sqrt{(1+c)(1+a)}}$ and $\frac{c}{1+c} \geq \frac{2}{\sqrt{(1+a)(1+b)}}$ $\qquad (5)$

Multiply the results of Inequalities (4), (5) to get

$$\left(\frac{a}{1+a}\right)\left(\frac{b}{1+b}\right)\left(\frac{c}{1+c}\right) \geq \left(\frac{2}{\sqrt{(1+b)(1+c)}}\right)\left(\frac{2}{\sqrt{(1+c)(1+a)}}\right)\left(\frac{2}{\sqrt{(1+a)(1+b)}}\right) \quad (6)$$

$\Rightarrow abc \geq 8$ (7)

Expand $F = (1 + a^2)(1 + b^2)(1 + c^2)$ to get

$$F = 1+(a^2+b^2+c^2)+(a^2b^2+b^2c^2+c^2a^2)+a^2b^2c^2$$

i.e., $F \geq 1+(3)(a^2b^2c^2)^{\frac{1}{3}}+3(a^4b^4c^4)^{\frac{1}{3}}+(a^2b^2c^2)$

i.e., $F \geq 1+(3)(2^2)+3(2^4)+(8)^2$ (as $abc \geq 8$, from Inequality (7))

i.e., $F \geq 1 + 12 + 48 + 64 = 125$.

Example 32 *x and y are positive real numbers; prove that*

$4x^4+4y^3+5x^2+y+1 \geq 12xy$.

Solution: Now, $4x^4+1 \geq 4x^2$ (AM–GM inequality)

and $4y^3+y \geq 4y^2$ (AM–GM inequality)

and hence,

$$4x^4+4y^3+5x^2+y+1 \geq 4x^2+4y^2+5x^2, \textit{ i.e., } 9x^2+4y^2$$

Again, taking AM–GM, $9x^2+4y^2 \geq 2\sqrt{36x^2y^2} = 12xy$.

$\Rightarrow 4x^4 + 4y^3 + 5x^2 + y + 1 \geq 12xy$.

Example 33 *Prove that, for all* $x, y, z \geq 0$, $x^2+xy^2+xyz^2 \geq 4xyz-4$.

Solution: $x^2+xy^2+xyz^2 \geq 4xyz-4 \Leftrightarrow x^2+xy^2+xyz^2+4 \geq 4xyz$

Now by AM–GM for x^2 and 4; $x^2+4 \geq 4x$

AM–GM for $4x$ and xy^2; $4x+xy^2 \geq 4xy$

AM–GM for $4xy$ and xyz^2; $4xy+xyz^2 \geq 4xyz$

$\Rightarrow x^2+xy^2+xyz^2+4 \geq 4xyz \quad \Rightarrow \quad x^2+xy^2+xyz^2 \geq 4xyz-4$.

Example 34 *Given real numbers a, b, c, d, e, all greater than unity, prove that,* $\frac{a^2}{c-1}+\frac{b^2}{d-1}+\frac{c^2}{e-1}+\frac{d^2}{a-1}+\frac{e^2}{b-1} \geq 20$.

Solution: We know that $(a-2)^2 \geq 0$, *i.e.*, $a^2-4a+4 \geq 0$

i.e., $$a^2 \geq 4(a-1) \quad (1)$$

Since, $a > 1$, we have $\frac{a^2}{a-1} \geq 4$ (2)

Similarly, $\frac{b^2}{b-1} \geq 4$; $\frac{c^2}{c-1} \geq 4$; $\frac{d^2}{d-1} \geq 4$; $\frac{e^2}{e-1} \geq 4$ (3)

By applying AM–GM inequality, we get,

$$\frac{a^2}{c-1}+\frac{b^2}{d-1}+\frac{c^2}{c-1}+\frac{d^2}{b-1}+\frac{e^2}{b-1} \geq 5\sqrt[5]{\frac{a^2b^2c^2d^2e^2}{(a-1)(b-1)(c-1)(d-1)(e-1)}}$$

$$\geq 5\sqrt[5]{4\cdot4\cdot4\cdot4\cdot4} = 5\times4 = 20.$$

Example 35 *If x, y, z are each greater than 1, prove that*

$$\frac{x^4}{(y-1)^2}+\frac{y^4}{(z-1)^2}+\frac{z^4}{(x-1)^2}\geq 48$$

Solution: Put $(x-1)=a$, so that $x=a+1$; similarly $y=b+1$; $z=c+1$ (1)

Thus, $\dfrac{x^4}{(y-1)^2}+\dfrac{y^4}{(z-1)^2}+\dfrac{z^4}{(z-1)^2}=\dfrac{(a+1)^4}{b^2}+\dfrac{(b+1)^4}{c^2}+\dfrac{(c+1)^4}{a^2}$

Apply AM–GM to the quantities, $\dfrac{(a+1)^4}{b^2},\dfrac{(b+1)^4}{c^2},\dfrac{(c+1)^4}{a^2}$; we get (2)

$$\therefore \frac{(a+1)^4}{b^2}+\frac{(b+1)^4}{c^2}+\frac{(c+1)^4}{a^2}\geq 3\left\{\frac{(a+1)^4(b+1)^4(c+1)^4}{a^2b^2c^2}\right\}^{\frac{1}{3}} \quad (3)$$

Also apply AM–GM for $a+1$, $b+1$, $c+1$;

Thus, $a+1\geq 2\sqrt{a}$, so that $(a+1)^4\geq(2\sqrt{a})^4=16a^2$

Similarly, $(b+1)^4\geq 16b^2$ and $(c+1)^4\geq 16c^2$ (4)

Thus the given expression $\geq 3\left\{\dfrac{16\cdot a^2\cdot 16\cdot b^2\cdot 16\cdot c^2}{a^2b^2c^2}\right\}^{\frac{1}{3}}=3\times 16=48.$

Example 36 *Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of the products of $a_1, a_2, \ldots, a_n$ taken k at a time. Show that*

$$S_kS_{n-k}\geq\binom{n}{k}^2 a_1a_2\ldots a_n$$

For $k=1, 2, \ldots, n-1$.

Solution: $S_k=\Sigma a_1a_2a_3\ldots a_k$

Note: Number of terms in S_k is $\binom{n}{k}$ and also a_1 is present in $\binom{n-1}{k-1}$ terms. Similarly $a_2, a_3, \ldots$ each one present in $\binom{n-1}{k-1}$ terms.

Apply AM $\geq$ GM

$$\frac{\sum a_1a_2\cdots a_k}{\binom{n}{k}}\geq\left(a_1^{\binom{n-1}{k-1}}\cdot a_2^{\binom{n-1}{k-1}}\cdots a_n^{\binom{n-1}{k-1}}\right)^{\frac{1}{\binom{n}{k}}}$$

$$\Rightarrow\quad S_k\geq\binom{n}{k}(a_1a_2\cdots a_n)^{\frac{\binom{n-1}{k-1}}{\binom{n}{k}}}$$

$$=\binom{n}{k}(a_1a_2\cdots a_n)^{\frac{k}{n}} \quad (1)$$

Similarly, $$S_{n-k}\geq\binom{n}{n-k}(a_1a_2\cdots a_n)^{\frac{n-k}{n}}=\binom{n}{k}(a_1a_2\cdots a_n)^{\frac{n-k}{n}} \quad (2)$$

Multiply Inequalities (1) and (2), we get

$$S_k \cdot S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 a_3 \cdots a_n.$$

Example 37 *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}.$$

Solution: Let $a = x + y$, $b = y + z$, $c = z + x$; $x, y, z > 0$, inequality becomes

$$\sqrt{2}(\sqrt{x}+\sqrt{y}+\sqrt{z}) \leq \sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}$$

Now,

$$x+y \geq 2\sqrt{xy} \qquad \text{(By AM} \geq \text{GM)}$$

$$\Rightarrow \quad 2(x+y) \geq x+y+2\sqrt{xy} = (\sqrt{x}+\sqrt{y})^2$$

$$\Rightarrow \quad \sqrt{2}\sqrt{x+y} \geq \sqrt{x}+\sqrt{y}$$

or
$$\sqrt{x+y} \geq \frac{1}{\sqrt{2}}(\sqrt{x}+\sqrt{y}) \tag{1}$$

Similarly
$$\sqrt{y+z} \geq \frac{1}{\sqrt{2}}(\sqrt{y}+\sqrt{z}) \tag{2}$$

and
$$\sqrt{z+x} \geq \frac{1}{\sqrt{2}}(\sqrt{z}+\sqrt{x}) \tag{3}$$

By adding Inequalities (1), (2) and (3), we get

$$\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x} \geq \sqrt{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right).$$

Example 38 *Let a, b, c be positive real numbers. Prove that*

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right).$$

Solution:

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

$$\Leftrightarrow \cancel{2}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{b}{a}+\frac{c}{b} \geq \cancel{2}+2\left(\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Let us prove that, $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a} \geq \dfrac{a+b+c}{\sqrt[3]{abc}}$ (1)

and
$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b} \geq \frac{a+b+c}{\sqrt[3]{abc}} \tag{2}$$

For Inequality (1), $2\dfrac{a}{b}+\dfrac{b}{c} \geq 3\left[\left(\dfrac{a}{b}\right)^2 \cdot \dfrac{b}{c}\right]^{\frac{1}{3}} = 3\dfrac{a^{\frac{2}{3}}}{(bc)^{\frac{1}{3}}} = \dfrac{3a}{\sqrt[3]{abc}}$

Similarly, $2\dfrac{b}{c}+\dfrac{c}{a} \geq \dfrac{3b}{\sqrt[3]{abc}}$ and $\dfrac{2c}{a}+\dfrac{a}{b} \geq \dfrac{3c}{\sqrt[3]{abc}}$

Add all three to get Inequality (1)

Similarly we can prove Inequality (2).

Build-up Your Understanding 3

1. If $a_1, a_2, ..., a_n$ are n positive real numbers, show that. $na_1a_2 \ldots a_n \le a_1{}^n + a_2{}^n + \cdots + a_n{}^n$.
2. Prove that if $a, b, c > 0$ then $a^2(b + c) + b^2(c + a) + c^2(a + b) \ge 6abc$.
3. If $a > 0$, prove that $(a^3 + a^2 + a + 1)^2 \ge 16a^3$.
4. If a, b, c are three distinct positive real numbers. Prove that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} > 6$ or, $bc(b + c) + ca(c + a) + ab\,(a + b) > 6abc$.
5. If a, b, c are three distinct positive real numbers, prove that $a^2\,(1 + b^2) + b^2\,(1 + c^2) + c^2\,(1 + a^2) > 6abc$.
6. If a, b, c, d are distinct positive real number, prove that $a^8(1 + b^8) + b^8\,(1 + c^8) + c^8(1 + d^8) + d^8(1 + a^8) > 8a^3\,b^3\,c^3\,d^3$.
7. If $x, y, z > 0$ and $x + y + z = 1$, prove that
   (a) $x^2 + y^2 + z^2 \ge \frac{1}{3}$
   (b) $x^2yz \le \frac{1}{64}$
8. If $x + y + z = 6\ (x, y, z > 0)$.
   (a) Find the maximum value of xyz.
   (b) Find the maximum value of x^2yz.
9. Show that, if a, b, c, d be four positive unequal quantities and $s = a + b + c + d$, then $(s - a)\,(s - b)\,(s - c)\,(s - d) > 81abcd$.
10. If a, b, c, d are distinct positive real numbers, such that $3s = a + b + c + d$, then prove that $abcd > 81(s - a)\,(s - b)\,(s - c)\,(s - d)$.
11. Prove that $(a + 1)^7\ \ (b + 1)^7\,(c + 1)^7 > 7^7\,a^4\,b^4\,c^4$, where $a, b, c \in \mathbb{R}^+$.
12. For every natural number greater than 1, prove that $2n - 1 \ge .\, n \cdot 2^{\frac{n-1}{2}}$.
13. Let $a, b, c, d \in \mathbb{R}^+$ such that $a + b + c + d = 1$. Prove that

$$\frac{a^3}{b+c} + \frac{b^3}{c+d} + \frac{c^3}{d+a} + \frac{d^3}{a+b} \ge \frac{1}{8}.$$

2.6 Weighted Means

Given any n positive real numbers $a_1, a_2, ..., a_n$, with their positive weights, $w_1, w_2, \ldots, w_n$ respectively the positive numbers A^*, G^* and H^*, defined by:

$$A^* = \frac{a_1w_1 + a_2w_2 + \cdots + a_nw_n}{w_1 + w_2 + \cdots + w_n},$$

$$G^* = (a_1^{w_1} \cdot a_2^{w_2} \cdots a_n^{w_n})^{\frac{1}{w_1+w_2+\cdots+w_n}} \quad \text{and} \quad H^* = \frac{w_1 + w_2 + \cdots + w_n}{\frac{w_1}{a_1} + \frac{w_2}{a_2} + \cdots + \frac{w_n}{a_n}}$$

are known as weighted AM, weighted GM and weighted HM respectively and we have

$$A^* \ge G^* \ge H^*$$

Equality holds in $A^* \ge G^* \ge H^*$ only when all the a_i are equal.

Example 39 *Prove that* $\left(\dfrac{a+b}{2}\right)^{a+b} > a^b \cdot b^a,\ a, b \in \mathbb{R}^+; a \neq b.$

Solution: Let us consider a with weight b and b with weight a

Then WAM > WGM

$$\Rightarrow \frac{ab+ab}{a+b} > (a^b b^a)^{\frac{1}{a+b}}$$

$$\Rightarrow \frac{2ab}{a+b} > (a^b b^a)^{\frac{1}{a+b}}$$

Now, $\dfrac{a+b}{2} > \dfrac{2ab}{a+b}$ (AM > HM)

$$\Rightarrow \left(\frac{a+b}{2}\right)^{a+b} > a^b \cdot b^a.$$

Example 40 *If a, b, α and β are positive real numbers, such that $\alpha + \beta = 1$, then prove that $a\alpha + b\beta \geq a^{\alpha}.b^{\beta}$. When does equality hold?*

Solution: Consider a with weight α and b with weight β. Now by weighted AM ≥ weighted GM, we have $\dfrac{a\alpha + b\beta}{\alpha+\beta} \geq (a^{\alpha} b^{\beta})^{\frac{1}{\alpha+\beta}}$

$\therefore \quad a\alpha + b\beta \geq a^{\alpha} \cdot b^{\beta}.$

Equlity holds when $a = b$

Build-up Your Understanding 4

1. For every positive real number $a \neq 1$ and for every positive integer n, prove that $\left(\dfrac{1+na}{1+n}\right)^{n+1} > a^n$.
2. For a and b positive real, prove that $\dfrac{a^3 b}{(a+b)^4} \leq \dfrac{27}{256}$.
3. Prove that $\left(\dfrac{a^2+b^2}{a+b}\right)^{a+b} > a^a b^b$.
4. Prove that $\left(\dfrac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z} > x^x y^y z^z > \left(\dfrac{x+y+z}{3}\right)^{x+y+z}$.
5. By assigning weights 1 and n to the numbers 1 and $\left(1+\dfrac{x}{n}\right)$ respectively, prove that if $x > -n$, then $\left(1+\dfrac{x}{n+1}\right)^{n+1} < \left(1+\dfrac{x}{n}\right)^{n}$.
6. If n is a positive integer, prove that $\{(n+1)!\}^{\frac{1}{n+1}} < 1 + \dfrac{n}{n+1}(n!)^{\frac{1}{n}}$.
7. If n is a positive integer, show that $\left(1-\dfrac{1}{n}\right)^{n} < \left(1-\dfrac{1}{n+1}\right)^{n+1}$.
8. Let $p, q \in \mathbb{R}^+$, $\dfrac{1}{p}+\dfrac{1}{q}=1$. Prove that $\dfrac{x^p}{p}+\dfrac{y^q}{q} \geq xy$ for $\forall\ x, y \in \mathbb{R}^+$.

2.7 Power Mean Inequality

Let $a_1, a_2, \ldots, a_n$ be n positive real numbers with their positive weights $w_1, w_2, \ldots, w_n$ respectively and let m be a non-zero real number, then

$$\text{WPM}_m = \left(\frac{w_1 a_1^m + w_2 a_2^m + \cdots + w_n a_n^m}{w_1 + w_2 + \cdots + w_n}\right)^{\frac{1}{m}}.$$

Now weighted power mean increases with increase in 'm', *i.e.*, for $p > q$, we have $\text{WPM}_p > \text{WPM}_q$

Equality holds when $a_1 = a_2 = \cdots = a_n$.

Note: $m = 1$, then $\text{WPM}_1 = A^*$ (weighted AM)

$m \to 0$, then $\text{WPM}_0 = G^*$ (weighted GM)

$m = -1$, then $\text{WPM}_{-1} = H^*$ (weighted HM)

$m = 2$, then $\text{WPM}_2 = QM^*$ (weighted quadratic mean).

$\Rightarrow A^* \le G^* \le H^* \le QM^*$

Example 41 *Prove that* $a^4 + b^4 + c^4 \ge abc(a + b + c)$, $[a, b, c > 0]$.

Solution: Using $\text{PM}_4 \ge \text{PM}_1$ inequality, we get

$$\left(\frac{a^4+b^4+c^4}{3}\right)^{\frac{1}{4}} \ge \left(\frac{a+b+c}{3}\right)^{\frac{1}{1}} \Rightarrow \frac{a^4+b^4+c^4}{3} \ge \left(\frac{a+b+c}{3}\right)^4$$

$$= \left(\frac{a+b+c}{3}\right)\left(\frac{a+b+c}{3}\right)^3 \ge \left(\frac{a+b+c}{3}\right)[(abc)^{\frac{1}{3}}]^3 \quad (\because \text{AM} \ge \text{GM})$$

or
$$\frac{a^4+b^4+c^4}{3} \ge \left(\frac{a+b+c}{3}\right)abc$$

$$\therefore\ a^4 + b^4 + c^4 \ge abc(a + b + c).$$

Example 42 *a, b, c, d and e are positive real numbers, such that* $a + b + c + d + e = 8$ *and* $a^2 + b^2 + c^2 + d^2 + e^2 = 16$, *find the range of e.*

Solution: Using $\text{PM}_1 \le \text{PM}_2$, we get

$$\left(\frac{a+b+c+d}{4}\right)^2 \le \frac{a^2+b^2+c^2+d^2}{4} \tag{1}$$

But, $a + b + c + d = 8 - e$ and $a^2 + b^2 + c^2 + d^2 = 16 - e^2$.

So, Eq. (1) becomes

$$\left(\frac{8-e}{4}\right)^2 \le \frac{16-e^2}{4}$$

$$\Rightarrow \qquad 4 - e + \frac{e^2}{16} \le 4 - \frac{e^2}{4}$$

$$\Rightarrow \qquad \frac{5e^2}{16} - e \le 0$$

$$\Rightarrow \qquad \frac{e}{16}(5e-16) \le 0$$

$$\Rightarrow \qquad 5e - 16 \le 0, \text{ since } e \ge 0.$$

Thus, $\qquad 0 < e \le \frac{16}{5}.$

Example 43 *Prove that* $\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c} \ge \frac{9}{2}$, *if* $s = a+b+c$, $[a, b, c > 0]$.

Solution: We have to prove that $\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \ge \frac{9}{2(a+b+c)}$

Using $PM_1 \ge PM_{-1}$ inequality for variables $\frac{1}{a+b}, \frac{1}{b+c}, \frac{1}{c+a}$, we get

$$\frac{(a+b)^{-1}+(b+c)^{-1}+(c+a)^{-1}}{3} \ge \left[\frac{a+b+b+c+c+a}{3}\right]^{-1}$$

or, $\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \ge \frac{9}{2(a+b+c)}$.

Aliter: AM $\ge$ HM

$$\Rightarrow \frac{(a+b)+(b+c)+(c+a)}{3} \ge \frac{3}{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}}$$

$$\Rightarrow \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \ge \frac{9}{2(a+b+c)}.$$

Example 44 *Find all non-zero real number triples* (x, y, z) *which satisfy*

$3(x^2+y^2+z^2) = 1;\ x^2y^2+y^2z^2+z^2x^2 = xyz(x+y+z)^3$.

Solution: Now, $\frac{x^2+y^2+z^2}{3} \ge \left(\frac{x+y+z}{3}\right)^2$ (Power mean inequality)

$\Rightarrow \quad 3(x^2+y^2+z^2) \ge (x+y+z)^2$

i.e., $1 \ge (x+y+z)^2$ or $(x+y+z)^2 \le 1$

$$\Rightarrow xyz(x+y+z)^3 \le xyz(x+y+z)$$

(As $xyz(x+y+z)$ is non-negative)

$$\Rightarrow \quad x^2y^2+y^2z^2+z^2x^2 \le x^2yz+y^2zx+z^2xy$$

$$\Rightarrow \quad (xy-yz)^2+(yz-zx)^2+(zx-xy)^2 \le 0$$

$$\Rightarrow \quad xy = yz = zx$$

$$\Rightarrow \quad x = y = z$$

$\therefore$ Solution is given by $(x, y, z) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $\left(\frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3}\right)$.

$$\left(\text{as } x^2 + y^2 + z^2 = \frac{1}{3} \Rightarrow 3x^2 = \frac{1}{3} \Rightarrow x^2 = \frac{1}{9} \Rightarrow x = \pm\frac{1}{3}\right)$$

When $x = y = z$.

Build-up Your Understanding 5

1. Let $a, b, c \in \mathbb{R}^+$ and $a^2 + b^2 + c^2 = 27$. Prove that $a^3 + b^3 + c^3 \geq 81$.
2. For $a, b, c, \in \mathbb{R}^+$, prove that $8(a^3 + b^3 + c^3)^2 \geq 9(a^2 + bc)(b^2 + ca)(c^2 + ab)$.

2.8 Rearrangement Inequality

Consider the followings illustration: There are five boxes containing ₹ 5, ₹10, ₹20, ₹50, ₹100 bills respectively. From each box you are allowed to take 2, 3, 4, 5 and 6 bills. How do you act to maximize the money you obtain?

Obviously you would take six ₹100 bills, five ₹50 bills, four ₹20 bills, three ₹10 bills, and two ₹5 bills and you will get 6 × 100 + 5 × 50 + 4 × 20 + 3 × 10 + 2 × 5 = ₹970.

Suppose you want to minimize the amount. In this case, you will take least possible number of units of highest denominations and you will get minimum

$$2 \times 100 + 3 \times 50 + 4 \times 20 + 5 \times 10 + 6 \times 5 = ₹510.$$

In rearrangement inequality we are using the same Idea.

Let $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b, \ldots, b_n$ be sequences of real numbers in ascending order and $b_{i1}, b_{i2}, \ldots, b_{in}$ is some permutation of $b_1, b_2, b_3, \ldots, b_n$ then

$$a_1b_n + a_2b_{n-1} + \cdots + a_nb_1 \leq a_1b_{i1} + a_2b_{i2} + \cdots + a_nb_{in} \leq a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

Let us define a notation for sum of product of corresponding terms of two sequences as,

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

Example 45 *Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.*

Solution: Let $a \leq b \leq c$

Now,
$$\begin{bmatrix} a & b & c \\ a & b & c \end{bmatrix} \geq \begin{bmatrix} a & b & c \\ b & c & a \end{bmatrix}$$

$$\Rightarrow \quad a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Example 46 *Prove that $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$ for positive real numbers a, b, c.*

Solution: Let $0 < a \leq b \leq c \Rightarrow a^2 \leq b^2 \leq c^2$

Now,
$$\begin{bmatrix} a^2 & b^2 & c^2 \\ a & b & c \end{bmatrix} \geq \begin{bmatrix} a^2 & b^2 & c^2 \\ b & c & a \end{bmatrix}$$

$$\Rightarrow a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a.$$

Example 47 *Prove that* $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\geq\dfrac{3}{2}$ *for* $a, b, c \in \mathbb{R}^+$.

Solution: Let $0<a\leq b\leq c \Rightarrow \dfrac{1}{b+c}\leq\dfrac{1}{c+a}\leq\dfrac{1}{a+c}$

Now
$$\begin{bmatrix} a & b & c \\ \dfrac{1}{b+c} & \dfrac{1}{c+a} & \dfrac{1}{a+b} \end{bmatrix} \geq \begin{bmatrix} a & b & c \\ \dfrac{1}{c+a} & \dfrac{1}{a+b} & \dfrac{1}{b+c} \end{bmatrix} \tag{1}$$

also
$$\begin{bmatrix} a & b & c \\ \dfrac{1}{b+c} & \dfrac{1}{c+a} & \dfrac{1}{a+b} \end{bmatrix} \geq \begin{bmatrix} a & b & c \\ \dfrac{1}{a+b} & \dfrac{1}{b+c} & \dfrac{1}{c+a} \end{bmatrix} \tag{2}$$

Adding Inequalities (1) and (2), we get

$$2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\geq 3$$

This is called **Nesbitt's** inequality.

Example 48 *Let* $a, b, c \in \mathbb{R}^+$, *such that* $abc = 1$. *Prove that* $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}\geq a+b+c$.

Solution: Let $\dfrac{a}{b}\leq\dfrac{b}{c}\leq\dfrac{c}{a}$

$$\Rightarrow \begin{bmatrix} \left(\dfrac{a}{b}\right)^{\frac{1}{3}} & \left(\dfrac{b}{c}\right)^{\frac{1}{3}} & \left(\dfrac{c}{a}\right)^{\frac{1}{3}} \\ \left(\dfrac{a}{b}\right)^{\frac{2}{3}} & \left(\dfrac{b}{c}\right)^{\frac{2}{3}} & \left(\dfrac{c}{a}\right)^{\frac{2}{3}} \end{bmatrix} \geq \begin{bmatrix} \left(\dfrac{b}{c}\right)^{\frac{1}{3}} & \left(\dfrac{c}{a}\right)^{\frac{1}{3}} & \left(\dfrac{a}{b}\right)^{\frac{1}{3}} \\ \left(\dfrac{a}{b}\right)^{\frac{2}{3}} & \left(\dfrac{b}{c}\right)^{\frac{2}{3}} & \left(\dfrac{c}{a}\right)^{\frac{2}{3}} \end{bmatrix}$$

$$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\left(\frac{a^2}{bc}\right)^{\frac{1}{3}}+\left(\frac{b}{ca}\right)^{\frac{1}{3}}+\left(\frac{c^2}{bc}\right)^{\frac{1}{3}}$$
$$= a+b+c. \quad (\text{using } abc = 1)$$

2.9 Chebyshev's Inequality

Let $x_i, y_i \in \mathbb{R}\ \forall\ i = 1, 2, 3, \ldots, n$ such that

$$x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n \text{ and } y_1 \leq y_2 \leq y_3 \leq \ldots \leq y_n, \text{ then}$$

$$\frac{x_1y_n+x_2y_{n-1}+\cdots+x_ny_1}{n}\leq\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)\left(\frac{y_1+y_2+\cdots+y_n}{n}\right)$$
$$\leq\frac{x_1y_1+x_2y_2+\cdots+x_ny_n}{n}$$

If one of the sequences is increasing and the other decreasing, then the direction of the inequality changes.

Corollary: Taking $a_i = b_i$ from right hand side inequality, we get

$$\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^2\leq\frac{a_1^{\,2}+a_2^{\,2}+\cdots+a_n^{\,2}}{n} \text{ which is known as QM inequality.}$$

Pafnuty Lvovich Chebyshev

16 May 1821–8 Dec 1894
Nationality: Russian

Example 49 *If $a, b, c \in \mathbb{R}^+$, prove that* $\dfrac{a^8+b^8+c^8}{a^3\cdot b^3\cdot c^3} \geq \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.

Solution: Applying Chebyshev's inequality, we get

$$3(a^8+b^8+c^8) \geq (a^6+b^6+c^6)(a^2+b^2+c^2)$$
$$\geq 3a^2\,b^2\,c^2\,(a^2+b^2+c^2) \quad \text{(By AM–GM)}$$
$$\geq 3a^2b^2c^2\,(ab+bc+ca) \quad \text{(Rearrangement)}$$
$$\Rightarrow \frac{a^8+b^8+c^8}{a^3b^3c^3} \geq \frac{ab+bc+ca}{abc} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Example 50 *If a, b, and c are positive real number, prove the inequality*

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a} \leq \frac{3(ab+bc+ca)}{2(a+b+c)}$$

Solution: Let $a \leq b \leq c$

$$\Rightarrow a+b \leq a+c \leq b+c \tag{1}$$

also we have $\dfrac{1}{c} \leq \dfrac{1}{b} \leq \dfrac{1}{a}$

$$\Rightarrow \frac{1}{b}+\frac{1}{c} \leq \frac{1}{c}+\frac{1}{a} \leq \frac{1}{b}+\frac{1}{a}$$

$$\Rightarrow \frac{1}{\frac{1}{b}+\frac{1}{a}} \leq \frac{1}{\frac{1}{c}+\frac{1}{a}} \leq \frac{1}{\frac{1}{b}+\frac{1}{c}}$$

$$\text{or} \quad \frac{ab}{a+b} \leq \frac{ac}{a+c} \leq \frac{bc}{b+c} \tag{2}$$

Using (1) and (2) and by applying Chebyshev's Inequality we get

$$3\left((a+b)\cdot\frac{ab}{a+b}+(a+c)\frac{ac}{a+c}+(b+c)\frac{bc}{b+c}\right)$$
$$\geq ((a+b)+(a+c)+(b+c))\left(\frac{ab}{a+b}+\frac{ac}{a+c}+\frac{bc}{b+c}\right)$$
$$\Rightarrow \frac{3(ab+bc+ca)}{2(a+b+c)} \geq \frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}$$

Build-up Your Understanding 6

1. Find the minimum of $\dfrac{\sin^3 x}{\cos x}+\dfrac{\cos^3 x}{\sin x}, x \in \left(0, \dfrac{\pi}{2}\right)$.
2. $a, b, c \in \mathbb{R}^+$, prove that $a^4+b^4+c^4 \geq a^2bc+b^2ca+c^2ab$
3. $a, b, c \in \mathbb{R}^+$, such that $a+b+c=3$. Prove that $\dfrac{a^2}{b+c}+\dfrac{b^2}{c+a}+\dfrac{c^2}{a+b} \geq \dfrac{a^2+b^2+c^2}{2}$.
4. $a, b, c \in \mathbb{R}^+$, prove that $\dfrac{a^2+b^2}{2c}+\dfrac{b^2+c^2}{2a}+\dfrac{c^2+a^2}{2b} \geq a+b+c$.

5. $a, b, c \in \mathbb{R}^+$, prove that $\dfrac{a^3}{bc}+\dfrac{b^3}{ca}+\dfrac{c^3}{ab} \ge \dfrac{a^2+b^2}{2c}+\dfrac{b^2+c^2}{2a}+\dfrac{c^2+a^2}{2b}$.

6. $a, b, c \in \mathbb{R}^+$, prove that $(a^2+b^2+c^2)(a^3+b^3+c^3) \le 3(a^5+b^5+c^5)$.

7. If a, b, and c are the lengths of the sides of a triangle, s its semiperimeter, and $n \ge 1$ an integer, prove that $\dfrac{a^n}{b+c}+\dfrac{b^n}{c+a}+\dfrac{c^n}{a+b} \ge \left(\dfrac{2}{3}\right)^{n-2} \cdot s^{n-1}$.

Augustin-Louis Cauchy

21 Aug 1789–23 May 1857
Nationality: French

2.10 Cauchy–Schwarz Inequality

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are $2n$ real numbers, then $(a_1b_1+a_2b_2+\cdots+a_n\,b_n)^2 \le (a_1{}^2+a_2{}^2+\cdots+a_n{}^2)\,(b_1{}^2+b_2{}^2+\cdots+b^2{}_n)$ with the equality holding if and only if, $\dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}=\cdots=\dfrac{a_n}{b_n}$.

Proof: Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. For every real x, we have

$$f(x) = (a_1x-b_1)^2+(a_2x-b_2)^2+\cdots+(a_nx-b_n)^2 \ge 0$$

$$= (\Sigma a_1{}^2)x^2 - 2(\Sigma a_1b_1)x + \Sigma b_1{}^2 \ge 0 \;\forall\, x \in \mathbb{R}$$

$$\Rightarrow D \le 0$$

$$\Rightarrow 4(\Sigma a_1b_1)^2 - 4\,\Sigma\, a_1{}^2\,\Sigma\, b_1{}^2 \le 0$$

$$\Rightarrow (a_1{}^2+a_2{}^2+\cdots+a_n{}^2)(b_1{}^2+b_2{}^2+\cdots+b_n{}^2) \ge (a_1\,b_1+a_2\,b_2+\cdots+a_n\,b_n)^2$$

Also equality holds, when $x=\dfrac{b_1}{a_1}=\dfrac{b_2}{a_2}=\cdots=\dfrac{b_n}{a_n}$.

Karl Hermann Amandus Schwarz

25 Jan 1843–30 Nov 1921
Nationality: Prussian

Corollary: An alternate form of Cauchy–Schwarz inequality usually known as Titu's inequality, is as follows:

For $x_1, x_2, x_3, \ldots, x_n \in \mathbb{R}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}^+$, we have

$$\frac{x_1^2}{\alpha_1}+\frac{x_2^2}{\alpha^2}+\cdots+\frac{x_n^2}{\alpha_n} \ge \frac{(x_1+x_2+\cdots+x_n)^2}{\alpha_1+\alpha_2+\cdots+\alpha_n}$$

Equality holds when $\dfrac{x_1}{\alpha_1}=\dfrac{x_2}{\alpha_2}=\cdots=\dfrac{x_n}{\alpha_n}$.

Proof: Take $a_i=\dfrac{x_i}{\sqrt{\alpha_i}}$ and $b_i=\sqrt{\alpha_i}$ and apply Cauchy–Schwarz inequality.

Titu Andreescu

12 Sep 1956 (age 60)
Nationality: Romania
Presently in USA

Example 51 *If a, b, and c are positive real numbers, prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{(a+b+c)^2}{ab+bc+ca}.$$

Solution:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} = \frac{a^2}{ab}+\frac{b^2}{bc}+\frac{c^2}{ca} \ge \frac{(a+b+c)^2}{ab+bc+ca} \text{ (By Titu's inquality).}$$

Example 52 *If* $p_1, p_2, \ldots, p_{2014}$ *be an arbitrary rearrangement of* 1, 2, 3, …, 2014, *prove the inequality:*

$$\frac{1}{p_1+p_2}+\frac{1}{p_2+p_3}+\frac{1}{p_3+p_4}+\cdots+\frac{1}{p_{2013}+p_{2014}}>\frac{2013}{2016}.$$

Solution: By Cauchy–Schwarz inequality,

$$\{(p_1+p_2)+(p_2+p_3)+\cdots+(p_{2013}+p_{2014})\}$$
$$\left\{\frac{1}{p_1+p_2}+\frac{1}{p_2+p_3}+\cdots+\frac{1}{p_{2013}+p_{2014}}\right\}\geq(2013)^2 \quad (1)$$

$$\therefore\ \frac{1}{p_1+p_2}+\frac{1}{p_2+p_3}+\cdots+\frac{1}{p_{2013}+p_{2014}}\geq\frac{(2013)^2}{2(p_1+p_2+\cdots+p_{2014})-p_1-p_{2014}}$$

$$=\frac{(2013)^2}{(2014)(2015)-p_1-p_{2014}}\geq\frac{(2013)^2}{(2014)(2015)-1-2}$$

$$\geq\frac{2013}{(2014)(2015)+2014-2015-1}$$

$$=\frac{(2013)^2}{(2014-1)(2015+1)}=\frac{(2013)^2}{(2013)(2016)}$$

$$=\frac{2013}{2016}.$$

Example 53 *Find all positive real numbers x, y, z, such that*

$$2x-2y+\frac{1}{z}=\frac{1}{2016};\ 2y-2z+\frac{1}{x}=\frac{1}{2016};\ 2z-2x+\frac{1}{y}=\frac{1}{2016}.$$

Solution: Now, $2xz-2yz+1=\dfrac{z}{2016}$ (1)

and $2yx-2zx+1=\dfrac{x}{2016}$ (2)

and $2zy-2xy+1=\dfrac{y}{2016}$ (3)

Adding Eqs. (1), (2) and (3), we get

$$3=\frac{z+x+y}{2016}$$

i.e., $x+y+z=3(2016)$ (4)

Similarly by adding given expressions, we get

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{3}{2016} \quad (5)$$

Now by Cauchy–Schwarz inequality, $(x+y+z)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)\geq(3)^2$

$$\text{i.e., } 3(2016)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9 \ \Rightarrow\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{93}{3\cdot 2016}\geq\frac{3}{2016}$$

But, $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{3}{2016}$ (From Eq. (5))

Hence, equality should hold $\Rightarrow$ $x=y=z$

As, $x+y+z=3(2013)\Rightarrow x=2016; y=2016; z=2016.$

Example 54 *If a, b, c are positive real numbers, prove the inequality:*

$$ab^3+bc^3+ca^3\geq abc(a+b+c)$$

Solution: Now, $\dfrac{ab^3}{abc}+\dfrac{bc^3}{abc}+\dfrac{ca^3}{abc}\geq(a+b+c)$

i.e., $\dfrac{b^2}{c}+\dfrac{c^2}{a}+\dfrac{a^2}{b}\geq(a+b+c)$

Now, $\text{LHS}=\dfrac{b^2}{c}+\dfrac{c^2}{a}+\dfrac{a^2}{b}\geq\dfrac{(b+c+a)^2}{c+a+b}$ (Titu's inequality)

$$=(a+b+c)=\text{RHS}.$$

Example 55 *If a, b, c, and d are positive, then prove that*

$$(a^3b+b^3c+c^3d+d^3a)(ab^3+bc^3+cd^3+da^3)\geq 16(abcd)^2.$$

Solution: Applying the Cauchy–Schwarz inequality

$$a^3b=a_1^2, b^3c=a_2^2, c^3d=a_3^2, d^3a=a_4^2$$

and $ab^3=b_1^2,\ bc^3=b_2^2, cd^3=b_3^2, da^3=b_4^2,$

we get, $a_1b_1=a^2b^2,\ a_2b_2=b^2c^2,\ a_3b_3=c^2d^2,\ a_4b_4=d^2a^2.$

$\Rightarrow$ $(a^3b+b^3c+c^3d+d^3a)(ab^3+bc^3+cd^3+da^3)\geq(a^2b^2+b^2c^2+c^2d^2+d^2a^2)^2$

Now, applying AM–GM inequality and taking square, we get

$$(a^2b^2+b^2c^2+c^2d^2+d^2a^2)^2\geq(4\sqrt[4]{a^4b^4c^4d^4})^2$$

$$=16\,a^2b^2c^2d^2.$$

Hence, $(a^3b+b^3c+c^3d+d^3a)(ab^3+bc^3+cd^3+da^3)\geq 16(abcd)^2.$

Example 56 *Given that* $x^2+y^2+z^2=8$, *prove that*

$x^3+\ y^3+\ z^3\geq 16\sqrt{\dfrac{2}{3}}.$

Solution: Applying Cauchy–Schwarz inequality with $x^{3/2}$, $y^{3/2}$, $z^{3/2}$ and $x^{1/2}$, $y^{1/2}$, $z^{1/2}$, we have

$$(x^2+y^2+z^2)^2\leq(x^3+y^3+z^3)(x+y+z)$$

Again, $x+y+z=x\times 1+y\times 1+z\times 1$

So, $(x+y+z)^2\leq(x^2+y^2+z^2)(1^2+1^2+1^2)$

$(x+y+z)\leq\sqrt{3\times 8}\ \Rightarrow\ \dfrac{1}{x+y+z}\geq\dfrac{1}{2\sqrt{6}}$

Hence,

$$(x^3+y^3+z^3) \geq \frac{(x^2+y^2+z^2)^2}{(x+y+z)} = \frac{64}{2\sqrt{6}}$$

$$\Rightarrow \qquad x^3+y^3+z^3 \geq 16\sqrt{\frac{2}{3}}.$$

Example 57 *If* $w^3+x^3+y^3+z^3=10$, *show that* $w^4+x^4+y^4+z^4 \geq \sqrt[3]{2500}$

Solution: Applying the Cauchy–Schwarz inequality for w^2, x^2, y^2, z^2 and w, x, y, z, we get

$$(w^3+x^3+y^3+z^3)^2 \leq (w^4+x^4+y^4+z^4)(w^2+x^2+y^2+z^2) \qquad (1)$$

Again, by applying the Cauchy–Schwarz inequality with w^2, x^2, y^2, z^2 and 1, 1, 1, 1, we get

$$(w^2+x^2+y^2+z^2)^2 \leq (w^4+x^4+y^4+z^4)^4$$

$$\Rightarrow \qquad (w^2+x^2+y^2+z^2) \leq (w^4+x^4+y^4+z^4)^2 \qquad (2)$$

$$\therefore (w^4+x^4+y^4+z^4) \geq \frac{(w^3+x^3+y^3+z^3)}{(w^2+x^2+y^2+z^2)} \qquad \text{(by Eq. (1))}$$

$$\geq \frac{(w^3+x^3+y^3+z^3)^2}{2(w^4+x^4+y^4+z^4)^{1/2}} \qquad \text{(by Eq.(2))}$$

$$\Rightarrow \quad (w^4+x^4+y^4+z^4)^{3/2} \geq \frac{100}{2} = 50$$

$$\Rightarrow \quad w^4+x^4+y^4+z^4 \geq 50^{2/3} \quad \text{or} \quad \sqrt[3]{2500}.$$

Build-up Your Understanding 7

1. (a) If $x_i > 0$, $(i = 1, 2, ..., n)$, then prove that

$$(x_1+x_2+...+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right) \geq n^2.$$

(b) If $a_1, a_2, ..., a_n$ are n non-zero real numbers, prove that

$$(a_1^{-2}+\cdots+a_n^{-2}) \geq \frac{n^2}{a_1^2+\cdots+a_n^2}.$$

2. If $a_i < 0$ for all $i = 1, 2, ..., n$, prove that

(a) $$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) > n^2.$$

(b) $(1-a_1+a_1{}^2)(1-a_2+a_2{}^2)\ldots(1-a_n+a_n{}^2) > 3^n(a_1 a_2 \ldots a_n)$ (where n is even).

3. If none of $b_1, b_2, \ldots, b_n$ is zero, prove that

$$\left(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}\right)^2 \leq (a_1{}^2+\cdots+a_n{}^2)(b_1{}^{-2}+\cdots+b_n{}^{-2}).$$

4. If $3x + 4y = 1$ for some $x, y \in \mathbb{R}$. Prove that $x^2 + y^2 \ge \dfrac{1}{25}$.

5. For $a, b, c \in \mathbb{R}$, prove that: $\dfrac{a^2}{2} + \dfrac{b^2}{3} + \dfrac{c^2}{6} \ge \left(\dfrac{a}{2} + \dfrac{b}{3} + \dfrac{c}{6}\right)^2$.

6. For $a, b, c \in \mathbb{R}^+$, prove that:

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \ge \frac{a}{3+a+b+c}.$$

7. $a, b, c, d \in \mathbb{R}^+$, prove that $\sqrt{(a+b)(c+d)} \ge \left(\sqrt{ac} + \sqrt{bd}\right)$.

8. $x, y, z, \in \mathbb{R}^+$, prove that $\dfrac{2}{x+y} + \dfrac{2}{y+z} + \dfrac{2}{z+x} \ge \dfrac{9}{x+y+z}$.

9. $a, b, c, \in \mathbb{R}^+$, prove that $\dfrac{a^2+b^2}{a+b} + \dfrac{b^2+c^2}{b+c} + \dfrac{c^2+a^2}{c+a} \ge a+b+c$.

2.11 Hölders Inequality

Otto Ludwig Hölder

22 Dec 1859–29 Aug 1937
Nationality: German

$(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}} (b_1^q + b_2^q + \cdots + b_n^q)^{\frac{1}{q}} \ge (a_1 b_2 + a_2 b_2 + \cdots + a_n b_n)$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $p, q > 0$; and a_i, b_i are non-negative real numbers.

This can be generalized to k set of variables:

$$(a_{11} + a_{12} + \cdots + a_{1n})^{\lambda_1} (a_{21} + a_{22} + \cdots + a_{2n})^{\lambda_2} \cdots (a_{k1} + a_{k2} + \cdots + a_{kn})^{\lambda_k}$$
$$\ge \left(a_{11}^{\lambda_1} a_{21}^{\lambda_2} a_{31}^{\lambda_3} \cdots a_{k1}^{\lambda_k} + a_{12}^{\lambda_1} a_{22}^{\lambda_2} a_{32}^{\lambda_3} \cdots a_{k2}^{\lambda_k} + \cdots + a_{1n}^{\lambda_1} a_{2n}^{\lambda_2} a_{3n}^{\lambda_3} \cdots a_{kn}^{\lambda_k}\right)$$

where, $a_{ij} > 0$, $\lambda_i > 0$ and $\Sigma \lambda_i = 1$

Another form of Hölder:

$$(a_{11} + a_{12} + \cdots + a_{1n})\,(a_{21} + a_{22} + \cdots + a_{2n}) \cdots (a_{k1} + a_{k2} + \cdots + a_{kn})$$
$$\ge \left(\sqrt[k]{a_{11} a_{21} \ldots a_{k1}} + \sqrt[k]{a_{12} a_{22} \cdots a_{k2}} + \cdots + \sqrt[k]{a_{1n} a_{2n} \cdots a_{kn}}\right)^k$$

Example 58 *Let $a, b, c \in \mathbb{R}^+$, prove that*

$$\frac{a^3}{(a+b)^2} + \frac{b^3}{(b+c)^2} + \frac{c^3}{(c+a)^2} = \frac{a+b+c}{4}.$$

Solution: Applying Hölder's inequality, we get

$$\left[(a+b)+(b+c)+(c+a)\right]\left[(a+b)+(b+c)+(c+a)\right]\left(\frac{a^3}{(a+b)^2} + \frac{b^3}{(b+c)^2} + \frac{c^3}{(c+a)^2}\right)$$
$$\ge \left(\sqrt[3]{(a+b)^2 \frac{a^3}{(a+b)^2}} + \sqrt[3]{(b+c)^2 \frac{b^3}{(b+c)^2}} + \sqrt[3]{(c+a)^2 \frac{c^3}{(c+a)^2}}\right)^3$$
$$= (a+b+c)^3$$

$$\Rightarrow \frac{a^3}{(a+b)^2} + \frac{b^3}{(b+c)^2} + \frac{c^3}{(c+a)^2} \ge \frac{a+b+c}{4}. \text{ (dividing by } 4(a+b+c)^2\text{)}$$

Example 59 Let $a, b, c \in \mathbb{R}^+$, prove that

$$\frac{a^2}{b^3}+\frac{b^2}{c^3}+\frac{c^2}{a^3} \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Solution: Applying Hölder's inequality, we get

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{a^2}{b^3}+\frac{b^2}{c^3}+\frac{c^2}{a^3}\right) \ge \left(\sqrt[3]{\frac{1}{a}\cdot\frac{1}{a}\cdot\frac{a^2}{b^3}}+\sqrt[3]{\frac{1}{b}\cdot\frac{1}{b}\cdot\frac{b^2}{c^3}}+\sqrt[3]{\frac{1}{c}\cdot\frac{1}{c}\cdot\frac{c^2}{a^3}}\right)^3$$

$$=\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)^3$$

$$\Rightarrow \frac{a^2}{b^3}+\frac{b^2}{c^3}+\frac{c^2}{a^3} \ge \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right). \left(\text{dividing by}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\right)$$

Example 60 *For $a, b, c \in \mathbb{R}^+$ and $a+b+c=1$, prove that* $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \ge 64.$

Solution: Applying Hölder's inequality, we get

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \ge \left(\sqrt[3]{1\cdot1\cdot1}+\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}\right)^3 = \left(1+\frac{1}{(abc)^{\frac{1}{3}}}\right)^3$$

Now it is sufficient to prove $\left(1+\frac{1}{(abc)^{\frac{1}{3}}}\right)^3 \ge 64$ or $abc \le \frac{1}{27}$

By AM $\ge$ GM, $\frac{a+b+c}{3} \ge (abc)^{\frac{1}{3}}$

$$\Rightarrow (abc)^{\frac{1}{3}} \le \frac{1}{3} \Rightarrow abc \le \frac{1}{27}.$$

Build-up Your Understanding 8

1. $a, b, c \in \mathbb{R}^+$, prove that $(1+a^3)(1+b^3)(1+c^3) \ge (1+abc)^3$.
2. $a, b, c, d \in \mathbb{R}^+$, prove that $(1+a^4)(1+b^4)(1+c^4)(1+d^4) \ge (1+abcd)^4$.
3. For $a, b, c \in \mathbb{R}_0^+$, prove that
   $(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge (ab+bc+ca)^2$
4. For $a, b, c \in \mathbb{R}^+$, prove that $3(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2) \ge (ab+bc+ca)^3$.
5. $\frac{9}{a}+\frac{24}{b}=1,\ a, b, \in \mathbb{R}^+$, prove that $a^2+b^2 \ge 9(4+\sqrt[3]{9})^3$.

2.12 Some Geometrical Inequalities

2.12.1 Ptolemy's Inequality

For any four points A, B, C, D we have, $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$.

Equality occurs if and only if, $ABCD$ is cyclic.

2.12.2 The Parallelogram Inequality

For any four points A, B, C, D not necessarily coplaner, we have, $AB^2 + BC^2 + CD^2 + DA^2 \geq AC^2 + BD^2$.

Equality occurs if and only if, $ABCD$ is a parallelogram.

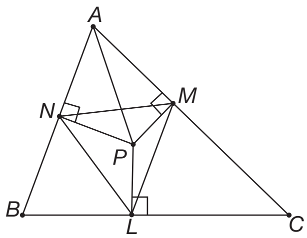
2.12.3 Torricelli's (or Fermat's) Point

For a given triangle ABC, the point P (In the plane of the triangle) for which $AP + BP + CP$ is minimal, is called Torricelli's (or Fermat) point. When all angles of ΔABC are less than 120° then at this point P all sides of the triangle subtends 120° angle. When any angle of the triangle is more than or equal to 120° then P is at that vertex

2.12.4 The Erodos–Mordell Inequality

Let P be a point in the interior of ΔABC and L, M, N projections of P onto BC, CA, AB respectively. Then $PA + PB + PC \geq 2(PL + PM + PN)$.

Equality holds iff, ΔABC is equilateral and P is its centroid.



Proof:
Let the sides of ΔABC be a opposite A, b opposite B and c opposite C; also let $PA = p$, $PB = q$, $PC = r$, dist $(P; BC) = x$, dist $(P; CA) = y$, dist $(P; AB) = z$.
Claim: $cr \geq ax + by$.

$$\Leftrightarrow \frac{c(r+z)}{2} \geq \frac{ax+by+cz}{2}.$$

The right side is the area of triangle ABC, but on the left side, $r + z$ is at least the height of the triangle; consequently, the left side cannot be smaller than the right side.

Now reflect P on the angle bisector at C.

We find that $cr \geq ay + bx$ for P's reflection.

Similarly, $bq \geq az + cx$ and $ap \geq bz + cy$.

$\Rightarrow r \geq (a/c)y + (b/c)\, x,$ (1)

and $q \geq (a/b)\, z + (c/b)\, x,$ (2)

and $p \geq (b/a)z + (c/a)y.$ (3)

Adding (1), (2) and (3), we get

$$p+q+r \geq \left(\frac{b}{c}+\frac{c}{b}\right)x + \left(\frac{a}{c}+\frac{c}{a}\right)y + \left(\frac{a}{b}+\frac{b}{a}\right)z \;\geq\; 2(x+y+z)$$

Claudius Ptolemy

c. AD 100–c. AD 170
Nationality: Greek

Evangelista Torricelli

15 Oct 1608–25 Oct 1647
Nationality: Italian

Paul Erdös

26 Mar 1913–20 Sep 1996
Nationality: Hungarian

Louis Joel Mordell

28 Jan 1888–12 Mar 1972
Nationality: British

(As the sum of a positive number and its reciprocal is at least 2 by AM–GM inequality)

Equality holds only for the equilateral triangle, where P is its centroid.

Gottfried Wilhelm Leibniz

1 July 1646–14 Nov 1716
Nationality: German

2.12.5 Leibniz's Theorem

Let P be any point in the plane of the ΔABC and G be centroid of the ΔABC. Then

$$AP^2 + BP^2 + CP^2 = \frac{1}{3}(AB^2 + BC^2 + CA^2) + 3PG^2$$

The point P for which $AP^2 + BP^2 + CP^2$ is minimal is the centroid of the triangle.

2.13 Jensen's Inequality

Let f be a real valued function, f defined on an interval $I \subset \mathbb{R}$, is called convex if for all $x_1, x_2 \in I$ and for $\lambda \in [0, 1]$, we have

Johan Ludwig William Valdemar Jensen

8 May 1859–5 Mar 1925
Nationality: Denmark

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

If f is convex over I and $x_1, x_2, x_3, \ldots, x_n \in I$, then

$$f\left(\frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots + w_n}\right) \le \frac{w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)}{w_1 + w_2 + \cdots + w_n}, \text{ where } w_i \in \mathbb{R}^+$$

Equality holds for $x_1 = x_2 = \ldots = x_n$.

In case of f concave, direction of inequality will change.

Note: For double differentiable functions, convex (or concave) $\Leftrightarrow f''(x) \ge 0$ (or ≤ 0).

Example 61 *Let $a, b, c \in \mathbb{R}^+$, prove that* $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b} \ge \dfrac{3}{2}$.

Solution: Let us normalize, the inequality with $a + b + c = 1$

$$\Rightarrow \frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \ge \frac{3}{2}$$

Consider, $f(x) = \dfrac{x}{1-x}$, $x \in (0, 1)$

$$\Rightarrow f'(x) = \frac{1}{(1-x)^2} \Rightarrow f''(x) = \frac{2}{(1-x)^2} \ge 0 \Rightarrow f \text{ is convex}$$

By Jensen's inequality,

$$f\left(\frac{a+b+c}{3}\right) \le \frac{f(a) + f(b) + f(c)}{3}$$

$$\Rightarrow \frac{\frac{1}{3}}{1-\frac{1}{3}} \le \frac{1}{3}\left(\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c}\right)$$

$$\Rightarrow \frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \ge \frac{3}{2}.$$

Build-up Your Understanding 9

If A, B and C are the angles of a triangle, prove the following:

1. $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\le\frac{1}{8}$.

2. $\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)\le\frac{3\sqrt{3}}{8}$.

3. $\cos A+\cos B+\cos C\le\frac{3}{2}$.

4. $\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)+\tan^2\left(\frac{C}{2}\right)\ge 1$.

5. $\sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}$.

6. In acute angle ΔABC, prove that $\frac{\sin A}{A}+\frac{\sin A}{B}+\frac{\sin C}{C}\le\frac{9\sqrt{3}}{2\pi}$.

7. Let a, b, c denote the measures of the sides of a triangle. Prove that $a^2(-a+b+c)+b^2(a-b+c)+c^2(a+b-c)\ge 3abc$

8. $a, b, c\in\mathbb{R}^+$, prove that $\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\ge 1$. **[IMO, 2001]**

9. $a, b, c\in\mathbb{R}^+$, prove that $\left(a+\frac{1}{a}\right)^{10}+\left(b+\frac{1}{b}\right)^{10}+\left(c+\frac{1}{c}\right)^{10}\ge\frac{10^{10}}{3^9}$.

10. $a_i\in\mathbb{R}^+$, prove that $\frac{a_1+a_2+\cdots+a_n}{n}\ge(a_1a_2\cdots a_n)^{\frac{1}{n}}$.

Solved Problems

Problem 1 *If $a, b, c, d, e, f>0$, prove that*

$$\frac{ab}{a+b}+\frac{cd}{c+d}+\frac{ef}{e+f}\le\frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f}.$$

Solution: Claim: $\frac{ab}{a+b}+\frac{cd}{c+d}\le\frac{(a+c)(b+d)}{a+b+c+d}$

Proof: Our claim is equivalent to,

$$[ab(c+d)+cd(a+b)](a+b+c+d)\le(a+c)(a+b)(b+d)(c+d)$$

$$\Leftrightarrow\ (cd+ab)(a+b)(c+d)+ab(c+d)^2+cd(a+b)^2\le(ab+cd+ad+bc)(a+b)(c+d)$$

$$\Leftrightarrow\ ab(c+d)^2+cd(a+b)^2\le(ad+bc)(a+b)(c+d)$$

$$\Leftrightarrow\ a(c+d)[d(a+b)-b(c+d)]+c(a+b)[b(c+d)-d(a+b)]\ge 0$$

$$\Leftrightarrow\ a(c+d)(ad-bc)+c(a+b)(bc-ad)\ge 0$$

$$\Leftrightarrow\ (ad-bc)[ac+ad-ac-bc]\ge 0$$

$$\Leftrightarrow\ (ad-bc)^2\ge 0\quad\text{(which is true)}$$

Now,

$$\frac{ab}{a+b}+\frac{cd}{c+d}+\frac{ef}{e+f}\le\frac{(a+c)(b+d)}{a+b+c+d}+\frac{ef}{e+f}\le\frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f}$$

Problem 2 *In an acute angle ΔABC, it is given that, $\Sigma \tan A \tan B = 9$. Find the size of $\angle A$.*

Solution: Let us first prove that $\cot A + \cot B + \cot C \ge \sqrt{3}$ (1)

Now, $(\cot A + \cot B + \cot C)^2 = \cot^2 A + \cot^2 B + \cot^2 C + 2(\cot A \cot B + \cot B \cot C + \cot C \cot A)$ (2)

i.e., $(\cot A + \cot B + \cot C)^2 = (\cot^2 A + \cot^2 B + \cot^2 C - \cot A \cot B - \cot B \cot C - \cot C \cot A) + 3(\cot A \cot B + \cot B \cot C + \cot C \cot A)$ (3)

But, $\Sigma \cot A \cot B = 1$, if $A + B + C = \pi$, (why?) (4)

$$\Rightarrow (\cot A+\cot B+\cot C)^2=\frac{1}{2}[(\cot A-\cot B)^2+(\cot B-\cot C)^2+(\cot C-\cot A)^2]+3 \qquad (5)$$

$$\Rightarrow \cot A+\cot B+\cot C\ge\sqrt{3} \qquad (6)$$

Dividing throughout by $\cot A \cot B \cot C$,

$$\frac{\cot A}{\cot A\cot B\cot C}+\frac{\cot B}{\cot A\cot B\cot C}+\frac{\cot C}{\cot A\cot B\cot C}\ge\left(\sqrt{3}\right)\tan A\tan B\tan C \qquad (7)$$

i.e., $\tan B\tan C+\tan C\tan A+\tan A\tan B\ge\left(\sqrt{3}\right)\tan A\tan B\tan C$ (8)

But, $\Sigma\tan A \tan B = 9$ (given)

$\Rightarrow \tan A\tan B\tan C\le 3\sqrt{3}$ (9)

But $\tan A\tan B\tan C=\tan A+\tan B+\tan C\ge 3\sqrt{3}$, (why?) (10)

From Eqs. (9) and (10) we conclude that all inequalities are equalities.

Thus, $A=B=C\Rightarrow\angle A=60°$.

Problem 3 *Find all real numbers in x, such that*

$$\frac{x^2}{x-1}+\sqrt{x-1}+\frac{\sqrt{x-1}}{x^2}=\frac{x-1}{x^2}+\frac{1}{\sqrt{x-1}}+\frac{x^2}{\sqrt{x-1}}$$

Solution: Let $\dfrac{x^2}{x-1}=a;\ \sqrt{x-1}=b;\ \dfrac{\sqrt{x-1}}{x^2}=c$

Now $\dfrac{x^2}{x-1}\times\sqrt{x-1}\times\dfrac{\sqrt{x-1}}{x^2}=1 \quad\Rightarrow abc=1$ (1)

$\therefore$ The given equation becomes,

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

i.e., $a+b+c=\dfrac{ab+bc+ca}{abc}$ or $(a+b+c)=(ab+bc+ca)$(as $abc=1$) (2)

We have, $1 - (a + b + c) + (ab + bc + ca) - abc = 0$ (3)

i.e., $(1 - a)(1 - b)(1 - c) = 0 \Rightarrow a = 1$ or $b = 1$ or $c = 1$ (4)

Thus, $\dfrac{x^2}{x-1}=1 \Rightarrow x^2-x+1=0 \Rightarrow$ no real root possible (5)

$\sqrt{x-1}=1 \Rightarrow x-1=1 \Rightarrow x=2,$ which satisfies the parent equation (6)

$\dfrac{\sqrt{x-1}}{x^2}=1 \Rightarrow \sqrt{x-1}=x^2 \Rightarrow x-1=x^4 \Rightarrow x^4-x+1=0 \Rightarrow$ no real root possible (7)

Thus the only solution to the above equation is $x=2$.

Problem 4 *Prove that, for all a, b, $c>0$,*

$$\frac{a+b+c}{\sqrt[3]{abc}}+\frac{8abc}{(a+b)(b+c)(c+a)}\geq 4.$$

Solution: Let $\alpha=\dfrac{a+b+c}{\sqrt[3]{abc}}$; then $\alpha^3=\dfrac{(a+b+c)^3}{abc} \Rightarrow \dfrac{1}{\alpha^3}=\dfrac{abc}{(a+b+c)^3}$ (1)

Also, $(a+b)(b+c)(c+a)\leq\left(\dfrac{(a+b)+(b+c)+(c+d)}{3}\right)^3$ (by AM-GM)

i.e., $(a+b)(b+c)(c+a)\leq\left(\dfrac{8}{27}\right)(a+b+c)^3$

$$\Rightarrow \quad \frac{8abc}{(a+b)(b+c)(c+a)}\geq\frac{(\cancel{8})(abc)(27)}{\cancel{8}(a+b+c)^3}=27\left(\frac{abc}{(a+b+c)^3}\right)=\frac{27}{\alpha^3} \quad (2)$$

Thus we have to prove that, $\alpha+\dfrac{27}{\alpha^3}\geq 4$.

Consider, AM–GM for the positive numbers, $\dfrac{\alpha}{3},\dfrac{\alpha}{3},\dfrac{\alpha}{3},\dfrac{27}{\alpha^3}$

Then, $\alpha+\dfrac{27}{\alpha^3}\geq 4\times\sqrt{\dfrac{\alpha}{3}\times\dfrac{\alpha}{3}\times\dfrac{\alpha}{3}\times\dfrac{27}{\alpha^3}}=4=\text{RHS}.$

Problem 5 *a, b, c, d are all positive reals. Also, its true that,*

$\dfrac{1}{1+a^4}+\dfrac{1}{1+b^4}+\dfrac{1}{1+c^4}+\dfrac{1}{1+d^4}=1.$ *Prove that, $abcd\geq 3$.*

Solution: Put $x=\dfrac{1}{1+a^4}$; $y=\dfrac{1}{1+b^4}$, $z=\dfrac{1}{1+c^4}$; $f=\dfrac{1}{1+d^4}$ (1)

Then, it is given that $x+y+z+f=1$ (2)

Now, $\dfrac{1}{1+a^4}=x \Rightarrow \dfrac{1+a^4}{1}=\dfrac{1}{x} \Rightarrow a^4=\dfrac{1}{x}-1=\dfrac{1-x}{x}$

Similarly, $b^4=\dfrac{1-y}{y}$; $c^4=\dfrac{1-z}{y}$; $d^4=\dfrac{1-f}{f}$

$\therefore$ We need to prove that, $a^4b^4c^4d^4=\left(\dfrac{1-x}{x}\right)\left(\dfrac{1-y}{y}\right)\left(\dfrac{1-z}{z}\right)\left(\dfrac{1-f}{f}\right)\geq 81$

$$\Leftrightarrow \left(\frac{y+z+f}{x}\right)\left(\frac{x+z+f}{y}\right)\left(\frac{x+y+f}{z}\right)\left(\frac{x+y+z}{f}\right) \geq 81$$

Apply AM-GM for these four terms on LHS individually,

$$\text{LHS} \geq \frac{(3)(yzf)^{\frac{1}{3}}}{x}\left((3)\frac{(xzf)^{\frac{1}{3}}}{z}\right)\left((3)\frac{(xyf)^{\frac{1}{3}}}{y}\right)\left((3)\frac{(xyz)^{\frac{1}{3}}}{f}\right) = 81$$

$\Rightarrow abcd \geq 3.$

Problem 6 *If a, b, c, d, and e are real numbers, prove that the roots of* $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ *cannot all be real if* $2a^2 < 5b$.

Solution: Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are the all real roots of the given equation.
Then, $\sum \alpha_i = -a$

$$\sum_{i<j} \alpha_i \alpha_j = b \tag{1}$$

$$(\sum \alpha_i)^2 = a^2$$

$$\Rightarrow \sum \alpha_i^2 + 2\sum_{i<j} \alpha_i \alpha_j = a^2$$

or $$\sum \alpha_i^2 = a^2 - 2b \quad \text{(from Eq. (1))}$$

By the power mean inequality, we have

$$(\sum \alpha_i)^2 \leq 5\sum \alpha_i^2$$

$$\Rightarrow a^2 \leq 5(a^2 - 2b) = 5a^2 - 10b$$

or, $$4a^2 \geq 10b \text{ or } 2a^2 \geq 5b$$

But, it is a contradiction because it is given that $2a^2 < 5b$. Hence, all the roots cannot be real.

Problem 7 *If x and y are real, solve the inequality* $\log_2 x + \log_x 2 + 2\cos y \leq 0$.

Solution: Here, $x > 0$ and $x \neq 1$
Let, $\log_2 x = p$ as $x \neq 1, p \neq 0$.

The given inequality becomes $p + \dfrac{1}{p} + 2\cos y \leq 0$

That is, $\dfrac{p^2 + 1 + 2p\cos y}{p} \leq 0.$

Case 1: When $p > 0$

$$p^2 + 1 + 2p\cos y \leq 0$$

$$\Rightarrow (p-1)^2 + 2p(1+\cos y) \leq 0 \tag{1}$$

Since $p > 0$, $1 + \cos y \geq 0$, and $(p-1)^2 \geq 0$

The only way Inequation (1) will be satisfied, when

$$(p-1)^2 = 0 \text{ and } 2p(1+\cos y) = 0,$$

$\therefore \quad p = 1$ and $\cos y = -1$

$\therefore \quad y = (2n+1)\pi$

Solution set is $x = 2$ and $y = (2n+1)\pi$

Case 2: When $p < 0$,

$$p^2 + 1 + 2p\cos y \geq 0$$
$$(p+1)^2 - 2p(1-\cos y) \geq 0$$

Which is true for all $p < 0$ as $1 - \cos y \geq 0$

$\Rightarrow \quad \text{Log}_2 x < 0 \quad \Rightarrow \quad 0 < x < 2^0 \quad \Rightarrow \quad 0 < x < 1$ and $y \in \mathbb{R}$.

Problem 8 *The positive number a, b and c satisfy* $a \geq b \geq c$ *and* $a + b + c \leq 1$. *Prove that* $a^2 + 3b^2 + 5c^2 \leq 1$.

Solution: As, $a + b + c \leq 1$

$\Rightarrow \quad (a+b+c)^2 \leq 1^2 = 1$

or $\quad 1 \geq (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

or $\quad 1 \geq a^2 + b^2 + c^2 + 2b^2 + 2c^2 + 2c^2 \qquad$ (Since, $a \geq b \geq c > 0$)

$\quad = a^2 + 3b^2 + 5c^2$

or $\quad a^2 + 3b^2 + 5c^2 \leq 1$.

Problem 9 *If a, b, c, and d are four non-negative real numbers and* $a + b + c + d = 1$, *show that* $ab + bc + cd \leq \dfrac{1}{4}$.

Solution: $(a+b+c+d)^2 - 4(ab+bc+cd)$

$= a^2 + b^2 + c^2 + d^2 - 2ab - 2bc - 2cd + 2ac + 2ad + 2bd$

$= a^2 - 2ab + b^2 + c^2 + d^2 - 2cd - 2bc + 2ac + 2ad + 2bd$

$= (a-b)^2 + (c-d)^2 + 2(a-b)(c-d) + 4ad$

$= [(a-b) + (c-d)]^2 + 4ad \geq 0$ ($\because$ $a, b, c, d, \geq 0$)

$\Rightarrow \quad 1 - 4(ab + bc + cd) \geq 0$

$\Rightarrow \quad 4(ab + bc + cd) \leq 1$

$\Rightarrow \quad ab + bc + cd \leq \dfrac{1}{4}$

Aliter: The above problem can be solved by using AM–GM inequality,

$(a + c) + (b + d) = 1$

$\Rightarrow \quad 2\sqrt{(a+c)(b+d)} \leq (a+c) + (b+d)$

$\Rightarrow \quad 2\sqrt{(a+c)(b+d)} \leq 1$

$\Rightarrow \quad 4(a+c)(b+d) \leq 1$

$\Rightarrow \quad ab + ad + bc + cd \leq \dfrac{1}{4}$

$\Rightarrow \quad ab + bc + cd \leq \dfrac{1}{4} - ad$

$\Rightarrow \quad ab + bc + cd \leq \dfrac{1}{4} \quad (\because a,\ d \geq 0)$

Equality holds for $a + c = b = \dfrac{1}{2}$ and $d = 0$ or $a = 0,\ c = b + d = \dfrac{1}{2}$.

Problem 10 *For* $n \in \mathbb{N}$, $n > 1$, *show that*

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2} > 1.$$

Solution: We have, $\frac{1}{n}+\underbrace{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n^2}}_{(n^2-n)\text{ terms}} > \frac{1}{n}+\underbrace{\left(\frac{1}{n^2}+\frac{1}{n^2}+\frac{1}{n^2}+\cdots+\frac{1}{n^2}\right)}_{(n^2-n)\text{ terms}}$

$$\Rightarrow \quad \frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n^2} > \frac{1}{n}+\frac{(n^2-n)}{n^2} = \frac{1}{n}+1-\frac{1}{n} = 1.$$

Problem 11 *What is the greatest integer n, for which there exists a simultaneous solution x to the inequalities* $k < x^k < k+1$, $k = 1, 2, 3, \ldots, n$.

Solution: If

$$k = 1;\ 1 < x < 2 \qquad (1)$$
$$k = 2;\ 2 < x^2 < 3 \qquad (2)$$
$$k = 3;\ 3 < x^3 < 4 \qquad (3)$$
$$k = 4;\ 4 < x^4 < 5 \qquad (4)$$
$$k = 5;\ 5 < x^5 < 6. \qquad (5)$$
$$\ldots$$
$$\ldots$$

Consider the inequality $2 < x^2 < 3$, then x should lie between $\sqrt{2}$ and $\sqrt{3}$

i.e., $\sqrt{2} < x < \sqrt{3}$

Now, $1 < \sqrt{2} < x < \sqrt{3} < \sqrt{4} = 2$ and hence, satisfies Eqs. (1) and (2) of the inequalities

$$\sqrt{2} < x < \sqrt{3}$$
$$\Rightarrow \quad (\sqrt{2})^3 < x^3 < (\sqrt{3})^3$$
$$\Rightarrow \quad 2\sqrt{2} < x^3 < 3\sqrt{3}$$

as $\quad 2\sqrt{2} < 3$ and $4 < 3\sqrt{3}$

Common solution of (1), (2), (3) are solution of (3)

From (3), $\quad 3 < x^3 < 4$

$$\Rightarrow \quad \sqrt[3]{3} < x < \sqrt[3]{4}$$
$$\Rightarrow \quad 3^4 < x^{12} < 4^4$$

From Inequality (4), $4^3 < x^{12} < 5^3$

Hence common solution of Inequalities (1), (2), (3), (4), is

$$3^4 < x^{12} < 5^3$$
$$\Rightarrow \quad \sqrt[3]{3} < x < \sqrt[4]{5}$$
$$\Rightarrow \quad 3^5 < x^{15} < \sqrt[4]{5^{15}}$$

But from 5th inequality we get $5^3 < x^{15} < 6^3$.

As $6^3 = 216 < 243 = 3^5$, common solution of (1), (2), (3), (4) has no solution common with (5), hence, the greatest n for which the rows of the given inequalities holds is 4 and for any x, such that $\sqrt[3]{3} < x < \sqrt[4]{5}$ will satisfy these inequalities.

Problem 12 *Determine the largest number in the infinite sequence;* $1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots, \sqrt[n]{n}$.

Solution: By checking the first four values, we find $3^{1/3}$ to be the largest number. We will prove that $(n^{1/n}), n \geq 3$ is a decreasing sequence.

$$n^{1/n} > (n+1)^{1/(n+1)}$$

$$\Leftrightarrow \quad n^{n+1} > (n+1)^n$$

$$\Leftrightarrow \quad n > \left(1+\frac{1}{n}\right)^n$$

Now, $\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2} \times \frac{1}{n^2} + \frac{n(n-1)(n-2)}{6} \cdot \frac{1}{n^3} + \cdots$

$$= 1 + 1 + \frac{1}{2}\left(1-\frac{1}{n}\right) + \frac{1}{6}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots < 3$$

or $$3 > \left(1+\frac{1}{n}\right)^n$$

$\therefore$ If $$n \geq 3, n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$$

i.e., $(n^{1/n})$ is decreasing for $n \geq 3$.

But, $3^{1/3}$ is also greater than 1 and $2^{1/2}$.
Hence, $3^{1/3}$ is the largest number.

Problem 13 *If a, b, and c are positive real numbers, such that* $abc = 1$, then. *prove that*

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \leq 1.$$

When does equality hold?

Solution:

$$\begin{aligned}
a^5 + b^5 &= (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) \\
&= (a+b)(a^4 + a^2b^2 + b^4 - ab(a^2 + b^2)) \\
&= (a+b)[(a^2 + ab + b^2)(a^2 - ab + b^2) - ab(a^2 + ab + b^2) + a^2b^2] \\
&= (a+b)\ [(a^2 + ab + b^2)(a^2 - 2ab + b^2) + a^2b^2] \\
&= (a+b)\ [(a^2 + ab + b^2)(a-b)^2 + a^2b^2] \\
&\geq (a+b) \times a^2b^2 \qquad [\because (a-b)^2(a^2 + ab + b^2) \geq 0]
\end{aligned}$$

i.e., $$a^5 + b^5 \geq a^2b^2(a+b)$$

and equality holds, if $a = b$.

Thus, $$\frac{ab}{a^5+b^5+ab} \leq \frac{ab}{a^2b^2(a+b)+ab}$$

$$= \frac{1}{ab(a+b)+1}$$

$$= \frac{1}{ab(a+b)+abc}$$

$$= \frac{1}{ab(a+b+c)}$$

$$= \frac{c}{(a+b+c)}$$

Similarly, $\dfrac{bc}{b^5+c^5+bc} \le \dfrac{a}{a+b+c}$

and $\dfrac{ca}{c^5+a^5+ca} \le \dfrac{b}{a+b+c}$

$$\therefore \quad \frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \le \frac{c}{a+b+c} + \frac{a}{a+b+c} + \frac{b}{a+b+c}$$

$$= \frac{a+b+c}{a+b+c} = 1$$

and the equality holds, if $a = b = c$ and since, $a \cdot b \cdot c = 1$, $a = b = c = 1$ implies equality.

Problem 14 *If $a_1 \ge a_2 \ge \ldots \ge a_n$ be real numbers, such that $a_1^k + a_2^k + \cdots + a_n^k \ge 0$ for all integers $k > 0$ and $p = \max\,[|a_1|, |a_2|, \ldots, |a_n|]$, prove that $p = |a_1| = a_1$ and that $(x - a_1)(x - a_2)\ldots(x - a_n) \le x^n - a_1{}^n$.*

Solution: Taking $k = 1$, since

$$a_1^k + a_2^k + \cdots + a_n^k \ge 0,$$

and for $k = 1$, we have

$$a_1 + a_2 + \cdots + a_n \ge 0 \tag{1}$$

and since, $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n$, $a_1 \ge 0$ (2)

and, if all a_i, $i = 1, 2, \ldots, n$ are positive, a_1 is the maximum of all a_i's

$$\therefore \quad p = |a_1| = a_1$$

Suppose that some of the a_i's are negative and $p \ne a_1$, then $a_n < 0$.
Hence,

$$p = |a_n|$$

Let, r be an index, such that

$$a_n = a_{n-1} = \cdots = a_{r+1} < a_r \le a_{r-1} \le \cdots \le a_1$$

Then, $a_1^k + a_2^k + \cdots + a_{r-1}^k + a_r^k + \cdots + a_n^k$

$$= a_n^k \left\{ \left(\frac{a_1}{a_n}\right)^k + \left(\frac{a_2}{a_n}\right)^k + \cdots + \left(\frac{a_{r-1}}{a_n}\right)^k + \left(\frac{a_r}{a_n}\right)^k + (n-r) \right\}$$

$$= a_n^k X$$

where the value of the second bracket is taken as X.

Since, $\left|\dfrac{a_1}{a_n}\right|, \left|\dfrac{a_2}{a_n}\right|, \ldots, \left|\dfrac{a_r}{a_n}\right|$ are all less than 1, so their kth powers are all less than these fractions and by taking k sufficiently large, which would make $X > 0$ and $Xa_n^k < 0$ for k odd, a contradiction and hence $p = a_1$.

Let, $x > a_1$, then by AM–GM inequality,

$$(x-a_2)(x-a_3)(x-a_4)\ \ldots\ (x-a_n) \le \left(\frac{\sum_{j=2}^{n}(x-a_j)}{n-1}\right)^{n-1}$$

$$\le \left(\frac{(n-1)x+a_1}{n-1}\right)^{n-1} \qquad \left[\because \sum_{i=1}^{b} a_i \ge 0\right]$$

$$= \left(x+\frac{a_1}{n-1}\right)^{n-1}$$

$$\le x^{n-1} + x^{n-2}\cdot a_1 + x^{n-2}a_1^2 + \cdots + a_1^{n-1} \left[\text{Here we have used} \binom{n-1}{r} \le (n-1)^r,\ r \ge 1.\right]$$

Multiplying both sides by $(x - a_1)$, we get

$$(x-a_1)(x-a_2)(x-a_3)\ \cdots\ (x-a_n) \le (x-a_1)(x^{n-1} + x^{n-2}a_1 + \cdots + a_1{}^{n-1}) = x^n - a^n{}_1.$$

Problem 15 *Let, $a > 2$ be given and define recursively $a_0 = 1, a_1 = a, a_{n+1} = \left(\dfrac{a_n^2}{a_{n-1}^2} - 2\right)a_n$. Show that for all $k \in \mathbb{N}$, we have*

$$\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} < \frac{1}{2}(2 + a - \sqrt{a^2-4}).$$

Solution: $a_0 = 1$ and $a_1 = a > 2$, so a can be written as $b + \dfrac{1}{b} = \dfrac{b^2+1}{b}$ for some real number $b > 1$ and $a^2 - 2 = b^2 + \dfrac{1}{b^2}$

Now, $$a_2 = \left(\frac{a_1^2}{a_0^2} - 2\right)a_1 = \left(\frac{a^2}{1} - 2\right)a = (a^2-2)a$$

$$= \left(b^2 + \frac{1}{b^2}\right)\left(b + \frac{1}{b}\right) = \frac{(b^2+1)(b^4+1)}{b^3}$$

Similarly, $$a_3 = \left[\left(\frac{a_2}{a_1}\right)^2 - 2\right]a_2 = \left[\left(b^2 + \frac{1}{b^2}\right)^2 - 2\right]a_2$$

$$= \left[\left(b^2 + \frac{1}{b^2}\right)^2 - 2\right]\left(b^2 + \frac{1}{b^2}\right)\left(b + \frac{1}{b}\right)$$

$$= \left(b^4 + \frac{1}{b^4}\right)\left(b^2 + \frac{1}{b^2}\right)\left(b + \frac{1}{b}\right)$$

$$= \left(b^{2^2} + \frac{1}{b^{2^2}}\right)\left(b^{2^1} + \frac{1}{b^{2^1}}\right)\left(b^{2^0} + \frac{1}{b^{2^0}}\right)$$

and proceeding in this manner, we get

$$a_n = \left(b^{2^{n-1}} + \frac{1}{b^{2^{n-1}}}\right)\left(b^{2^{n-2}} + \frac{1}{b^{2^{n-2}}}\right)\cdots\left(b + \frac{1}{b}\right)$$

Hence,

$$\text{LHS} = \sum_{i=0}^{n} \frac{1}{a_i} = 1 + \frac{b}{b^2+1} + \frac{b^3}{(b^2+1)(b^4+1)} + \frac{b^7}{(b^4+1)(b^4+1)(b^8+1)}$$
$$+\cdots+\frac{b^{2^n-1}}{(b^2+1)(b^4+1)\cdots(b^{2^n}+1)}$$

The right-hand side of the inequality is

$$\frac{1}{2}(a+2-\sqrt{a^2-4}) = \frac{1}{2}\left[b+\frac{1}{b}+2-\left(b-\frac{1}{b}\right)\right] = \left(\frac{1}{b}+1\right)$$

Now,

$$\text{LHS} = \frac{1}{b}\left[b + \frac{b^2}{b^2} + \frac{b^4}{(b^2+1)(b^4+1)} + \cdots + \frac{b^{2^n}}{(b^2+1)\ldots(b^{2^n}+1)}\right]$$
$$= 1 + \frac{1}{b}\left[\frac{b^2}{b^2+1} + \cdots + \frac{b^{2^n}}{(b^2+1)\ldots(b^{2^n}+1)}\right]$$

and, clearly $\dfrac{b^2}{b^2+1} + \dfrac{b^4}{(b^2+1)(b^4+1)} + \cdots + \dfrac{b^{2^n}}{(b^2+1)\ldots(b^{2^n}+1)}$

$$= \sum_{i=1}^{n} \frac{b^{2^i}}{(1+b^2)\ldots(1+b^{2^i})} = 1 - \frac{1}{(1+b^2)\ldots(1+b^{2^n})}$$

Here we used, $\displaystyle\sum_{j=1}^{n} \frac{a_j}{(1+a_1)\ldots(1+a_j)} = 1 - \frac{1}{(1+a_1)\ldots(1+a_n)}$

[This result is obtained by using partial fractions]

So, the LHS $\displaystyle = \sum_{i=0}^{n} \frac{1}{a_i} = 1 + \frac{1}{b}\left[\sum_{i=1}^{n} \frac{b^2}{(1+b^2)\cdots(1+b^{2^i})}\right]$

$$= 1 + \frac{1}{b}\left(1 - \frac{1}{(1+b^2)\cdots(1+b^{2^n})}\right)$$
$$= 1 + \frac{1}{b} - \frac{1}{b(1+b^2)(1+b^4)\ldots(1+b^{2^n})} < 1 + \frac{1}{b}$$
$$= \text{RHS}$$

And hence, is the result.

Problem 16 *A sequence of numbers a_n, $n = 1, 2, \ldots$, is defined as follows: $a_1 = \dfrac{1}{2}$ and for each $n \geq 2$, $a_n = \left(\dfrac{2n-3}{2n}\right)a_{n-1}$*

Prove that $\displaystyle\sum_{k=1}^{n} a_k < 1$ for all $n \geq 1$.

Solution: Given: $a_1 = \dfrac{1}{2}$ for $n \geq 2$

So, $a_k = \dfrac{2k-3}{2k} a_{k-1}$ for $k \geq 2$

$$\begin{aligned}
\text{or} \quad & 2ka_k = (2k-3)a_{k-1} \\
\Rightarrow \quad & 2ka_k - (2k-3)a_{k-1} = 0 \\
\Rightarrow \quad & 2ka_k - 2(k-1)a_{k-1} + a_{k-1} = 0 \\
\Rightarrow \quad & 2ka_k - 2(k-1)a_{k-1} = -a_{k-1} \qquad (1)
\end{aligned}$$

Now, adding up Eq. (1) from $k = 2$ to $k = (n+1)$, we have

$$\begin{aligned}
4a_2 - 2a_1 &= -a_1 \\
6a_3 - 4a_2 &= -a_2 \\
8a_4 - 6a_3 &= -a_3 \\
\vdots \quad \vdots \quad &\ \ \vdots \\
2na_n - 2(n-1)a_{n-1} &= -a_{n-1} \\
2(n+1)a_{n+1} - 2na_n &= -a_n.
\end{aligned}$$

Summing we get, $2(n+1)a_{n+1} - 2a_1 = -\sum_{k=1}^{n} a_k$

$$\Rightarrow \quad \sum_{k=1}^{n} a_k = 2a_1 - 2(n+1)a_{n+1} = 1 - 2(n+1)a_{n+1}$$

Now $a_n = \left(1 - \dfrac{3}{2n}\right)a_{n-1} \Rightarrow a_n$ is positive as $\left(1 - \dfrac{3}{2n}\right)$ is positive for all $n \geq 2$, and a_1 is positive.

Hence, $\sum_{k=1}^{n} a_k = 1 - 2(n+1)a_{n+1} < 1$. $\quad [\because \ 2(n+1)a_{n+1} > 0]$

Check Your Understanding

1. Show that the real number 'r' where $r = \dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3+\sqrt{5}}}$ satisfy the inequality $\sqrt{2} < r < 2$.
2. If $abcd = 1$ and $a, b, c, d \in \mathbb{R}^+$, prove the inequality $(1+a)(1+b)(1+c)(1+d) \geq 16$.
3. Find the smallest value of the expression $\dfrac{4x^2 + 8x + 13}{6(1+x)}$ for $x \geq 0$.
4. If x, y, z are positive reals such that $x^3y^2z = 7$, prove that $2x + 5y + 3z \geq 9(525/2^7)^{1/9}$.
5. If x, y, z are positive real numbers, such that $x < y < z$, show that
$$\frac{x^2}{z} < \frac{x^2+y^2+z^2}{x+y+z} < \frac{z^2}{x}.$$
6. By considering the sequence $1, a^2, a^4, \ldots, a^{2n}, \ldots$, where $0 < a < 1$, prove that
   (a) $1 - a^{2n} > na^{n-1}(1 - a^2)$
   (b) $1 - a^{2n} < n(1 - a^2)$.

7. If a, b, c, are positive real numbers, prove that
$6abc \le \Sigma a^2(b+c) \le 2(a^3+b^3+c^3)$.

8. Let $x_1\, x_2$ be the roots of the equation $x^2 + px - \dfrac{1}{2p^2} = 0$ where x is unknown and p is a real parameter. Prove that $x_1^4 + x_2^4 \ge 2+\sqrt{2}$.

9. Prove that $n^n\left(\dfrac{n+1}{2}\right)^{2n} > (n!)^3$.

10. Show that $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} < n\sqrt{\dfrac{n+1}{2}} < (n+1)^{3/2}$.

11. If $n^5 < 5^n$ for a fixed positive integer $n \ge 6$, show that $(n+1)^5 < 5^{n+1}$.

12. Show that for any real number x, $x^2\sin x + x\cos x + x^2 + \dfrac{1}{2} > 0$.

13. $a, b, c \in \mathbb{R}$ (*i.e.*, a, b, c are real numbers), such that $a^2+b^2+c^2 = 1$, then prove that $-\dfrac{1}{2} \le ab+bc+ca \le 1$.

14. Show that if the real numbers a_1, b_1, c_1, and a_2, b_2, c_2 satisfy
$a_1c_2 - 2b_1b_2 + c_1a_2 = 0$ and $a_1c_1 - b^2{}_1 > 0$, then $a_2c_2 - b_2{}^2 \le 0$.

15. If a, b, c, d are four real numbers, such that, $a + 2b + 3c + 4d \ge 30$, prove that $a^2+b^2+c^2+d^2 \ge 30$.

16. Let a, b, c, d be positive real numbers. Prove that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{4}{c}+\dfrac{16}{d} \ge \dfrac{64}{a+b+c+d}$.

17. If a, b, c are all greater than zero and distinct, then prove that
$a^4+b^4+c^4 > abc(a+b+c)$.

18. If a, b, c, d are positive real numbers, prove that
$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3} \ge \frac{1}{abc}+\frac{1}{bcd}+\frac{1}{cda}+\frac{1}{dab}.$$

19. Given that x, y, z are positive reals, satisfying the conditions that, $xyz = 32$, find the minimum value of the expression $x^2 + 4xy + 4y^2 + 2z^2$, as an integer.

20. Prove that $0 \le yz + zx + xy - 2xyz \le \dfrac{7}{27}$, where, x, y, z are non-negative real numbers and $x+y+z = 1$.

21. Prove, in a triangle the following inequality holds:
$$\frac{a\cos A + b\cos B + c\cos C}{a\sin B + b\sin C + c\sin A} \le \frac{a+b+c}{9R}.$$

22. Prove that the following inequality holds:
In any acute angled triangle ΔABC, $\cot^2 A + \cot^2 B + \cot^2 C \ge 1$.

23. In an acute angled triangle ABC, show that, $\tan^2 A + \tan^2 B + \tan^2 C \ge 9$. When does the equality occur?

24. If x, y, z are real numbers such that, $x+y+z=4$, $x^2+y^2+z^2=6$, then show that, each of x, y, z lies in the closed interval $\left[\dfrac{2}{3}, 2\right]$. Can '$x$' take the extreme values? Justify.

25. Prove that $\dfrac{1}{\sqrt{2n+1}} > \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} > \dfrac{\sqrt{n+1}}{2n+1}, n \in \mathbb{N}$.

26. Show that $\dfrac{1}{3}+\dfrac{8}{7}+\cdots+\dfrac{n^3}{n^2+n+1} < \dfrac{n(3n+5)}{6}$.

27. If a, b, c, are real numbers, such that

(a) $a+b+c>0$

(b) $ab+bc+ca>0$

(c) $abc>0$

Prove that a, b, c all are positive.

28. Suppose that $0 \le x_i \le 1$ for $i = 1, 2, \ldots, n$, prove that $2^{n-1}(1+x_1x_2\ldots x_n) \ge (1+x_1)(1+x_2)\ldots(1+x_n)$ with equality, if and only if, $(n-1)$ of the x_i's are equal to 1.

29. x, y, z are positive numbers, such that, $x = \dfrac{2y}{1+y}$, $y = \dfrac{2z}{1+z}$ and $z = \dfrac{2x}{1+x}$. Prove that $x = y = z$.

30. Let a, b, c, d be real numbers, such that $a < b < c < d$. Prove the inequality; $(a+b+c+d)^2 > 8(ac+bd)$.

31. Prove the following inequalities:

(a) $5 < \sqrt{5}+\sqrt[3]{5}+\sqrt[4]{5}$

(b) $8 > \sqrt{8}+\sqrt[3]{8}+\sqrt[4]{8}$

(c) $n > \sqrt{n}+\sqrt[3]{n}+\sqrt[4]{n}$, specifying conditions, if any, to be fulfilled.

32. Prove that, without using tables or calculators, $19^{93} > 13^{99}$.

33. Let a, b, c, d be positive real numbers, such that $a+b+c+d=1$. Prove the inequality: $\dfrac{a^2}{a+b}+\dfrac{b^2}{b+c}+\dfrac{c^2}{c+d}+\dfrac{d^2}{d+a} \ge \dfrac{1}{2}$. When does the equality hold?

34. Find all pairs (x, y) of real numbers, such that $16^{x^2+y}+16^{y^2+x}=1$.

35. If a, b, c be non-negative reals and $\dfrac{1}{1+a}+\dfrac{1}{1+b}+\dfrac{1}{1+c}=1$, prove that $abc \ge 8$.

36. If a, b, c are positive real numbers, such that, $a+b>c$, prove; $\dfrac{a}{1+a}+\dfrac{b}{1+b} \ge \dfrac{c}{1+c}$.

Challenge Your Understanding

1. Prove that, for a, b, c and $d \in \mathbb{R}$, $(1+ab)^2+(1+cd)^2+a^2b^2+c^2d^2 \ge 1$.

2. Let P be an interior point in ΔABC. Let x, y, z be the perpendicular distance of P from BC, CA, AB, respectively. Prove that

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \le \sqrt{\frac{a^2+b^2+c^2}{2\mathrm{R}}}.$$

3. With the same notation as in the previous problem, find the point P, such that $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}$ is least.

4. Prove that $\dfrac{4^n}{2\sqrt{n}} \le \dbinom{2n}{n} \le \dfrac{4^n}{\sqrt{3n+1}} \; \forall n \in \mathbb{N}$.

5. If a, b, c, d are positive real numbers such that $a + b + c + d = 1$, prove that $\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} + \sqrt{4d+1} < 6$; when does the equality hold?

6. In $\Delta\, ABC$, prove in the usual notation that $a^2 + b^2 + c^2 > 4\sqrt{3}\Delta$, where Δ is the area of $\Delta\, ABC$. When does the equality hold? (Weitzenböck's Inequality).

7. If $a > 0$, $b > 0$, prove that $4\left(\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2}\right) - 20\left(\dfrac{a}{b} + \dfrac{b}{a}\right) + 33 \le 0$ implies $a = 2b$ or $b = 2a$.

8. Let x, y, z be three positive real numbers, each less than 4. Prove that at least one of the numbers $\dfrac{1}{x} + \dfrac{1}{4-y}, \dfrac{1}{y} + \dfrac{1}{4-z} + \dfrac{1}{z} + \dfrac{1}{4-x}$ is greater than or equal to units.

9. Let ΔABC be an acute angled triangle and let H be its orthocentre. Let h_{max} denote the largest altitude of ΔABC. Prove the inequality; $AH + BH + CH < 2\, h_{\text{max}}$.

10. Suppose a and b are real numbers, such that, the roots of the cubic equation $ax^3 - x^2 + bx - 1 = 0$ are all positive real numbers, prove the following:
   (a) $0 < 3ab \le 1$
   (b) $b \ge \sqrt{3}$

11. Let a, b, c, be the lengths of the sides of a triangle and r its inradius; then show that $3r\,(a + b + c) < a^2 + b^2 + c^2$.

12. If a, b, c, are sides of a triangle and a, b, c, are integers, prove the inequality
$$\left(1 + \frac{b-c}{a}\right)^a \cdot \left(1 + \frac{c-a}{b}\right)^b \cdot \left(1 + \frac{a-b}{c}\right)^c < 1.$$

13. If a, b, c are sides of triangle, prove that $\dfrac{a}{c+a-b} + \dfrac{b}{a+b-c} + \dfrac{c}{b+c-a} \ge 3$.

14. If a, b, c are three positive real numbers, prove the inequality $\dfrac{a^2+1}{b+c} + \dfrac{b^2+1}{c+a} + \dfrac{c^2+1}{a+b} > 3$.

15. Given positive real numbers a, b, c such that, $a + b + c = 1$; prove that $a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \le 1$.

16. For positive real numbers a, b, c and d, show that, $a \cdot d^{b-c} + b \cdot d^{c-a} + c \cdot d^{a-b} \ge a + b + c$.

17. If a, b, c are sides of a triangle and p, q, r are positive real numbers, prove the following inequality:
$$a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q) \ge 0.$$

18. Let b, c be the legs of a right angled triangle, satisfying the following inequality:
$$\sqrt{b^2 - 6b\sqrt{2} + 19} + \sqrt{c^2 - 4c\sqrt{3} + 16} \le 3.$$
Find its hypotenuse as well as its area.

19. Show that $\dfrac{xyz}{x^3 + y^3 + xyz} + \dfrac{xyz}{y^3 + z^3 + xyz} + \dfrac{xyz}{z^3 + x^3 + xyz} \le 1$, where x, y, z are positive real numbers.

20. For arbitrary positive numbers a, b, c, prove that,

$$\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\geq 1.$$

21. $x_1, x_2, x_3, \ldots, x_n$ $(n \geq 2)$ are real numbers satisfying

$$\frac{1}{x_1+2011}+\frac{1}{x_2+2011}+\frac{1}{x_3+2011}+\cdots+\frac{1}{x_n+2011}=\frac{1}{2011}.$$

Prove that, $\dfrac{\sqrt[n]{x_1\cdot x_2\cdot x_3\cdots x_n}}{n-1}\geq 2011.$

22. α, β, γ, δ are positive angles, each being less than $\dfrac{\pi}{2}$. Also it is given that, $\alpha+\beta+\gamma+\delta=180°$. Prove the inequality:

$$\sqrt{2}(\tan\alpha+\tan\beta+\tan\gamma+\tan\delta)\geq \sec\alpha+\sec\beta+\sec\gamma+\sec\delta.$$

23. The positive real numbers a, b, c with $(a+b+c)=1$ are given. Prove the inequality:

$$\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}\leq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

24. Prove the inequality:

$$3(x^2+y^2+xy)(y^2+z^2+yz)(z^2+x^2+zx)\geq (x+y+z)^2(xy+yz+zx)^2$$

25. Let a, b, c, x, y, z be positive real numbers, such that, $a+b+c=x+y+z$ and $abc=xyz$. Further, suppose, $a\leq x\leq y\leq z\leq c$ and $a<b<c$. Prove that $a=x$; $b=y$; $c=z$.

26. Prove that, if a, b, c, are positive real numbers, then, the expression $\left\{\dfrac{1}{2}(a+b+c)-\dfrac{bc}{b+c}-\dfrac{ca}{c+a}-\dfrac{ab}{b+a}\right\}$ is always non-negative. Find also the condition that this expression is void.

27. Find all positive real numbers a, b, c, d satisfying the following conditions:
(a) $a+b+c+d=12$
(b) $abcd=27+ab+ac+ad+bc+bd+cd$

28. If x, y, z are all positive and $x+y+z=6$, prove that

$$\left(x+\frac{1}{y}\right)^2+\left(y+\frac{1}{z}\right)^2+\left(z+\frac{1}{x}\right)^2\geq\frac{75}{4}.$$

29. If a, b, c, d are positive real numbers, prove that

$$\frac{a^2+b^2+c^2}{a+b+c}+\frac{b^2+c^2+d^2}{b+c+d}+\frac{c^2+d^2+a^2}{c+d+a}+\frac{d^2+a^2+b^2}{d+a+b}\geq a+b+c+d.$$

30. Let a, b, c, d, be positive real numbers. Show that,

$$\frac{ab+bc+ca}{a^3+b^3+c^3}+\frac{ab+bd+da}{a^3+b^3+d^3}+\frac{ac+cd+da}{a^3+b^3+c^3}+\frac{bc+cd+db}{b^3+c^3+d^3}$$

$$\leq \min\left[\frac{a^2+b^2}{(ab)^{\frac{3}{2}}}+\frac{c^2+d^2}{(cd)^{\frac{3}{2}}},\frac{a^2+c^2}{(ac)^{\frac{3}{2}}}+\frac{b^2+d^2}{(bd)^{\frac{3}{2}}},\frac{a^2+d^2}{(ad)^{\frac{3}{2}}}+\frac{b^2+c^2}{(bc)^{\frac{3}{2}}}\right].$$

31. Determine all possible values of the expression

$$S = \frac{a}{d+a+b} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{c+d+a},$$

for arbitrary positive reals, a, b, c, d.

32. a, b, c are real numbers, such that, $abc + a + c = b$ and $ac \neq 1$. Find the greatest value of the expression: $\left(\frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1}\right)$.

33. Let a, b, c be positive real numbers, such that, $abc = 1$; prove the inequality

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

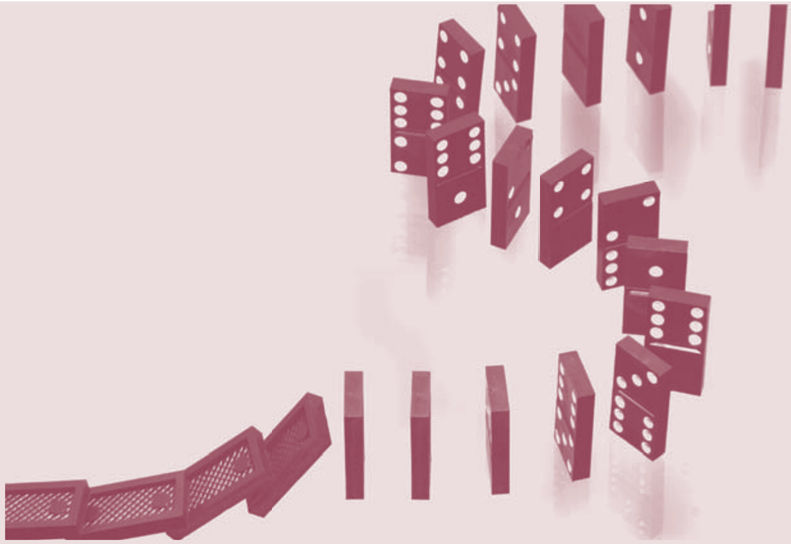
34. Given that a, b, c are positive real numbers, show that

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}, \text{ if } a^2 + b^2 + c^2 = 3abc.$$

35. Let, $x_n = \sqrt[2]{2 + \sqrt[3]{3 + \sqrt[4]{4 \cdots + \sqrt[n]{n}}}}$; prove that $x_{n+1} - x_n < \frac{1}{n!}, n = 2, 3, \ldots.$

Chapter

3



Mathematical Induction

3.1 Introduction

The process of deducing particular results from a general result is called deduction. The process of establishing a valid general result from particular results is called induction. The word induction means the method of reasoning about a general statement from the conclusion of particular cases. Inductions starts with observations. It may be true but then it must be so proved by the process of reasoning. Else it may be false but then it must be shown by finding a counter example where the conjecture fails.

In mathematics there are some results or statements that are formulated in terms of n, where $n \in \mathbb{N}$. To prove such statements we use a well suited method, based on the specific technique, which in known as **principle of mathematical induction**.

Giuseppe Peano

27 Aug 1858–20 Apr 1932
Nationality: Italian

3.1.1 Proposition

A statement which is either true or false is called a proposition or statement. $P(n)$ denotes a proposition whose truth value depends on natural variable 'n'.

For example, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ is a proposition whose truth value depends on natural number n.

We write, $P(n): 1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$,

where $P(5)$ means $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \dfrac{5(5+1)\ (10+1)}{6}$.

To prove the truth of proposition $P(n)$ depending on natural variable n, we use mathematical induction.

Consider the statement:
$P(n)$: '$n(n + 1)$ is even'. We wish to show that this statement is true for all $n \in \mathbb{N}$.

For $n = 1$, $P(1) = 1 \times 2 = 2$ (even)
For $n = 2$, $P(2) = 2 \times 3 = 6$ (even) and so on.
Alternatively, we can prove by stating that for n even, $n(n + 1)$ is even and for n odd, $n + 1$ is even and thus $n(n + 1)$ is even. But all statement may not be that simple, e.g., $P(n)$: '$3^n > n$'.
For $n = 1$, $P(1) : 3 > 1$ is true.
If we assume that the result is true for $n = r$, then $P(r) : 3^r > r$ is true.
For $n = r + 1$, $P(r + 1)$: $3^{r+1} = 3^r \times 3 > 3r > r + 1$ for $r \in \mathbb{N}$.
Hence, $P(r + 1)$ is true.

So what we got here? Nothing more than whenever $P(r)$ is true $P(r + 1)$ is true! But if we combined it with $P(1)$ is true, we see the domino effect!! As $P(1)$ true $\Rightarrow P(2)$ true!! Now $P(2)$ true $\Rightarrow P(3)$ true and so on. We can go on up to any length so result is true for all n. This process is called induction. There are two kind of Inductions.

3.2 First (or Weak) Principle of Mathematical Induction

The statement $P(n)$ is true for all $n \in \mathbb{N}$, if

1. $P(1)$ is true.
2. $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

The above statement can be generalized as $P(n)$ is true for all $n \in \mathbb{N}$ and $n \geq k$, if

1. $P(k)$ is true.
2. $P(m)$ is true $(m \geq k)$ $\Rightarrow$ $P(m+1)$ is true.

3.2.1 Working Rule

To prove any statement $P(n)$ to be true for all $n \geq k$ with the help of first principle of mathematical induction we follow the following procedure:

Step 1 (verification): Check if the statement is true or false for $n = k$.
Step 2 (assumption): Assume the statement be true for $n = m$, $m \geq k$.
Step 3 (Induction): Prove the statement is true for $n = m+1$ using the assumption.

We proceed to illustrate the use of the above principle by means of a few examples.

3.2.2 Problems of the Divisibility Type

If $f(n)$ is divisible by a number x and it is to be proved that $f(n + 1)$ is divisible by x, some times it is easier to show that $f(n + 1) - f(n)$ is divisible by x.

Example 1 *Show that* $7^{2n} + (2^{3n-3})(3^{n-1})$ *is divisible by* 25 *for all natural numbers* n.

Solution: Let $P(n) = 7^{2n} + (2^{3n-3})(3^{n-1})$

(a) $P(1) = 7^2 + (2^{3-3})(3^{1-1}) = 49 + 1 \cdot 1 = 50$, which is divisible by 25.
(b) Let $P(k)$ be true, *i.e.*, $7^{2k} + (2^{3k-3})(3^{k-1})$ is divisible by 25.
(c) We have to prove that $P(k + 1)$ is true, *i.e.*,

$$\begin{aligned} P(k+1) &= 7^{2(k+1)} + (2^{3(k+1)-3})(3^{k+1-1}) = 7^{2K} \cdot 7^2 + (2^{3k-3} \cdot 2^3)(3^{k-1} \cdot 3) \\ &= 49 \cdot 7^{2k} + 24(2^{3k-3})(3^{k-1}) = (25 + 24)7^{2k} + 24(2^{3k-3})(3^{k-1}) \\ &= 24(7^{2k} + 2^{3k-3}\, 3^{k-1}) + 25 \cdot 7^{2k} = 24\, P(k) + 25 \cdot 7^{2k} \end{aligned}$$

But we know that $P(k)$ is divisible by 25. Also, $25 \cdot 7^{2k}$ is clearly divisible be 25. Hence, $P(k + 1)$ is divisible by 25. Hence, by mathematical induction, the result is true for all n.

Example 2 *Show that* $11^{n+2} + 12^{2n+1}$ *is divisible by* 133 *for every natural number* n.

Solution: Let $P(n) = 11^{n+2} + 12^{2n+1}$

$P(1) = 11^3 + 12^3 = 3059 = 133 \times 23$, which is divisible by 133.

Let $P(k) = 11^{k+2} + 12^{2k+1}$ be divisible by 133.

$$\begin{aligned} P(k+1) &= 11^{k+3} + 12^{2k+3} \\ &= 11^{k+2} \cdot 11 + 12^{2k+1}\ 144 \\ &= 11 \cdot 11^{k+2} + (133 + 11)\ 12^{2k+1} \\ &= 11[11^{k+2} + 12^{2k+1}] + 133 \cdot 12^{2k+1} = 11 \cdot P(k) + 133 \cdot 12^{2k+1} \end{aligned}$$

$P(k)$ is divisible by 133 and so is $133 \cdot 12^{2k+1}$. Hence, $P(k+1)$ is also divisible by 133.

Hence, by mathematical induction, the result is true for all n.

Example 3 *Show that* $10^{2n-1} + 1$ *is divisible by* 11 *for all natural numbers* n.

Solution: Let $P(n) = 10^{2n-1} + 1$.

$P(1) = 10^1 + 1 = 11$ which is clearly divisible by 11.

Let $P(k) = 10^{2k-1} + 1$ be divisible by 11.

$$\begin{aligned} P(k+1) &= 10^{2k+1} + 1 = 10^{2k-1} \cdot 10^2 + 1 = [10^{2k-1} + 1] + 99 \cdot 10^{2k-1} \\ &= 1 \cdot P(k) + 99 \cdot 10^{2k-1} \end{aligned}$$

which is divisible by 11. Hence $P(k+1)$ is divisible by 11.

Hence, by mathematical induction, the result is true for all n.

Build-up Your Understanding 1

1. Use mathematical induction to prove the following $\forall n \in \mathbb{N}$:
   (a) $7^n - 3^n$ is divisible by 4.
   (b) $2 \cdot 7^n + 3 \cdot 5^n - 5$ is divisible by 24.
   (c) $3^{2n} - 1$ is divisible by 8.
   (d) $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9.
   (e) $5^{2n+1} + 2^{n+4} + 2^{n+1}$ is divisible by 23.
   (f) $7^{2n} - 1$ is divisible by 8.
   (g) $3^{2n+2} - 8n - 9$ is divisible by 8.
   (h) $41^n - 14^n$ is a multiple of 27.
   (i) $15^{2n-1} + 1$ is a multiple of 16.
   (j) $5^{2n+1} + 3^{n+2} \cdot 2^{n-1}$ is divisible by 19.
   (k) $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9.
   (l) $9^n - 8n - 1$ is divisible by 64.
2. Use mathematical induction to prove the following $\forall n \in \mathbb{N}$:
   (a) $n^3 + 3n^2 + 5n + 3$ is divisible by 3.
   (b) $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.
   (c) $n(n+1)(n+5)$ is a multiple of 3.
   (d) $(n+1)(n+2)(n+3)(n+4)(n+5)$ is divisible by 120.
   (e) $n(n+1)(n+2)$ is a multiple of 6.
   (f) $n(n+1)(2n+1)$ is divisible by 6.
   (g) $n^5 - n$ is a multiple of 5.

3. Use mathematical induction to prove the following $\forall n \in \mathbb{N}$:
   (a) $x^n - y^n =$ is divisible by $(x - y)$, where $x - y \neq 0$.
   (b) $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$, where $x + y \neq 0$.
   (c) $(1 + x)^n - nx - 1$ is divisible by x^2, where $x \neq 0$.
4. Use mathematical induction to prove that $\forall n \in \mathbb{N}$, $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a positive integer.
5. Use mathematical induction to prove the following:
   (a) For $n \in$ odd positive integers, $n(n^2 - 1)$ is divisible by 24,
   (b) For $n \in$ even positive integers, $n(n^2 + 20)$ is divisible by 48.
6. Show that $2^{2n} + 1$ or $2^{2n} - 1$ is divisible by 5 according as n is odd or even positive integer.
7. Prove that $5^{2n} + 1$ is divisible by 13 if n is odd. Hence, deduce that 5^{99} leaves a remainder 8 when divided by 13.
8. Show that $4 \cdot 6^n + 5^{n+1}$ leaves remainder 9 when divided by 20.
9. Show that $3^n + 8^n$ is not divisible by 5 for $n \in \mathbb{N}$.
10. Prove by induction that the last digit of $P(n) = 2^{2^n} + 1$ is 7 $\forall$ $(n > 1)$.

3.2.3 Problems Based on Summation of Series

Example 4 *Prove that* $S_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Solution: $S_1 = \frac{1 \times 2}{2} = 1$ is true.

Let,
$$S_k = \sum_{t=1}^{k} t = \frac{k(k+1)}{2}$$
$$S_{k+1} = \sum_{t=1}^{k+1} t = \left(\sum_{t=1}^{k} t\right) + k + 1$$
$$= \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{(k+1)(k+2)}{2}$$

Hence, the identity is true for all n by induction.

Example 5 *Use mathematical induction to show that* $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ *is true for all natural numbers* n.

Solution: Let $P(n) = 1 + 3 + 5 + \cdots + (2n - 1) = n^2$
$P(1) = 1 = 1$, which is true.

Assume that $P(k)$ holds good.

$$\Rightarrow \quad P(k) = 1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$
$$P(k + 1) = [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1)$$
$$= P(k) + 2k + 1 = k^2 + 2k + 1 = (k + 1)^2$$

Hence, $P(k + 1)$ is true.

Hence, by mathematical induction, the result is true for all n.

Example 6 *Show that*

$$\frac{1}{a+d}+\frac{a}{(a+d)(a+2d)}+\cdots+\frac{a}{[a+(n-1)d](a+nd)}=\frac{n}{a+nd}.$$

Solution: Let $P(n):\dfrac{1}{a+d}+\dfrac{a}{(a+d)(a+2d)}+\cdots+\dfrac{a}{(a+(n-1)d)(a+nd)}=\dfrac{n}{a+nd}$

for $n = 1$, LHS $=\dfrac{1}{a+d}$; RHS $=\dfrac{1}{a+d}$

$\Rightarrow$ $P(1)$ is true

Assume that

$$P(k):\frac{1}{a+d}+\frac{a}{(a+d)(a+2d)}+\cdots+\frac{a}{[a+(k-1)d](a+kd)}=\frac{k}{(a+kd)}$$

$$P(k+1):\frac{1}{a+d}+\frac{a}{(a+d)(a+2d)}+\cdots+\frac{a}{[a+(k-1)d]\cdot(a+kd)}+\frac{a}{(a+kd)\cdot[a+(k+1)d]}$$

$$=\frac{k}{a+kd}+\frac{a}{(a+kd)[a+(k+1)d]}$$

$$=\frac{k[a+(k+1)d]+a}{(a+kd)[a+(k+1)d]}$$

$$=\frac{a(k+1)+k(k+1)d}{(a+kd)[a+(k+1)d)]}$$

$$=\frac{(k+1)(a+kd)}{(a+kd)[a+(k+1)d]}$$

$$=\frac{(k+1)}{a+(k+1)d}$$

Thus, $P(1)$ holds, $P(k) \Rightarrow P(k+1)$, hence $P(n)$ holds for all $n \in \mathbb{N}$, by the principle of mathematics induction.

Example 7 *Prove, using mathematical induction, that*

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)} \quad \forall n \in \mathbb{N}.$$

Solution: We have to prove that $p(k)+t_{k+1}=p(k+1)$
or $p(k+1)-p(k)=t_{k+1}$

$$P(k+1)-P(k)=\frac{(k+1)(k+4)}{4(k+2)(k+3)}-\frac{k(k+3)}{4(k+1)(k+2)}$$

$$=\frac{1}{4(k+2)}\left[\frac{(k+1)(k+4)}{k+3}-\frac{k(k+3)}{k+1}\right]$$

$$= \frac{1}{4(k+2)}\left[\frac{(k^2+5k+4)(k+1)-k(k^2+6k+9)}{(k+1)(k+3)}\right]$$

$$= \frac{1}{4(k+2)(k+1)(k+3)}\left[k^3+5k^2+4k+k^2+5k+4-k^3-6k^2-9k\right]$$

$$= \frac{1}{(k+2)(k+1)(k+3)} = t_{k+1}$$

$\Rightarrow \quad p(k+1)$ is true.

Example 8 *Show by using principle of mathematical induction that*

$$1\cdot 3+2\cdot 3^2+3\cdot 3^3+\cdots+n\cdot 3^n = \frac{(2n-1)3^{n+1}+3}{4}.$$

Solution: Let $P(n): 1\cdot 3+2\cdot 3^2+3\cdot 3^3+\cdots+n\cdot 3^n = \dfrac{(2n-1)3^{n+1}+3}{4}$

When $n = 1$, LHS $= 1 \cdot 3 = 3$

and RHS $= \dfrac{(2n-1)3^{n+1}+3}{4} = \dfrac{(2\cdot 1-1)3^2+3}{4} = \dfrac{12}{4} = 3$

Hence, $P(1)$ is true.

Let $P(m)$ be true

$$\Rightarrow \quad 1\cdot 3+2\cdot 3^2+3\cdot 3^3+\cdots+m\cdot 3^m = \frac{(2m-1)3^{m+1}+3}{4} \tag{1}$$

To prove $P(m+1)$ is true, *i.e.*,

$$1\cdot 3+2\cdot 3^2+\cdots+m\cdot 3^m+(m+1)\cdot 3^{m+1} = \frac{(2m+1)3^{m+2}+3}{4}$$

Adding $(m+1).3^{m+1}$ to both sides of Eq. (1), we get

$$1\cdot 3+2\cdot 3^2+\cdots+m\cdot 3^m+(m+1)\cdot 3^{m+1} = \frac{(2m-1)3^{m+1}+3}{4}+(m+1)\cdot 3^{m+1}$$

$$= \frac{\{2m-1+4(m+1)\}\cdot 3^{m+1}+3}{4}$$

$$= \frac{(2m+1)3^{m+2}+3}{4}$$

Hence, $P(m+1)$ is true whenever $P(m)$ is true.

It follows that $P(n)$ is true for all natural numbers n.

Example 9 *Prove the following theorem of Nicomachus by induction:*

$$1^3 = 1,\ 2^3 = 3+5,\ 3^3 = 7+9+11,\ 4^3 = 13+15+17+19, \text{ etc.}$$

Solution: From the given pattern $1^3 = 1$, $2^3 = 3+5$, $3^3 = 7+9+11$, $4^3 = 13+15+17+19$, … note that the first term on the RHS are 1st, 2nd, 4th, 7th, … odd numbers. So the RHS of the nth identity to be proved has $\left[\dfrac{(n-1)n}{2}+1\right]$ st odd number as first term. Which is

$$2\left(\frac{n(n-1)}{2}+1\right)-1=n(n-1)+1$$

Hence, the nth identity to be proved is

$$n^3=(n(n-1)+1)+(n(n-1)+3)+\cdots n \text{ odd terms.}$$

i.e., $$n^3=\underbrace{(n^2-n+1)+(n^2-n+3)+\cdots+(n^2+n-1)}_{n \text{ terms}}$$

Assume this is true for n.

Then, RHS of $(n + 1)$th identity

$$=\underbrace{(n^2+n+1)+(n^2+n+3)+\cdots+(n^2+n+2n+1)}_{n+1 \text{ terms}}$$

$$=\underbrace{(n^2-n+1)+(n^2-n+3)+\cdots+(n^2+n-1)}_{n \text{ terms}}+2n^2$$

$$+(n^2+n+2n+1)$$

$$=n^3+2n^2+n^2+3n+1=n^3+3n^2+3n+1$$

$$=(n+1)^3$$

Note: Now adding both the sides of n rows, we get

$1^3 + 2^3 + 3^3 + \cdots + n^3 = 1 + 3 + 5 + \cdots + (2n - 1) + \cdots + (n^2 + n - 1)$.

Thus, on the right side there are

$$\frac{(n^2+n-1)+1}{2}=\frac{n(n+1)}{2} \text{ odd numbers are starting from 1.}$$

$$\text{So, } 1^3+2^3+3^3+\cdots+n^3=\frac{1}{2}\left(\frac{n(n+1)}{2}\right)(1+n^2+n-1)$$

$$=\left(\frac{n(n+1)}{2}\right)^2$$

Also observe sum of the first n odd numbers $= n^2$.

Example 10 *Using mathematical induction, show that* $\sum_{r=0}^{n} r\,{}^nC_r = n\cdot 2^{n-1}$.

Solution: Let $P(n) = 1\cdot{}^nC_1 + 2\cdot{}^nC_2 + \cdots + n\cdot{}^nC_n = n\cdot 2^{n-1}$

$$P(1) = 1\cdot{}^1C_1 = 1 = 1\cdot 2^{1-1} = 1.$$

Hence $P(1)$ holds true.

Assume that $P(k)$ is true

$$\Rightarrow \quad 1.{}^kC_1 + 2{}^kC_2 + \cdots + k{}^kC_k = k2^{k-1}$$

To prove that $P(k + 1)$ is true, we write

$$\sum_{r=0}^{k+1} r\,{}^{k+1}C_r = 1\cdot{}^{k+1}C_1 + 2\cdot{}^{k+1}C_2 + \cdots + (k+1){}^{k+1}C_{k+1}$$

$$= \sum_{r=0}^{k} r^{k+1}C_r + (k+1) = \sum_{r=0}^{k} r\left[{}^kC_r + {}^kC_{r-1}\right] + (k+1)$$

$$= \sum_{r=0}^{k} r^kC_r + \sum_{r=0}^{k} r^kC_{r-1} + (k+1) = P(k) + \sum_{r=0}^{k} r^kC_{r-1} + (k+1)$$

$$= P(k) + \sum_{r=1}^{k} r^kC_{r-1} + (k+1)^kC_k$$

Changing $r - 1$ to r, we get

$$P(k+1) = P(k) + \sum_{r=0}^{k-1} (r+1)^kC_r + (k+1)^kC_k$$

$$= k \cdot 2^{k-1} + \sum_{r=0}^{k} (r+1).^kC_r = k.2^{k-1} + \sum_{r=0}^{k} r.^kC_r + \sum_{r=0}^{k} {}^kC_r$$

$$= k \cdot 2^{k-1} + P(k) + 2^k = 2k \cdot 2^{k-1} + 2^k = k \cdot 2^k + 2^k = 2^k(k+1).$$

Hence, the result is true for $P(k+1)$.

Hence, by mathematical induction, the result is true for all n.

Example 11 *Using mathematical induction, show that*

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2}.$$

Solution: Let $P(n) \equiv \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2}$

$$\text{LHS of } P(1) = \left(1-\frac{1}{2^2}\right) = \frac{3}{4} = \text{RHS}$$

Hence, $P(1)$ is true.

Assume that $P(k)$ is true,

$$\Rightarrow \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$$

For $P(k+1)$, the LHS becomes

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right)$$

$$= P(k)\left(1-\frac{1}{(k+2)^2}\right) = \frac{k+2}{2k+2}\left(\frac{k^2+4k+3}{(k+2)^2}\right)$$

$$= \frac{k+2}{2k+2}\,\frac{(k+1)(k+3)}{(k+2)^2} = \frac{(k+2)(k+1)(k+3)}{2(k+1)(k+2)^2}$$

$$= \frac{k+3}{2(k+1)+2}$$

$$\Rightarrow P(k+1) \text{ is true}$$

Hence, by mathematical induction, the result is true for all n.

Build-up Your Understanding 2

1. Use mathematical induction to prove the following $\forall n \in \mathbb{N}$:

(a) $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n-1)}{2}$.

(b) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

(c) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$.

(d) $1^2 - 3^2 + 5^2 - 7^2 + \cdots + (4n-3)^2 - (4n-1)^2 = -8n^2$.

(e) $1^3 + 2^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$.

(f) $3 \cdot 6 + 6 \cdot 9 + 9 \cdot 12 + \cdots + 3n(3n+3) = 3n(n+1)(n+2)$.

(g) $\displaystyle\sum_{r=1}^{n} r(2r+1) = \frac{1}{6}n(n+1)(4n+5)$.

(h) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

(i) $a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] = \dfrac{n}{2}[2a + (n-1)d]$.

(j) $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(r^n - 1)}{r-1}$ for $r \neq 1$.

(k) $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

(l) $\dfrac{1}{1\cdot 4} + \dfrac{1}{4\cdot 7} + \dfrac{1}{7\cdot 10} + \cdots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{(3n+1)}$.

(m) $7 + 77 + 777 + \cdots + \underbrace{777\ldots7}_{n \text{ digits}} = \dfrac{7}{81}(10^{n+1} - 9n - 10)$.

(n) $1 + \dfrac{1}{(1+2)} + \dfrac{1}{(1+2+3)} + \cdots + \dfrac{1}{(1+2+3+\cdots+n)} = \dfrac{2n}{(n+1)}$.

2. Use mathematical induction to prove that

(a) $\tan\alpha + 2\tan 2\alpha + 2^2 \tan 2^2\alpha + \cdots n \textit{ terms} = \cot\alpha - 2^n \cot 2^n\alpha$.

(b) $\sin x + \sin 2x + \sin 3x + \cdots + \sin nx = \sin\left(\dfrac{n+1}{2}x\right)\sin\dfrac{nx}{2}\operatorname{cosec}\dfrac{x}{2}$.

3.2.4 Problems Involving Inequations

Example 12 *Prove by induction that if* $n \geq 10$, *then* $2^n > n^3$.

Solution: For $n = 10$, we have $2^{10} = 1024 > 10^3 = 1000$.

So the statement is true for $n = 10$.

Supposing that this statement is true for $n = k \geq 10$, *i.e.*, $2^k > k^3$.

For $n = k + 1$, $2^{k+1} > 2 \times k^3$.
Now, $2k^3 - (k^3 + 3k^2 + 3k + 1) = k^3 - 3k^2 - 3k - 1$
$= (k-1)^3 - 6k$.

Let $k = 10 + a$, where $a \geq 0$.
Then $(k-1)^3 - 6k = (10 + a - 1)^3 - 6\,(10 + a)$
$= (9 + a)^3 - 60 - 6a$
$= 729 + 243a + 27a^2 + a^3 - 60 - 60a$
$= 669 + 183a + 27a^2 + a^3 \geq 0$ $[\because\ a \geq 0]$

$\Rightarrow\quad 2k^3 > (k+1)^3$
$\Rightarrow\quad 2^{k+1} > (k+1)^3$.
Hence, the inequality is true for all $n \geq 10$.

Example 13 *Using mathematical induction show that* $\tan n\alpha > n \tan \alpha$ *where* $0 < \alpha < \dfrac{\pi}{4(n-1)}\ \forall$ *natural numbers,* $n > 1$.

Solution: Since $n > 1$ we start with $n = 2$.

$$\Rightarrow\quad \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} > 2\tan\alpha,\ \text{since } 1 - \tan^2\alpha < 1.$$

Hence, the result holds for $n = 2$.
Suppose it holds for $n = k$
$\Rightarrow\quad \tan k\alpha > k \tan \alpha$.
For $n = k + 1$,

$$\tan(k+1)\alpha = \frac{\tan k\alpha + \tan\alpha}{1-\tan k\alpha \tan\alpha}$$

$$> \frac{k\tan\alpha + \tan\alpha}{1-\tan k\alpha\tan\alpha} > (k+1)\tan\alpha, \text{since } 1-\tan k\alpha\tan\alpha < 1.$$

Hence, the result holds for $n = k + 1$.
Hence, by mathematical induction, the result is true for all n.

3.2.4.1 *Use of Transitive Property*

Sppose it is given $F(n) > G(n)$ or $\dfrac{F(n)}{G(n)} > 1$ (Where $G(n) > 0$)

We have to prove that,

$$F(n+1) > G(n+1) \text{ or } \frac{F(n+1)}{G(n+1)} > 1$$

If possible, we may aim to prove,

$$\frac{F(n+1)}{G(n+1)} > \frac{F(n)}{G(n)} > 1$$

or

$$\frac{F(n+1)}{F(n)}\,\frac{G(n)}{G(n+1)} > 1.$$

Example 14 $n! < \left(\frac{n+1}{2}\right)^n$, $n > 1$.

Solution: Let $P(n) = \left(\frac{n+1}{2}\right)^n > n!$

For $n = 2$, LHS $= \left(\frac{3}{2}\right)^2 = \frac{9}{4}$, RHS $= 2! = 2$

$\frac{9}{4} > 2$, Hence $P(2)$ is true.

Here, $F(n) = \left(\frac{n+1}{2}\right)^n$ $G(n) = n!$

$$F(n+1) = \left(\frac{n+2}{2}\right)^{n+1} \quad G(n+1) = (n+1)!$$

Let $P(n)$ is true, *i.e.*, $F(n) > G(n)$

$$\Rightarrow \quad \frac{F(n+1)}{F(n)}\,\frac{G(n)}{G(n+1)} = \frac{1}{2}\frac{(n+2)^{n+1}}{(n+1)^n}\cdot\frac{n!}{(n+1)!}$$

$$= \frac{1}{2}\left(\frac{n+2}{n+1}\right)^{n+1} = \frac{1}{2}\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$> \frac{2}{2} = 1 \qquad \left\{\left(1+\frac{1}{m}\right)^m > 2\right\}$$

$$\Rightarrow \quad \frac{F(n+1)}{G(n+1)} > \frac{F(n)}{G(n)} > 1$$

$$\Rightarrow \quad F(n+1) > G(n+1) \quad \Rightarrow \quad p(n+1) \text{ is true .}$$

Example 15 *Show, using mathematical induction, that*

$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$ *for all natural numbers* n.

Solution: Let us test for $n = 1$.

$$\Rightarrow \quad \frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{6+4+3}{12} = \frac{13}{12} > 1.$$

Hence, the result is true for $n = 1$.

Let us assume that the result holds for $n = k$.

That is $\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k+1} > 1$

For $n = k + 1$,

$$\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4}$$

$$=\left[\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{3k+1}\right]+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}-\frac{1}{k+1}$$

$$>1+\frac{1}{3k+2}+\frac{1}{3k+4}-\frac{2}{3k+3}$$

Now, if $1+\frac{1}{3k+2}+\frac{1}{3k+4}-\frac{2}{3k+3}>1.$

then we are throught. Or if $\frac{1}{3k+2}+\frac{1}{3k+4}-\frac{2}{3k+3}>0,$

$$\text{LHS}=\frac{(3k+4)\ (3k+3)+(3k+2)\ (3k+3)-2[(3k+2)\ (3k+4)]}{(3k+2)\ (3k+4)\ (3k+3)}$$

$$=\frac{3k+4-3k-2}{(3k+2)(3k+4)(3k+3)}$$

which is positive. Hence, the result is true for $n = k + 1$.
Hence, by mathematical induction, the result is true for all n.

Example 16 *Using mathematical induction, show that*

$1+\frac{1}{4}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$, *for all natural numbers n greater than* 1.

Solution: For $n = 2$,

$$\text{LHS}=1+\frac{1}{4}=\frac{5}{4}\text{ and RHS}=2-\frac{1}{2}=\frac{3}{2}$$

$$\frac{5}{4}<\frac{3}{2}.$$

Hence it holds for $n = 2$

Assume the result to hold for $n=k\ \Rightarrow\ 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}<2-\frac{1}{k}$

For $n=k+1,\ \left(1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}\right)+\frac{1}{(k+1)^2}<2-\frac{1}{k}+\frac{1}{(k+1)^2}$

Now, if we show that

$2-\frac{1}{k}+\frac{1}{(k+1)^2}<2-\frac{1}{(k+1)}$ or $\frac{1}{k}-\frac{1}{(k+1)^2}>\frac{1}{(k+1)}$ then we are through.

$$\Rightarrow\ \frac{1}{k}-\frac{1}{(k+1)^2}-\frac{1}{k+1}>0$$

$$\Rightarrow\ \frac{(k+1)^2-k-k(k+1)}{k(k+1)^2}$$

$$=\frac{k^2+2k+1-k-k^2-k}{k(k+1)^2}=\frac{1}{k(k+1)^2}>0.$$

Hence, the result is true for $n = k + 1$.
Hence, by mathematical induction, the result is true for all n.

Build-up Your Understanding 3

1. Use mathematical induction to prove the following $\forall n \in \mathbb{N}$:
   (a) $(2n+7) < (n+3)^2$.
   (b) $2^n > n$.
   (c) $1+2+3+\cdots+n < \frac{1}{8}(2n+1)^2$.
   (d) $1^2+2^2+\cdots+n^2 > \frac{n^3}{3}$.
2. Prove the following inequalities by mathematical induction:
   (a) $2^n > n^2$ for $n \ge 5, n \in \mathbb{N}$.
   (b) $\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}$ for $n > 1, n \in \mathbb{N}$.
   (c) $n^n < (n!)^2,\ n \ge 3, n \in \mathbb{N}$.
   (d) $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} > \frac{13}{24}$ for $n > 1,\ n \in \mathbb{N}$,
3. Prove by the principle of mathematical induction that $(1+x)^n > 1+nx$, $n > 1$, $n \in \mathbb{N}$ and $x > -1$, $x \neq 0$.

3.3 Second (or Strong) Principle of Mathematical Induction

The set of statements,

$$\{P(n): n \in \mathbb{N}\}$$

is true for each natural number $n \ge 1$ provided that:

1. $P(1)$ is true.
2. $P(n)$ is true for $n \le m$ (where $m \ge 1$) $\Rightarrow P(n)$ is true for $n = m + 1$.

The above statement can be generalized as $P(n)$ is true for all $n \in \mathbb{N}$ and $n \ge k$, if

1. $P(k)$ is true.
2. $P(n)$ is true for $n \le m$ (where $m \ge k$) $\Rightarrow$ $P(m+1)$ is true.

This is also called **extended principle of Mathematical Induction.**

3.3.1 Working Rule

Step 1: Verify that $P(n)$ is true for $n = k$, $n = k + 1$.
Step 2: Assume that $P(n)$ is true for $n \le m$ (where $m \ge k$).
Step 3: Prove that $P(n)$ is true for $n = m + 1$.

Once Step 3 is completed after Steps 1 and 2, we are through. That is, $P(n)$ is true for all natural numbers $n \ge k$.

(This method is to be used when $P(n)$ can be expressed as a combination of $P(n-1)$ and $P(n-2)$. In case $P(n)$ turns out to be a combination of $P(n-1)$, $P(n-2)$, and $P(n-3)$, we verify for $n = k + 2$ also in Step 1).

Example 17 *In a sequence* 1, 4, 10, …, $t_1 = 1$, $t_2 = 4$, *and* $t_n = 2t_{n-1} + 2t_{n-2}$ *for* $n \geq 3$. *Show by mathematical induction that*

$$t_n = \frac{1}{2}[(1+\sqrt{3})^n + (1-\sqrt{3})^n] \text{ for all } n \in \mathbb{N}.$$

Solution: Let us assume that the result is true for t_k for all $k < n$.

$$t_1 = \frac{1}{2}[(1+\sqrt{3})^1 + (1-\sqrt{3})^1]$$
$$= \frac{1}{2}(1+\sqrt{3}+1-\sqrt{3})$$
$$= \frac{1}{2}\times 2 = 1 \text{ is true}$$

$$t_2 = 4 = \frac{1}{2}[(1+\sqrt{3})^2 + (1-3)^2] = \frac{1}{2}(8) = 4 \text{ is also true.}$$

Now, we have to prove that

$$t_n = \frac{1}{2}[(1+\sqrt{3})^n + (1-\sqrt{3})^n]$$

Since,

$$t_n = 2\,[t_{n-1} + t_{n-2}]$$
$$= 2\left[\frac{1}{2}\{(1+\sqrt{3})^{n-1} + (1-\sqrt{3})^{n-1}\} + \frac{1}{2}\{(1+\sqrt{3})^{n-2} + (1-\sqrt{3})^{n-2}\}\right]$$
$$= [(1+\sqrt{3})^{n-1} + (1+\sqrt{3})^{n-2} + (1-\sqrt{3})^{n-1} + (1-\sqrt{3})^{n-2}]$$
$$= [(1+\sqrt{3})^{n-2}(2+\sqrt{3}) + (1-\sqrt{3})^{n-2}(2-\sqrt{3})]$$
$$= \left[(1+\sqrt{3})^{n-2}\frac{(1+\sqrt{3})^2}{2} + (1-\sqrt{3})^{n-2}\frac{(1-\sqrt{3})^2}{2}\right]$$
$$= \frac{1}{2}[(1+\sqrt{3})^n + (1-\sqrt{3})^n]$$

Thus, $t_n = \frac{1}{2}[(1+\sqrt{3})^n + (1-\sqrt{3})^n]$

So, by the second principle of mathematical induction, the formula is true for all natural numbers.

Example 18 *It is given that* $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$ *for* $n \geq 1$.

Use mathematical induction to prove that $u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.

Solution: For $n = 1$, and 2, we have

$$u_1 = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = 1$$
$$u_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = 1$$

$\Rightarrow$ The result is true for $n = 1, 2$.

Assume the result to be true for $n \le k$.

Then $$u_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

From the given relation

$$u_{k+1} = u_k + u_{k-1}$$

$$\Rightarrow u_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\frac{1+\sqrt{5}}{2}+1\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\frac{1-\sqrt{5}}{2}+1\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\frac{3+\sqrt{5}}{2}\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\frac{3-\sqrt{5}}{2}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\left(\frac{1+\sqrt{5}}{2}\right)^2\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1}{\sqrt{5}}\right)\left[\left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

Hence, the result is true for $n = k + 1$.

Hence, by mathematical induction, the result is true for all n.

Example 19 *If $x + y = a + b$, $x^2 + y^2 = a^2 + b^2$,*

prove by mathematical induction that $x^n + y^n = a^n + b^n$ for all natural numbers n.

Solution: Let $P(n) \equiv x^n + y^n = a^n + b^n$

$$P(1) \equiv x + y = a + b \qquad (1)$$

$$P(2) \equiv x^2 + y^2 = a^2 + b^2 \qquad (2)$$

Hence, $P(1)$ and $P(2)$ are true. Assume the result to be true for $n \le k$.

$$\Rightarrow \quad x^{k-1} + y^{k-1} = a^{k-1} + b^{k-1} \text{ and } x^k + y^k = a^k + b^k$$

In order to prove that $P(k + 1)$ is true, we write

$$\begin{aligned} x^{k+1} + y^{k+1} &= x(a^k + b^k - y^k) + y(a^k + b^k - x^k) \\ &= (a^k + b^k)(x + y) - xy\,(x^{k-1} + y^{k-1}) \\ &= (a^k + b^k)(a + b) - xy\,(a^{k-1} + b^{k-1}) \end{aligned}$$

Now from Eqs. (1) and (2) $xy = ab$

$$\Rightarrow x^{k+1} + y^{k+1} = a^{k+1} + b^{k+1}$$

which is the desired RHS for $P(k + 1)$.

Hence, by mathematical induction, the result is true for all n.

Example 20 *For* $x^3 = x + 1$, $a_n = a_{n-1} + b_{n-1}$, $b_n = a_{n-1} + b_{n-1} + c_{n-1}$, $c_n = a_{n-1} + c_{n-1}$, *prove that* $x^{3n} = a_n x + b_n + c_n x^{-1}$ $\forall\, n \in \mathbb{N}$ and $a_o = 0$, $b_o = 1$, $c_o = 0$.

Solution: We prove the result for $n = 1$, first. Accordingly, we should have

$$x^{3(1)} = a_1\,x + b_1 + c_1\,x^{-1}.\text{ Also } a_1 = a_o + b_o = 0 + 1 = 1$$
$$b_1 = a_o + b_o + c_o = 0 + 1 + 0 = 1;$$
$$c_1 = a_o + c_o = 0 + 0 = 0.$$
$$\Rightarrow \quad x^3 = 1x + 1 = x + 1, \text{ which is true.}$$

Assume the result to be true for $n = k$

$$\Rightarrow \quad x^{3k} = a_k x + b_k + c_k \cdot x^{-1}$$

For $n = k + 1$, $x^{3(k+1)} = x^{3k} \cdot x^3$

$$\begin{aligned}
&= (a_k x + b_{k\,+}\, c_k\, x^{-1})(x^3) = (a_k x + b_k + c_k\, x^{-1})(1 + x). \qquad (\text{since } x^3 = 1 + x)\\
&= a_k x + a_k x^2 + b_k + b_k x + c_k x^{-1} + c_k\\
&= x\,[a_k + b_k] + a_k\, x^{-1}\, x^3 + b_k + c_k\, x^{-1} + c_k\\
&= x\,[a_k + b_k] + a_k\, x^{-1}\,(1 + x) + b_k + c_k\, x^{-1} + c_k \qquad (\text{since } x^3 = 1 + x)\\
&= x\,[a_k + b_k] + a_k\, x^{-1} + a_k + b_k + c_k\, x^{-1} + c_k\\
&= x\,[a_k + b_k] + a_k + b_k + c_k + x^{-1}\,[a_k + c_k]\\
&= a_{k+1}\, x + b_{k+1} + c_{k+1}\, x^{-1}
\end{aligned}$$

Hence, the result is true for $n = k + 1$.

Hence, by mathematical induction, the result is true for all n.

Example 21 *Prove that, for all natural numbers* $n, (3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ *is divisible by* 2^n.

Solution: Let T_n be the statement that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is divisible by 2^n.

T_1: $(3+\sqrt{5}) + (3-\sqrt{5}) = 6$ is divisible by 2^1 is true.

T_2: $(3+\sqrt{5})^2 + (3-\sqrt{5})^2 = 28$ is divisible by 2^2 is true. Let us take that T_k is true for all $k < n$ for some n.

To prove T_n: $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is divisible by 2^n.

Now, for $n - 1 < n$,

$(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}$ is divisible by 2^{n-1}.

$$\begin{aligned}
&(3+\sqrt{5})^n + (3-\sqrt{5})^n\\
&= [(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}](3+\sqrt{5}+3-\sqrt{5})\\
&\quad - [(3+\sqrt{5})(3-\sqrt{5})^{n-1} + (3-\sqrt{5})(3+\sqrt{5})^{n-1}]\\
&= 6[(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}] - [4(3-\sqrt{5})^{n-2} + 4(3+\sqrt{5})^{n-2}]\\
&= 3 \times 2[(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}] - 4[(3+\sqrt{5})^{n-2} + (3-\sqrt{5})^{n-2}]
\end{aligned}$$

Here, $2[(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}]$ is divisible by $2 \times 2^{n-1} = 2^n$, and $4[(3+\sqrt{5})^{n-2} + (3-\sqrt{5})^{n-2}]$ is divisible by $4 \times 2^{n-2} = 2^n$.

Thus, $(3+\sqrt{5})^n+(3-\sqrt{5})^n$ is divisible by 2^n, *i.e.*, T_n is true if T_{n-1} and T_{n-2} are true. As, T_1 and T_2 are true, by the second principle of mathematical induction, T_n is true for all $n \in N$.

Build-up Your Understanding 4

1. If $a=\dfrac{1+\sqrt{5}}{2}$, $b=\dfrac{1-\sqrt{5}}{2}$ and $u_n=\dfrac{a^n-b^n}{\sqrt{5}}$, show that $u_n=u_{n-1}+u_{n-2}$.

Hence show that u_n is a positive integer for all $n \in \mathbb{N}$.

2. If $u_1=u_2=1$ and $u_n=u_{n-1}+u_{n-2}$ for $n>2$, prove that

(a) $u_{2n+2}=u_1+u_3+\cdots+u_{2n+1}$.

(b) $u_n^2-u_{n+1}\cdot u_{n-1}=(-1)^{n+1}$.

(c) $u_{2n+1}=1+u_2+u_4+\cdots+u_{2n}$.

(d) $u_{n+p-1}=u_{n-1}\cdot u_{p-1}+u_n\cdot u_p$.

(e) $u_n \mid u_{nk}\ \forall n,k \in \mathbb{N}$.

Solved Problems

Problem 1 *If n is a positive integer, prove that*

$$\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}.$$

Solution: Let $U_n=\dfrac{1}{n}+\dfrac{1}{n+1}+\cdots+\dfrac{1}{2n-1}$,

and $V_n=1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\cdots+\dfrac{1}{2n-1}$.

Now, we should prove that $U_n=V_n$ for all $n \in N$.

1. For $n=1$, $U_1=\dfrac{1}{1}=1$ and $V_n=1$ and hence, the statement is true for $n=1$.
2. Let the statement be true for $n=k$.

Now,

$$U_{k+1}-U_k=\left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k+1}\right)-\left(\frac{1}{k}+\frac{1}{k+1}+\cdots+\frac{1}{2k-1}\right)$$
$$=\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{k}=\frac{1}{2k+1}-\frac{1}{2k},$$

and

$$V_{k+1}-V_k=\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2k+1}\right)-\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2k-1}\right)$$
$$=-\frac{1}{2k}+\frac{1}{2k+1}=\frac{1}{2k+1}-\frac{1}{2k},$$

and so, $U_{k+1}-U_k=V_{k+1}-V_k$.

But $V_k=U_k$ by assumption and so $U_{k+1}=V_{k+1}$.

Thus, by the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$.

Problem 2 *Prove that*

$$\frac{1}{15} < \frac{1\cdot 3\cdot 5\cdot 7\cdots 99}{2\cdot 4\cdot 6\cdot 8\cdots 100} < \frac{1}{10}.$$

Solution: Let $P = \dfrac{1\cdot 3\cdot 5\cdot 7\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot 2n}$

Here we will prove that the product P_n is actually lesser than $\dfrac{1}{\sqrt{3n+1}}$ for $n > 1$ and greater than $\dfrac{1}{\sqrt{4n+1}}$.

$$P_2 = \frac{1\cdot 3}{2\cdot 4} = \frac{3}{8}$$

As $\dfrac{1}{\sqrt{4\times 2+1}} = \dfrac{1}{3} < \dfrac{3}{8} < \dfrac{1}{\sqrt{7}} = \dfrac{1}{\sqrt{3\times 2+1}} \Rightarrow P_2$ is true.

Now let $P_n^2 = \dfrac{1^2\cdot 3^2\cdot 5^2\cdots(2n-1)^2}{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}$.

We use mathematical induction to prove our assertion.

$\dfrac{1}{\sqrt{4n+1}} < P_n < \dfrac{1}{\sqrt{3n+1}}$ or equivalently $\dfrac{1}{4n+1} < P_n^2 < \dfrac{1}{3n+1}$

Let us assume that this result is true for $n = m$.

i.e., $$\frac{1}{4m+1} < P_m^2 < \frac{1}{3m+1}$$

i.e., $$\frac{1}{4m+1} < \frac{1^2\cdot 3^2\cdots(2m-1)^2}{2^2\cdot 4^2\cdots(2m)^2} < \frac{1}{3m+1}$$

$$P_{m+1}^2 = \frac{1^2\cdot 3^2\cdots(2m-1)^2\cdot(2m+1)^2}{2^2\cdot 4^2\cdots(2m)^2\cdot(2m+2)^2}$$

$$P_{m+1}^2 = P_m^2\cdot\frac{(2m+1)^2}{(2m+2)^2}$$

$$\Rightarrow \frac{1}{4m+1}\frac{(2m+1)^2}{(2m+2)^2} < P_{m+1}^2 < \frac{1}{3m+1}\cdot\frac{(2m+1)^2}{(2m+2)^2}$$

Now $\dfrac{1}{(3m+1)}\times\dfrac{(2m+1)^2}{2^2(m+1)^2} = \dfrac{4m^2+4m+1}{4(3m+1)(m^2+2m+1)}$

$= \dfrac{4m^2+4m+1}{12m^3+28m^2+20m+4} < \dfrac{4m^2+4m+1}{12m^3+28m^2+19m+4}$, where m is positive

$= \dfrac{(4m^2+4m+1)}{(4m^2+4m+1)(3m+4)} = \dfrac{1}{3m+4} = \dfrac{1}{3(m+1)+1}$.

Also $\dfrac{1}{4m+1}\times\dfrac{(2m+1)^2}{(2m+2)^2}$

$$=\frac{4m^2+4m+1}{(4m+1)(4m^2+8m+4)}$$
$$=\frac{(4m^2+4m+1)}{16m^3+36m^2+24m+4}$$
$$=\frac{(4m^2+4m+1)}{(4m^2+4m+1)(4m+5)-1}$$
$$>\frac{4m^2+4m+1}{(4m^2+4m+1)(4m+5)}=\frac{1}{4m+5}=\frac{1}{4(m+1)+1}$$

Thus, $\dfrac{1}{4(m+1)+1}<P_{m+1}^2<\dfrac{1}{3(m+1)+1}$

As P_2 is true and the truth of P_m implies the truth of P_{m+1}, so P_n is true for all $n \geq 2$.

$$\therefore \quad \frac{1}{4n+1}<P_n^2<\frac{1}{3n+1} \quad \forall n\geq 2$$

or $$\frac{1}{\sqrt{4n+1}}<P_n<\frac{1}{\sqrt{3n+1}} \quad \forall n\geq 2$$

In the problem, we have $n = 50$.

So $\dfrac{1\cdot 3\cdots(2\times 50-1)}{2\cdot 4\cdots(2\times 50)}<\dfrac{1}{\sqrt{150+1}}=\dfrac{1}{\sqrt{151}}<\dfrac{1}{\sqrt{100}}=\dfrac{1}{10}.$

Also, $\dfrac{1\cdot 3\cdot 5\cdots 99}{2\cdot 4\cdot 6\cdots 100}>\dfrac{1}{\sqrt{4\cdot 50+1}}=\dfrac{1}{\sqrt{201}}>\dfrac{1}{\sqrt{225}}=\dfrac{1}{15}.$

Problem 3 *Prove the rule of exponents $(ab)^n = a^n b^n$ by using principle of mathematical induction for every natural number.*

Solution: Let $P(n)$ be the given statement, *i.e.*, $P(n): (ab)^n = a^n b^n$

We note that $P(n)$ is true for $n = 1$ since $(ab)^1 = a^1 b^1$

Let $P(k)$ be true, *i.e.*, $(ab)^k = a^k b^k$ (1)

We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true.

Now, we have $(ab)^{k+1} = (ab)^k (ab)$

$$=(a^k b^k)(ab) \quad \text{[by Eq. (1)]}$$
$$=(a^k\cdot a^1)(b^k\cdot b^1)=a^{k+1}\cdot b^{k+1}$$

Therefore, $P(k+1)$ is also true whenever $P(k)$ is true.

Hence, by principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Problem 4 *Prove that $7^{2n}+(2^{3n-3})\cdot 3^{n-1}$ is divisible by 25, for $n \in \mathbb{N}$.*

Solution: Let $P(n)$ be the statement that '$7^{2n}+(2^{3n-3})\cdot 3^{n-1}$ is divisible by 25'.

For $n = 1$, $7^{2n}+(2^{3n-3})\cdot 3^{n-1} = 7^2+(1)\cdot 1 = 50$, which is divisible by 25.

Let $P(r)$ be true for $n = r$, *i.e.*,

$$7^{2r} + (2^{3r-3})\cdot 3^{r-1} \text{ (is divisible by 25)} = 25k,\ k \in \mathbb{N}. \qquad (1)$$

For $n = r + 1$, $P(r+1): 7^{2r+2} + (2^{3(r+1)-3})\cdot 3^{r+1-1}$

$= 49\cdot 7^{2r} + (2^{3r})\cdot 3^{r} = 49\cdot 7^{2r} + 8\cdot(2^{3r-3})\cdot 3\cdot 3^{r-1}$

$= 25\cdot 7^{2r} + 24[7^{2r} + (2^{3r-3})3^{r-1}] = 25\cdot 7^{2r} + 24\cdot 25k$ (using Eq. (1))

$= 25[7^{2r} + 24k]$. Hence, $P(r + 1)$ is also true.

Hence by mathematical induction, the result is true for all $n \in \mathbb{N}$.

Problem 5 *Given* $n^4 < 10^n$ *for a fixed positive integer* $n \geq 2$, *prove that* $(n+1)^4 < 10^{n+1}$.

Solution: The given statement is

$P(n)$: $n^4 < 10^n$, $n \geq 2$. For $n = 2$, this is obviously true. Now

$$(n+1)^4 - 10n^4 = -9n^4 + 4n^3 + 6n^2 + 4n + 1$$

$$= -\left[n^3(2n-4) + \frac{3}{2}n^2(n^2-4) + n(n^3-4) + \frac{9}{2}n - 1\right] \qquad (1)$$

For $n \geq 2$, each term on the RHS of (i) is ≤ 0.

Hence, $(n+1)^4 - 10n^4 < 0$, $\quad n \geq 2 \Rightarrow (n+1)^4 < 10n^4 < 10\cdot 10^n$ (given)

Hence, $(n+1)^4 < 10^{n+1}$ for $n \geq 2$.

Hence, by mathematical induction, the result is true for all $n \geq 2$.

Peoblem 6 *Show that* $\dfrac{1^2}{1\cdot 3} + \dfrac{2^2}{3\cdot 5} + \cdots + \dfrac{n^2}{(2n-1)(2n+1)} = \dfrac{n(n+1)}{2(2n+1)}$ *for* $n \geq 1$.

Solution: Let $P(n) \equiv \dfrac{1^2}{1\cdot 3} + \dfrac{2^2}{3\cdot 5} + \cdots + \dfrac{n^2}{(2n-1)(2n+1)} = \dfrac{n(n+1)}{2(2n+1)}$

For $n = 1$, LHS is $\dfrac{1^2}{1\cdot 3} = \dfrac{1}{3}$; RHS is $\dfrac{1(2)}{2(3)} = \dfrac{1}{3}$.

$\Rightarrow$ The result is true for $n = 1$.

Let us assume it to be true for $n = k$. *i.e.*,

$$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \cdots + \frac{k^2}{(2k-1)(2k+1)} = \frac{k(k+1)}{2(2k+1)},$$

Let us examine $P(k + 1)$. Then

$$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \cdots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$

$$= P(k) + \frac{(k+1)^2}{(2k+1)(2k+3)} = \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+1}\left[\frac{k}{2} + \frac{k+1}{2k+3}\right] = \frac{(k+1)}{(2k+1)}\left[\frac{2k^2+3k+2k+2}{2(2k+3)}\right]$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{2k^2+5k+2}{2(2k+3)}\right] = \frac{(k+1)}{(2k+1)}\left[\frac{2k^2+4k+k+2}{2(2k+3)}\right]$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{2k(k+2)+1(k+2)}{2(2k+3)}\right] = \frac{(2k+1)(k+1)(k+2)}{2(2k+3)(2k+1)}$$

$$= \frac{(k+1)(k+2)}{2(2k+3)}$$

$\Rightarrow P\left(k+1\right)$ is true.

Hence, by mathematical induction, the result is true for all n.

Problem 7 *Show that* $H_1 + H_2 + \cdots + H_n = (n+1)\,H_n - n$.

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\ \forall n \in \mathbb{N}$.

Solution: Let $P(n) \equiv H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$
For $n = 1$, we have $H_1 = \text{LHS} = 1$

RHS is $2H_1 - 1 = 2 - 1 = 1$. Hence $P(1)$ is true.

Assume that $P(k)$ is true. Thus, $H_1 + H_2 + \cdots + H_k = (k+1)\,H_k - k$.

For $n = k + 1$,

$P(k+1) \equiv H_1 + H_2 + \cdots + \mathrm{H}_k + \mathrm{H}_{k+1} = (k+1)H_k + H_{k+1} - k$.

$$= (k+1)\left[H_{k+1} - \frac{1}{k+1}\right] + H_{k+1} - k = H_{k+1}[k+1+1] - 1 - k$$

$$= (k+2)\,H_{k+1} - (k+1)$$

which is the desired RHS. Hence, we are through.

Hence, by mathematical induction, the result is true for all n.

Problem 8 *Show that* ${}^{2n}C_n < 4^n\ \forall\ n \in\ \mathbb{N}$.

Solution: Let $P(n) \equiv {}^{2n}C_n < 4^n$

For $n = 1$, LHS $= {}^{2}C_1 = 2$, RHS $= 4^1 = 4$.

$2 < 4$, Hence, $P(1)$ is true. Assume that $P(k)$ is true.

$\Rightarrow {}^{2k}C_k < 4^k$

For $n = k + 1$, ${}^{2k+2}C_{k+1} = \dfrac{(2k+2)(2k+1)}{(k+1)(k+1)} \cdot {}^{2k}C_k < \dfrac{(2k+2)(2k+1)}{(k+1)(k+1)}4^k$

If we show that $\dfrac{(2k+2)(2k+1)}{(k+1)(k+1)}4^k \le 4^{k+1}$, we are through.

Hence, we prove that $\dfrac{2(2k+1)}{k+1} \le 4$

That is, $2k + 1 \le 2k + 2$ or $1 \le 2$, which is correct. Hence, $P(k + 1)$ is shown to be true.
Hence, by mathematical induction, the result is true for all n.

Problem 9. *Show that* $1 + 2x + 3x^2 + \cdots + nx^{n-1} = \dfrac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$ *for all* $n \in \mathbb{N}$.

Solution: We write $P(n) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}$.
Let us start with $P(1)$, LHS $= 1$.

$$\text{RHS} = \frac{1-2x+x^2}{(1-x)^2} = \frac{(x-1)^2}{(1-x)^2} = 1$$

$\Rightarrow$ $P(1)$ is true

Assume that $P(k)$ is true.

$$\Rightarrow\ 1+2x+3x^2+\cdots+kx^{k-1} = \frac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}$$

Let us examine $P(k+1)$, *i.e.*,

$$1+2x+3x^2+\cdots+kx^{k\text{-}1}+(k+1)x^k = P(k)+(k+1)x^k$$

$$= \frac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2} + \frac{(k+1)x^k}{1}$$

$$= \frac{1-(k+1)x^k+kx^{k+1}+(k+1)x^k[1+x^2-2x]}{(1-x)^2}$$

$$= \frac{1-(k+1)x^k+kx^{k+1}+(k+1)x^k+(k+1)x^{k+2}-2(k+1)x^{k+1}+0}{(1-x)^2}$$

$$= \frac{1+(k-2k-2)x^{k+1}+(k+1)x^{k+2}}{(1-x)^2} = \frac{1-(k+2)x^{k+1}+(k+1)x^{k+2}}{(1-x)^2}$$

which is the RHS of $P(k+1)$.

$\Rightarrow P(k+1)$ is true.

Hence, by mathematical induction, the result is true for all n.

Problem 10 *Show that* $\cos a \cos 2a \cos 4a \ldots \cos(2^{n-1} a) = \dfrac{\sin 2^n a}{2^n \sin a}$ $\forall\ n \in \mathbb{N}$.

Solution: Let $P(n) = \cos a \cos 2a \cos 4a \ldots \cos(2^{n-1}a)$

For $P(1)$, LHS is $\cos a$

$$\text{RHS is } \frac{\sin 2a}{2\sin a} = \frac{2\sin a\ \cos a}{2\sin a} = \cos a$$

Hence $P(1)$ is true.

Assume the result to be true for $P(k)$,

$$\textit{i.e.},\ \cos a \cos 2a\ \cos 4a \ldots \cos(2^{k-1}a) = \frac{\sin 2^k a}{2^k \sin a}$$

Now $P(k+1)$

$$= \cos a\ \cos 2a\ \cos 4a \ldots \cos(2^{k-1}a)\ \cos(2^k a) = P(k)\cdot\cos(2^k a)$$

$$= \frac{\sin 2^k a\ \cos 2^k a}{2^k \sin a} = \frac{\sin 2^{k+1} a}{2^{k+1}\sin a} \Rightarrow P\left(k+1\right) \text{ is true.}$$

Hence, by mathematical induction, the result is true for all n.

Problem 11 *Show that*

$$\tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right)+\tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right)+\cdots+\tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right)$$

$$= \tan^{-1}(n+1)x - \tan^{-1} x \quad \forall n \in \mathbb{N}.$$

Solution: Let $P(n) =$

$$P(n) = \tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right) + \tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right) + \cdots + \tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right)$$

For $n = 1$,

$$\tan^{-1}\left(\frac{x}{1+2x^2}\right) = \text{LHS and RHS} = \tan^{-1} 2x - \tan^{-1} x$$

$$= \tan^{-1}\left[\frac{2x-x}{1+2x\cdot x}\right] = \tan^{-1}\frac{x}{1+2x^2}$$

$\Rightarrow$ LHS = RHS

Let us assume that $P(k)$ is true.

$$\Rightarrow \tan^{-1}\left(\frac{x}{1+1\cdot 2x^2}\right) + \tan^{-1}\left(\frac{x}{1+2\cdot 3x^2}\right) + \cdots + \tan^{-1}\left(\frac{x}{1+k(k+1)x^2}\right)$$

$$= \tan^{-1}(k+1)x - \tan^{-1} x$$

Now, $P(k+1)$

$$= \tan^{-1}\left(\frac{x}{1+1\cdot 2x^2}\right) + \tan^{-1}\left(\frac{x}{1+2\cdot 3x^2}\right) + \cdots + \tan^{-1}\left[\frac{x}{1+k(k+1)x^2}\right]$$

$$+ \tan^{-1}\left[\frac{n}{1+(k+1)(k+2)x^2}\right]$$

$$= P(k) + \tan^{-1}\frac{x}{1+(k+1)(k+2)x^2}$$

$$= \tan^{-1}(k+1)x + \tan^{-1}\frac{x}{1+(k+1)(k+2)x^2} - \tan^{-1} x$$

$$= \tan^{-1}\left[\frac{(k+1)x + (k+1)^2(k+2)x^3 + x}{1+(k+1)^2(x)^2}\right] - \tan^{-1} x$$

$$= \tan^{-1}\left[\frac{(k+2)x + (k+1)^2(k+2)x^3}{1+(k+1)^2(x)^2}\right] - \tan^{-1} x$$

$$= \tan^{-1}\left[\frac{(k+2)x[1+(k+1)^2x^2]}{1+(k+1)^2x^2}\right] - \tan^{-1} x = \tan^{-1}(k+2)x - \tan^{-1} x$$

$\Rightarrow P(k+1)$ is true.

Hence, by mathematical induction, the result is true for all n.

Problem 12 *Prove, using Mathematical induction, that*

$$\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n\text{ times}} = 2\cos\frac{\pi}{2^{n+1}} \quad \forall\, n \in \mathbb{N}.$$

Solution: Let $P(n) = \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n\text{ times}}$

For $n = 1$.

$$\text{LHS} = \sqrt{2} \text{ and RHS} = 2\cos\left(\frac{\pi}{2^2}\right) = 2\cos\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

LHS = RHS, hence $P(1)$ is true.

Assume that $P(k)$ is true.

$$\Rightarrow \quad \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{k \text{ times}} = 2\cos\left(\frac{\pi}{2^k+1}\right)$$

Now, $P(k+1)$

$$= \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{k+1 \text{ times}} = \sqrt{2+\Big(\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{k \text{ times}}\Big)}$$

$$= \sqrt{2+P(k)} = \sqrt{2+2\cos\left(\frac{\pi}{2^k+1}\right)}$$

$$= \sqrt{2}\sqrt{1+\cos\left(\frac{\pi}{2^{k+1}}\right)} = \sqrt{2}\sqrt{2\cos^2\frac{\pi}{2^{k+2}}}$$

$$= 2\cos\frac{\pi}{2^{k+2}}$$

$\Rightarrow P(k+1)$ is true.

Hence, by mathematical induction, the result is true for all n.

Problem 13 *Use mathematical induction to prove that*

$$\cos x + \cos 2x + \cdots + \cos nx = \cos\frac{n+1}{2}x \cdot \sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}.$$

Solution: For $n = 1$, RHS = $\cos x \sin\frac{x}{2}\operatorname{cosec}\frac{x}{2} = \cos x = \text{LHS}$

Hence, the result is true for $n = 1$. Let the result be true for $n = r$, *i.e.*,

$$\cos x + \cos 2x + \cdots + \cos rx = \cos\frac{r+1}{2}x\sin\frac{rx}{2}\operatorname{cosec}\frac{x}{2} \tag{1}$$

For $n = r + 1$,

$$\text{LHS} = \cos x + \cos 2x + \cdots + \cos rx + \cos(r+1)x = \cos\frac{r+1}{2}x\sin\frac{rx}{2}\operatorname{cosec}\frac{x}{2} + \cos(r+1)x$$

$$= \operatorname{cosec}\frac{x}{2}\left[\cos\frac{r+1}{2}x\sin\frac{rx}{2} + \cos(r+1)x\sin\frac{x}{2}\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\sin\frac{2r+1}{2}x - \sin\frac{x}{2} + \sin\frac{2r+3}{2}x - \sin\frac{2r+1}{2}x\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\sin\frac{2r+3}{2}x - \sin\frac{x}{2}\right] = \operatorname{cosec}\frac{x}{2}\cdot\sin\frac{r+1}{2}x\cos\frac{r+2}{2}x,$$ so that the result is true for $n = r + 1$.

Hence, by mathematical induction, the result is true for all $n \geq 1$.

Problem 14 *Let* $0 < A_i < \pi$ *for* $i = 1, 2, \ldots, n$. *Use mathematical induction to prove that*

$$\sin A_1 + \sin A_2 + \cdots + \sin A_n \le n \sin\left(\frac{A_1 + A_2 + \cdots + A_n}{n}\right)$$

where $n \ge 1$ *is a natural number.*
{You may use the fact that $p \sin x + (1-p)\sin y \le \sin[px + (1-p)y]$
where $0 \le p \le 1$ *and* $0 \le x, y \le p$*}.*

Problem 15 *Using mathematical induction, prove that for every integer* $n \ge 1$, $(3^{2^n} - 1)$ *is divisible by* 2^{n+2} *but not by* 2^{n+3}.

Solution: Let $P(n) = 3^{2^n} - 1$

$P(1) = 3^{2^1} - 1 = 8 = 1 \cdot 8$ is divisible by 2^3 but not by 2^4

$P(2) = 3^{2^2} - 1 = 80 = 5 \cdot 2^4$ is divisible by 2^4 but not by 2^5
$\Rightarrow P(1)$ and $P(2)$ are true.

Assume that $P(k) = 3^{2^k} - 1$ is divisible by 2^{k+2} but not by 2^{k+3}

$\Rightarrow 3^{2^k} - 1 = A \cdot 2^{k+2}$ where A is an odd integer.

Now, $P(k+1) = 3^{2^{k+1}} - 1$

$$\begin{aligned}
&= (3^{2^k})^2 - 1 \\
&= (A \cdot 2^{k+2} + 1)^2 - 1 \\
&= A^2\, 2^{2k+4} + 2.A \cdot 2^{k+2} \\
&= 2^{k+3}(A^2 \cdot 2^{k+1} + A) \\
&= 2^{k+3} \cdot \text{an odd integer} = 2^{k+3} \cdot B
\end{aligned}$$

$\Rightarrow P(k+1)$ is divisible by 2^{k+3} but not by 2^{k+4} because B is an odd integer.
$\Rightarrow P(k+1)$ is true.
Hence, by mathematical induction, the result is true for all n.

Problem 16 *Let p be a prime and m a positive integer. By mathematical induction on m, prove that whenever r is an integer such that p does not divide r, p divides* ${}^{mp}C_r$.

Solution: For $m = 1$, ${}^{p}C_r = \dfrac{p(p-1)(p-2)\cdots(p-r+1)}{1 \cdot 2 \cdot 3 \cdots r}$ where $1 \le r \le p - 1$.

Since p is a prime number it cannot be divisible by any of the numbers 2, 3, …, r.
Bring a positive integer ${}^{p}C_r$ is divisible by p. Hence, the statement is true for $m = 1$.
Let the statement be true for $m = n$, *i.e.*, ${}^{np}C_r$ is divisible by p.
Now, $(1+x)^{(n+1)p} = (1+x)^p\,(1+x)^{np}$
Coefficient of x^r on LHS = Coefficient of x^r on RHS
$\Rightarrow {}^{(n+1)p}C_r = {}^{p}C_0 \cdot {}^{np}C_r + {}^{p}C_1\, {}^{np}C_{r-1} + {}^{p}C_2 \cdot {}^{np}C_{r-2} + \cdots + {}^{p}C_r\, {}^{np}C_0.$
All the terms on RHS are divisible by p as ${}^{np}C_r, {}^{p}C_1, {}^{p}C_2, \ldots, {}^{p}C_r$ are divisible by p.
$\Rightarrow {}^{(n+1)p}C_r$ is divisible by p.
Hence, by the principle of mathematical induction the statement is true for all m.

Problem 17 *Prove by using mathematical induction or otherwise,*

$\dfrac{n^7}{7} + \dfrac{n^5}{5} + \dfrac{2n^3}{3} - \dfrac{n}{105}$ *is an integer.*

Solution:

Using induction: Let $M(n)$ be the statement that $15n^7 + 21n^5 + 70n^3 - n$ is divisible by 105 for $n = 1$.

$M(1) = 15 \times 1^7 + 21 \times 1^5 + 70 \times 1^3 - 1 = 105$ is divisible by 105.

So, $M(1)$ is true.

Assume that $M(k)$ is true, *i.e.*,

$$M(k) = 15k^7 + 21k^5 + 70k^3 - k = 105s.$$

Now,

$$\begin{aligned} M(k+1) &= 15(k+1)^7 + 21(k+1)^5 + 70(k+1)^3 - (k+1) \\ &= (15k^7 + 21k^5 + 70k^3 - k) \\ &\quad + 15\{(k+1)^7 - k^7\} + 21\{(k+1)^5 - k^5\} \\ &\quad + 70\{(k+1)^3 - k^3)\} - \{(k+1) - k\} \\ &= 105s + 15\left(\binom{7}{1}k^6 + \binom{7}{2}k^5 + \cdots + \binom{7}{6}k + \binom{7}{7}\right) \\ &\quad + 21\left(\binom{5}{1}k^4 + \binom{5}{2}k^3 + \cdots + \binom{5}{4}k + \binom{5}{5}\right) \\ &\quad + 70\{3k^2 + 3k + 1\} - 1 \\ &= 105s + 15 \times 7p + 15 + 21 \times 5q + 21 + 70 \times 3r + 70 - 1 \end{aligned}$$

where $\binom{7}{1}k^6 + \binom{7}{2}k^3 + \cdots + \binom{7}{6}k$ is a multiple of 7 and hence, taken as $7p$,

$\binom{5}{1}k^4 + \binom{5}{2}k^3 + \cdots + \binom{5}{4}k$ is a multiple of 5 and hence, written as $5q$ and clearly $3k^2 + 3k$ is a multiple of 3 and hence is, $3r$.

So,

$$\begin{aligned} M(k+1) &= 105s + 105p + 105q + 105 \times 2r + 15 + 21 + 70 - 1 \\ &= 105(s + p + q + 2r) + 105 \\ &= 105(s + p + q + 2r + 1) \text{ is divisible by } 105. \end{aligned}$$

So, $M(k)$ implies $M(k+1)$

$\because$ $M(1)$ is true, hence, the statement $15n^7 + 21n^5 + 70n^3 - n$ is divisible by 105 for all $n \in \mathbb{N}$.

So, $\dfrac{(15n^7 + 21n^5 + 70n^3 - n)}{105}$ is an integer.

Aliter: $\dfrac{1}{105}[15n^7 + 21n^5 + 70n^3 - n]$.

Let, $f(n) = 15n^7 + 21n^5 + 70n^3 - n$.

We will show that

$f(n) = 15n^7 + 21n^5 + 70n^3 - n$ is divisible by 105.

$105 = 7 \times 5 \times 3$.

We will prove that $f(n)$ is divisible by 3, 5, 7 for all n and hence, by 105.

Consider 7:

$n^7 \equiv n \pmod 7$ [by F.L.T.]

$\therefore$ $15n^7 \equiv 15n \pmod 7 \equiv n \pmod 7$

$\therefore$ $f(n) \equiv (n + 0 + 0 - n) \bmod 7 = 0 \pmod 7$

for all n.

$\therefore$ $7 / f(n)$.

Consider 5:

$5 | (15n^7 + 70n^3)$

$n^5 \equiv n \pmod 5$ [by FLT]

$\therefore$ $f(n) \equiv (0 + 21n + 0 - n) \pmod 5 \equiv 0 \pmod 5$

$\therefore$ $5 | f(n)$.

Consider 3:

$n^3 \equiv n \pmod 3$ [by FLT]

$f(n) = (0 + 0 + 70n - n) \pmod 3 = 0 \pmod 3$

$\therefore$ $3 / f(n)$

Hence, $105 | f(x)$.

Hence, the given expression is an integer.

Problem 18 *Show that* $3^{2n+5} + 160n^2 - 56n - 243$ *is divisible by* 512.

Solution: Here we use mathematical induction. Let $M(n)$ be the statement that

$M(n) = 3^{2n+5} + 160n^2 - 56n - 243$ is divisible by 512.

$M(1) = 3^7 + 160 - 56 - 243 = 2048 = 512 \times 4$ and hence, $M(1)$ is true.

Let us assume that $M(k)$ is true

$$\begin{aligned} M(k+1) &= 3^{2(k+1)+5} + 160(k+1)^2 - 56(k+1) - 243 \\ &= 3^{2k+7} + 160k^2 + 264k - 139 \\ &= 3^2(3^{2k+5} + 160k^2 - 56k - 243) - 8 \times 160k^2 + 768k + 2048 \\ &= 3^2(3^{2k+5} + 160k^2 - 56k - 243) - 256(5k^2 - 3k - 8) \\ &= 3^2(3^{2k+5} + 160k^2 - 56k - 243) - 256(5k - 8)(k+1). \end{aligned}$$

By $M(k)$, $3^{2k+5} + 160k^2 - 56k - 243$ is divisible by 512.

Also $(5k - 8)(k + 1)$ is even for all k. Since if k is even, $(5k - 8)$ is even, if k is odd, $(k + 1)$ is even and so, $-256(5k - 8)(k + 1)$ is divisible by $256 \times 2 = 512$.

So $3^2(3^{2k+5} + 160k^2 - 56k - 243) - 256(5k - 8)(k + 1)$ is divisible by 512, which implies that $M(k + 1)$ is true. Thus, $M(1)$ is true, $M(k)$ implies $M(k + 1)$.

$\therefore$ $M(n)$ is true for all $n \in \mathbb{N}$ and hence, the result.

Aliter: $3^{2n+5} = 3^5 \cdot 3^{2n} = 243(1+8)^n$

$$\begin{aligned} &= 243(1 + 8n + n(n-1)32 + \tbinom{n}{3}8^3 + \cdots) \\ &= 243[1 - 24n + 32n^2 + 512\lambda] \end{aligned}$$

$$\begin{aligned} \Rightarrow\ & 3^{2n+5} + 160n^2 - 56n - 243 \\ &= 243 \times 512\lambda + 32 \times 248n^2 - 5888n \\ &= 243 \times 512\lambda + 256\underbrace{n(31n - 23)}_{\text{even}} \end{aligned}$$

$$\Rightarrow\ 512 \,\big|\, (3^{2n+5} + 160n^2 - 56n - 243)$$

Problem 19 $a_1, a_2, a_3, \ldots$ *are natural numbers such that* $a_1 = 6$, $a_2 = 9$ *and such that* $a_n = 3a_{n-1} + 18a_{n-2}$ *for* $n > 2$. *Show that* $a_n = \frac{1}{2} \times 6^n - (-3)^n$ *for all* $n \geq 1$.

Solution: Here we use the second principle of mathematical induction. That is, we have to verify if the statement is true for $n = 1$, *i.e.*, $M(1)$ is true.

Then, we should prove that, if the statement is true for all $n \leq k$, a fixed natural number (say), then the statement is true for $(k + 1)$. Then, the statement is true for all n.

$$a_n = \frac{1}{2} \times 6^n - (-3)^n$$

$$\Rightarrow \quad a_1 = \frac{1}{2} \times 6^1 - (-3)^1 = 3 - (-3) = 6.$$

So, $M(1)$ is true.

$$a_2 = \frac{1}{2} \times 36 - (-3)^2 = 18 - 9 = 9.$$

$M(2)$ is also true.

Let the statement be true for $2, 3, \ldots, k$.

So, $a_k = \frac{1}{2} \times 6^k - (-3)^k$ is true.

Since $a_n = 3a_{n-1} + 18a_{n-2}$, we have $a_{k+1} = 3a_k + 18a_{k-1}$.

But since the formula is true for all $n \leq k$, we have

$$\begin{aligned}
a_{k+1} &= 3\left\{\frac{1}{2} \times 6^k - (-3)^k\right\} + 18\left\{\frac{1}{2} \times 6^{k-1} - (-3)^{k-1}\right\} \\
&= \frac{3}{2} \times 6^k - 3(-3)^k + \frac{3}{2} \times 6 \times 6^{k-1} - 2 \times 3^2 (-3)^{k-1} \\
&= \frac{3}{2} \times 6^k + (-3)^{k+1} + \frac{3}{2} \times 6^k - 2(-3)^{k+1} \\
&= 3 \times 6^k - (-3)^{k+1} \\
&= \frac{1}{2} \times 6 \times 6^k - (-3)^{k+1} \\
&= \frac{1}{2} \times 6^{k+1} - (-3)^{k+1}.
\end{aligned}$$

Thus, the formula is true for a_{k+1}, whenever it is true for all $n \leq k$.

It is true for $n = 1$, $n = 2$.

Thus, this formula is true for all $n \in \mathbb{N}$.

Problem 20 *There must be something wrong with the following proof: What is it?*

Theorem: *Let a be a positive number. For all positive integers n, we have* $a^{n-1} = 1$.

Proof: If $n = 1$, $a^{n-1} = a^{1-1} = a^0 = 1$.

Assume that this statement is true for $n \leq k$, *i.e.*, $a^{n-1} = 1$ for all $n \leq k$.

If $k \geq 1$ now for $n = k + 1$, we have

$$a^{(k+1)-1} = a^k = \frac{a^{k-1} \times a^{k-1}}{a^{k-2}} = \frac{1 \times 1}{1} = 1.$$

So the theorem is true for $n = k + 1$ whenever the theorem is true for $n \le k$ and hence, by the second principle of mathematical induction, the theorem is true for all natural numbers, n.

Solution:

Fallacy, for this explanation: When we have written $a^{(k+1)-1}$ as $\dfrac{a^{k-1} \times a^{k-1}}{a^{k-2}}$, we have assumed that the theorem is true for $n \le k$ and we have verified that it is true for $n =$ 1. For example, taking $k = 1$; the denominator becomes $a^{1-2} = -a^{-1}$ but we have not proved that $a^{-1} = 1$; neither it can be proved. Therefore the proof has a loophole here.

Check Your Understanding

1. Prove the following by mathematical induction:

(a) $\dfrac{1}{2}\tan\dfrac{x}{2} + \dfrac{1}{2^2}\tan\dfrac{x}{2^2} + \cdots + \dfrac{1}{2^n}\tan\dfrac{x}{2^n} = \dfrac{1}{2^n}\cot\dfrac{x}{2^n} - \cot x,\ \forall n \in \mathbb{N}.$

(b) $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{7} + \cdots + \tan^{-1}\dfrac{1}{n^2+n+1} = \tan^{-1}\dfrac{n}{n+2},\ \forall\, n \in \mathbb{N}.$

(c) $\cot^{-1}3 + \cot^{-1}5 + \cdots + \cot^{-1}(2n+1)$

$= \tan^{-1}2 + \tan^{-1}\dfrac{3}{2} + \cdots + \tan^{-1}\dfrac{n+1}{n} - \dfrac{n\pi}{4}\ \forall\, n \in \mathbb{N}.$

(d) $5 + 55 + \cdots + \underbrace{55\ldots5}_{n \text{ times}} = \dfrac{5}{81}(10^{n+1} - 9n - 10)\ \forall n \in \mathbb{N}.$

2. Show by mathematical induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,\ n \in \mathbb{N}$.
3. If $\theta_1, \theta_2, \ldots, \theta_n$ are real numbers, use the principal of mathematical induction to show the following:
$(\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n)^2 + (\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_n)^2 \le n^2$ for all $n \in \mathbb{N}$.
4. Show that $\sum_{k=0}^{n} k^2\ {}^nC_k = n(n+1)2^{n-2}$ for $n \ge 1$.
5. Prove by the method of induction, that $I_n = 10^n - (5+\sqrt{17})^n - (5-\sqrt{17})^n$ is divisible by 2^{n+1} for all $n > 1$.
6. Using mathematical induction to show that $p^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$ for all $n \in \mathbb{N}$.
7. Prove by induction that the integer next to greater than $(3+\sqrt{5})^n$ is divisible by 2^n for all $n \in \mathbb{N}$.
8. Prove the following inequalities by mathematical induction:

(a) $\dfrac{(2n)!}{(n!)^2 4^n} > \dfrac{1}{2\sqrt{n}}$ *for* $n > 1$ (b) $1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$ *for* $n > 1$

9. If $a, b > 0$, show that $(a+b)^n < 2^n(a^n + b^n)$ for all $n \in \mathbb{N}$.
10. Show for any n, $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$.
11. Show that $(m+\sqrt{p})^n + (m-\sqrt{p})^n$ is an integer for all $n \in \mathbb{N}$, where p is a prime number and m is an integer.

12. Prove that $3^{n^2} > (n!)^4 \ \forall n \in \mathbb{N}$.

13. Prove that $\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{n^2} \geq \dfrac{7}{12}-\dfrac{1}{n+1} \ \forall n \in \mathbb{N}\backslash\{1\}$.

14. Let α be some real number such that $\alpha+\dfrac{1}{\alpha} \in \mathbb{Z}$, prove that $\alpha^n+\dfrac{1}{\alpha^n} \in \mathbb{Z} \ \forall n \in \mathbb{N}$.

15. Prove that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \forall n \in \mathbb{N}$, where $F_0 = 0$, $F_1 = 1$,

$F_{n+2} = F_{n+1} + F_n$, $n \geq 0$, $\quad n \in \mathbb{N}_0$. You may use

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{21}+a_{22}b_{22} \end{bmatrix}$$

Challenge Your Understanding

1. If $-1 < a_i < 0$ for all i, prove that

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

Hence show that if x_i are arbitrary positive numbers satisfying

$$x_1 + x_2 + \cdots + x_n \leq \frac{1}{2},$$

then $(1-x_1)(1-x_2)\ldots(1-x_n) \geq \dfrac{1}{2} \forall \ n \in \mathbb{N}$.

2. Using mathematical induction, show that

$$\sum_{m=0}^{k}(n-m)\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left(\frac{n}{r+1}-\frac{k}{r+2}\right)$$

where n, m, r and k are non-negative integers.

3. If $p \geq 3$ be an integer and α, β be the roots of $x^2 - (p + 1)\,x + 1 = 0$, using mathematical induction show that $\alpha^n + \beta^n$ is

(a) an integer. (b) is not divisible by p.

4. If $u_1 = 0$, $u_2 = 1$ and $u_n = (n - 1)(u_{n-1} + u_{n-2})$ prove that

$$u_n = n!\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+\frac{(-1)^n}{n!}\right] \ \forall \ n \in \mathbb{N}.$$

5. Prove that sequence $\{a_n\}$, where $a_n = \dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\sqrt{2n+1}$ is a monotonic decreasing sequence.

6. If $a_1 = \dfrac{1}{2}\left(a_0 + \dfrac{A}{a_0}\right)$, $a_2 = \dfrac{1}{2}\left(a_1 + \dfrac{A}{a_1}\right)$ and $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{A}{a_n}\right)$ for $n \geq 2$ where

$a_i > o$, $A > o$, prove by mathematical induction that $\dfrac{a_n - \sqrt{A}}{a_n + \sqrt{A}} = \left(\dfrac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{n-1}}$.

7. Define a sequence (a_n), $n \geq 1$ by $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = 2a_{n+1} - a_{n+2}$, for $n \geq 1$. Prove that, for any m, $a_m a_{m+1}$ is also a term in the sequence. **[INMO, 1996]**

8. Prove that $\sum_{k=0}^{n}\binom{n+k}{k}\frac{1}{2^k} = 2^n \ \forall n \in \mathbb{N}$.

9. Prove that $(1+a)^n > (1+a^{n+1})(1+a^{n+2})\cdots(1+a^{2n}) \forall n \in \mathbb{N}$, where $a \in \left(0, \frac{7}{12}\right]$.

10. Let $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, prove that $\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}} \ \forall n \in \mathbb{N}$.

11. Let a and b be positive integer with $(a, b) = 1$ and a, b having different parities. Let the set S have the following properties:
(i) $a, b \in S$
(ii) If $x, y, z \in S$ then, $x + y + z \in S$.
Prove that all integers greater than $2ab$ are in S. **[China MO, 2008]**

12. There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

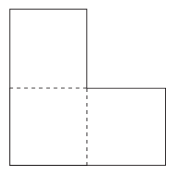
[USA MO, 2010]

13. Prove that $(\sqrt{2}-1)^n = \sqrt{m} - \sqrt{m-1} \ \forall n \in \mathbb{N}$ for a certain suitable positive integer m.

[Polish MO, 1953]

14. The area of union of several circles equals 1. Prove that it is possible to choose several of them that do not intersect each other and whose total area is greater than $\frac{1}{9}$. **[Moscow MO, 1979]**

15. Consider a square of size $2^n \times 2^n$. It is sub-divided in unit squares of sizes 1×1. Prove that we can tile it with L-shaped triominos (as shown in the figure) provided one unit square is removed.


L-shaped Friominos

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Chapter

4

Fibonacci discovered his famous sequence while looking at how generations of rabbit breed

At the start, there is just one pair. **Month 0** **1 Pair**

After the first month, the initial pair mates, but have no young. **Month 1** **1 Pair**

After the second month, the initial pair give birth to a pair of bables. **Month 2** **2 Pair**

After the third month, the initial pair give birth to second pair, Month 3 and their first-boms mate but have not yet given birth to any young. **Month 3** **3 Pair**

After the fourth month, the initial pair give birth to another pair and their first-born pair also produces a pair of their own. **Month 4** **5 Pair**

After the fifth month, the initial pair give birth to another pair, their first born pair produces another pair, and the second-born pair produce a pair of their own **Month 5** **8 Pair**

The process continues...
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, etc
...the Fibonacci Sequence

Recurrence Relation

4.1 Introduction

A recurrence relation is an equation that recursively defines a sequence whose next term is a function of the previous terms.

In general, $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-m})$; $n \geq m+1$, is called recurrence relation for sequence $\{a_n\}$, $n \geq 1$.

For example, consider the sequence,

1, 1, 2, 3, 5, 8, …

This sequence is known as Fibonacci sequence. Its each term governed by the relation $a_{n+2} = a_{n+1} + a_n \ \forall \ n \in \mathbb{N}$; $a_1 = 1$, $a_2 = 1$. Later in this chapter we will prove that

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

We can observe the immediate advantage of recurrence over explicit formula for a_n, the former is easy to apply/remember. There are only few types of recurrence relation which can be solved in closed form, *i.e.*, any term in the sequence can be evaluated by plugging numbers into an equation ($a_n = f(n)$) instead of having to calculate entire sequence.

Leonardo Fibonacci

C. 1175–C. 1240–50
Nationality: Italian

4.2 Classification

Let us classify the recurrence relation: These relations are classified by the ways in which terms are combined, the nature of coefficients involved, and the number and the nature of previous terms used.

Let us observe the following table:

Order	Linear Or Non-linear	Homogeneous Or Non-homogeneous	Coefficient (Variable or Constant)	Example
First order	Linear	Homogeneous	Constant	$a_n + 3a_{n-1} = 0$
First order	Linear	Homogeneous	Variable	$a_n + na_{n-1} = 0$
First order	Linear	Non-homogeneous	Constant	$a_n - 2a_{n-1} = 1$
First order	Linear	Non-homogeneous	Variable	$a_n - na_{n-1} = (-1)^n$
First order	Non-linear	Homogeneous	Constant	$a_n a_{n-1} + a_n = 0$
First order	Non-linear	Non-homogeneous	Constant	$a_n a_{n-1} + a_n = 1$
Second order	Linear	Homogeneous	Constant	$a_n - a_{n-1} - a_{n-2} = 0$
Second order	Linear	Non-homogeneous	Constant	$a_n - a_{n-1} - a_{n-2} = 5$

In general consider the following:

$$f_0(n)a_n + f_1(n)a_{n-1} + \cdots + f_r(n)a_{n-r} = g(n)$$

Where, $f_i(n)$ and $g(n)$ are some arbitrary known functions of 'n'.

If $f_r \neq 0$ and $f_0 \neq 0$, then it is called rth order recurrence relation.

If $g = 0$, it is called linear homogeneous recurrence relation.

If $g = 0$, f_i = constant, it is called linear homogeneous recurrence relation with constant coefficient, which are specially nice to handle.

Example 1 *Classify the following recurrence relations:*

(a) $a_n + 3a_{n-1} - 2a_{n-2} = 0$

(b) $a_n + 4a_{n-2} = n!$

(c) $a_n + \sqrt{n}a_{n-1} = n^n$

(d) $a_n + a_{\left\lfloor \frac{n}{2} \right\rfloor} = 2n$

(e) $a_n + 3a_{n-1} - 2\sqrt{n}a_{n-2} + 2\sqrt{n}a_{n-2}a_{n-3} = f(n);\ \ f(n) \neq 0$

(f) $a_n + 3a_{n-1} - 2\sqrt{n}a_{n-2} + 2\sqrt{n}a_{n-2}a_{n-3} = 0$

(g) $a_n^2 + 2a_n a_{n-1} + a_{n-1}^2 = 0$

(h) $a_n - a_{n-1}a_{n-2} - \sqrt{a_{n-2}} = 0$

(i) $a_n + a_{n-1}a_{n-2} = 1$

(j) $a_n - a_{\left\lfloor \frac{n}{2} \right\rfloor} - a_{\left\lceil \frac{n}{2} \right\rceil} = n$

Solution:

(a) Linear, homogeneous, with constant coefficient and of order '2'.
(b) Linear, non-homogeneous, with constant coefficient of order '2'.
(c) Linear, non-homogeneous, with variable coefficient of order '1'.
(d) Linear, non-homogeneous, with constant coefficient, order not defined.
(e) Non-linear, non-homogeneous, with variable coefficient of order '3'.
(f) Non-linear, homogeneous, with variable coefficient of order '3'.
(g) Non-linear, homogeneous, with constant coefficient of order '1'.
(h) Non-linear, homogeneous, with constant coefficient of order '2'.

(i) Non-linear, non-homogeneous, with constant coefficient of order '2'.
(j) Linear, non-homogeneous, with constant coefficient and order not defined.

4.3 First Order Linear Recurrence Relation

Let us consider first order linear and non-homogeneous,

$$a_n = f(n)a_{n-1} + g(n), \quad n \geq 2,\ a_1 = \alpha$$

Where $f(n)$ and $g(n)$ are known functions of 'n' and $f(n) \neq 0$.

Divide whole equation by $p_n = f(1)\cdot f(2)\cdots f(n)$ and rewrite as

$$\frac{a_n}{p_n} - \frac{a_{n-1}}{p_{n-1}} = \frac{g(n)}{p_n}$$

Consider, $\dfrac{a_n}{p_n} = v_n$

then $v_n - v_{n-1} = \dfrac{g(n)}{p_n}$

Now plug, $n = 2, 3, \ldots, n$ and add all, we get,

$$\Rightarrow v_n - v_1 = \sum_{r=2}^{n} \frac{g(r)}{p_r}$$

$$\Rightarrow \frac{a_n}{p_n} - \frac{a_1}{p_1} = \sum_{r=2}^{n} \frac{g(r)}{p_r}$$

$$\Rightarrow a_n = p_n\left(\frac{\alpha}{f(1)} + \sum_{r=2}^{n} \frac{g(r)}{p_r}\right)$$

Example 2 *Let* $a_n = \dfrac{2}{3}a_{n-1} + n^2 - 15, n \geq 2, a_1 = 1$. *Find* a_n.

Solution:

$$a_n = \frac{2}{3}a_{n-1} + n^2 - 15$$

Compare it with $a_n = f(n)a_{n-1} + g(n)$

$$\Rightarrow f(n) = \frac{2}{3}$$

$$\Rightarrow f(1)f(2)\cdots f(n) = \left(\frac{2}{3}\right)^n$$

By dividing whole equation by $\left(\dfrac{2}{3}\right)^n$ we get,

$$\frac{a_n}{\left(\frac{2}{3}\right)^n} - \frac{a_{n-1}}{\left(\frac{2}{3}\right)^{n-1}} = \frac{n^2 - 15}{\left(\frac{2}{3}\right)^n}$$

Let, $\dfrac{a_n}{\left(\frac{2}{3}\right)^n} = b_n$

$$\Rightarrow b_n - b_{n-1} = (n^2-15)\left(\frac{3}{2}\right)^n$$

Plugging, $n = 2, 3, 4, \ldots, n$ and adding all we get

$$b_n - b_1 = \sum_{r=2}^{n}(r^2-15)\left(\frac{3}{2}\right)^r$$

$$\Rightarrow \frac{a_n}{\left(\frac{2}{3}\right)^n} - \frac{3}{2} = S \quad \text{(say)}$$

where,

$$S = (2^2-15)\left(\frac{3}{2}\right)^2 + (3^2-15)\left(\frac{3}{2}\right)^3 + \cdots + (n^2-15)\left(\frac{3}{2}\right)^n$$

$$\frac{3}{2}S = \qquad +(2^2-15)\left(\frac{3}{2}\right)^3 + \cdots + \left((n-1)^2-15\right)\left(\frac{3}{2}\right)^n + (n^2-15)\left(\frac{3}{2}\right)^{n+1}$$

$$- \qquad - \quad - \qquad -$$

$$-\frac{1}{2}S = -11\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right)^3 + 7\left(\frac{3}{2}\right)^4 + 9\left(\frac{3}{2}\right)^5 + \cdots + (2n-1)\left(\frac{3}{2}\right)^n - (n^2-15)\left(\frac{3}{2}\right)^{n+1}$$

$$-\frac{3}{4}S = \qquad -11\left(\frac{3}{2}\right)^3 + 5\left(\frac{3}{2}\right)^4 + 7\left(\frac{3}{2}\right)^5 + \cdots + (2n-3)\left(\frac{3}{2}\right)^n + (2n-1)\left(\frac{3}{2}\right)^{n+1} - (n^2-15)\left(\frac{3}{2}\right)^{n+2}$$

$$+ \qquad + \quad - \quad - \qquad - \qquad - \qquad +$$

$$\frac{1}{4}S = -11\left(\frac{3}{2}\right)^2 + 16\left(\frac{3}{2}\right)^3 + 2\left(\frac{3}{2}\right)^4 + 2\left(\frac{3}{2}\right)^5 + \cdots + 2\left(\frac{3}{2}\right)^n - (n^2+2n-16)\left(\frac{3}{2}\right)^{n+1} + (n^2-15)\left(\frac{3}{2}\right)^{n+2}$$

$$\Rightarrow \frac{1}{4}S = \frac{-99}{4} + 54 + 2\left(\frac{3}{2}\right)^4\left(\frac{\left(\frac{3}{2}\right)^{n-3} - 1}{\frac{3}{2} - 1}\right) - (n^2+2n-16)\left(\frac{3}{2}\right)^{n+1} + (n^2-15)\left(\frac{3}{2}\right)^{n+2}$$

$$= \frac{117}{4} + 4\left(\frac{3}{2}\right)^{n+1} - \frac{81}{4} + \left(\frac{3}{2}\right)^{n+1}\left[\frac{1}{2}n^2 - 2n - \frac{13}{2}\right]$$

$$\Rightarrow \frac{1}{4}S = 9 + \left(\frac{3}{2}\right)^{n+1} \cdot \frac{1}{2}(n^2-4n-5)$$

$$\Rightarrow S = 36 + \left(\frac{3}{2}\right)^n (3n^2-12n-15)$$

Now,

$$\frac{a_n}{\left(\frac{2}{3}\right)^n} - \frac{3}{2} = 36 + \left(\frac{3}{2}\right)^n (3n^2 - 12n - 15)$$

$$\Rightarrow a_n = \frac{75}{2} \times \left(\frac{2}{3}\right)^n + 3n^2 - 12n - 15$$

$$\Rightarrow a_n = 25 \cdot \left(\frac{2}{3}\right)^{n-1} + 3n^2 - 12n - 15$$

4.3.1 First Order Linear Homogeneous

$$a_n = f(n)a_{n-1},\ n \geq 2,\ a_1 = \alpha$$

$$\Rightarrow \frac{a_n}{a_{n-1}} = f(n)$$

$$\Rightarrow \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} = f(n) \cdot f(n-1) \cdots f(2)$$

$$\Rightarrow a_n = [f(n)f(n-1) \cdots f(2)]\alpha$$

Example 3 *Let* $a_n = na_{n-1}$, $a_1 = 1$. *Find* a_n.

Solution: Let us rewrite the recurrence as $\dfrac{a_n}{a_{n-1}} = n$

$$\Rightarrow \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_{n-2}}{a_{n-3}} \cdots \frac{a_2}{a_1} = n(n-1)(n-2) \cdots 2$$

$$\Rightarrow \frac{a_n}{a_1} = n!$$

$$\Rightarrow a_n = n!$$

4.3.2 First Order Linear, Non-homogeneous with Constant Coefficients

$a_n = c_1 a_{n-1} + c_2$, $n \geq 2$, $a_1 = \alpha$, (where c_1, c_2 constant and $c_1 \neq 1$)

Let,

$$a_n = b_n + \lambda \quad (\lambda \text{ some constant})$$

$$\Rightarrow b_n + \lambda = c_1 b_{n-1} + c_1\lambda + c_2$$

$$\Rightarrow b_n = c_1 b_{n-1} + (c_1 - 1)\lambda + c_2$$

By taking $\lambda = \dfrac{c_2}{1 - c_1}$, we get $b_n = c_1 b_{n-1}$

Which is a geometric progression with common ratio 'c_1'.

$$b_n = c_1^{n-1} \cdot b_1$$

$$\Rightarrow \quad a_n = c_1^{n-1}(\alpha - \lambda) + \lambda$$

Note: In case of $c_1 = 1$,

$$a_n = c_1 a_{n-1} + c_2 \quad \Rightarrow \quad a_n - a_{n-1} = c_2$$

Which is an arithmetic progression, with common difference c_2.

$$\Rightarrow a_n = a_1 + (n-1)c_2.$$

Example 4 *Let $\{a_n\}$ be a sequence such that $a_1 = 4$, and sum of first n terms is S_n and $S_{n+1} - 3S_n - 2n - 4 = 0 \quad \forall n \in \mathbb{N}$, find a_n.*

Solution:

We know that $a_{n+1} = S_{n+1} - S_n \quad \forall \in n \geq 0 \quad (\text{as } S_0 = 0)$

Now, $S_{n+1} - S_n = [3S_n + 2n + 4] - [3S_{n-1} + 2(n-1) + 4]$

$$\Rightarrow a_{n+1} = 3a_n + 2 \quad \forall n \geq 1, a_1 = 4$$

Let,

$$a_n = b_n + \lambda$$
$$\Rightarrow b_{n+1} = 3b_n + 2\lambda + 2$$

Make $2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = -1$

$$\Rightarrow b_{n+1} = 3b_n, b_1 = 5$$
$$\Rightarrow b_n = 3^{n-1} \cdot b_1 = 5 \cdot 3^{n-1} \qquad (\text{As } b_1 = a_1 + 1 = 4 + 1 = 5)$$
$$\Rightarrow a_n = 5 \cdot 3^{n-1} - 1 \quad \forall n \in \mathbb{N}.$$

Build-up Your Understanding 1

1. Find the n^{th} term of the sequence $\{a_n\}$ such that $a_1 = 2$, $a_{n+1} = 2a_n + 1$ $(n = 1, 2, 3, \ldots)$.
2. Find the n^{th} term of the sequence $\{a_n\}$ such that

   $$a_1 = 1, \ a_{n+1}^2 = -\frac{1}{4}a_n^2 + 4 \, (a_n > 0, \ n \geq 1)$$
3. Find the n^{th} term of the sequence $\{a_n\}$ such that

   $$\frac{a_1 + a_2 + \cdots + a_n}{n} = n + \frac{1}{n} \quad (n = 1, 2, 3, \ldots).$$
4. The positive sequence $\{a_n\}$ satisfies the following conditions (a), (b)

   (a) $a_1 = 1$

   (b) $\log a_n - \log a_{n-1} = \log (n-1) - \log (n+1)$, $n \geq 2$.

   Find $\sum_{k=1}^{n} a_k$
5. Find the n^{th} term of the sequence $\{a_n\}$ such that

   $$a_1 = 1, \ a_{n+1} = \frac{1}{2}a_n + \frac{n^2 - 2n - 1}{n^2(n+1)^2} \quad (n = 1, 2, 3, \ldots).$$
6. Let $a_1 = 1$, $a_n = (n-1)a_{n-1} + 1$. Find n such that $n | a_n$.
7. Let $a_0 = 1$, $a_n = n\, a_{n-1} + (n+1)!\, 2^{-n}$. Find a_n
8. Let $a_1 = 1$, $(n+1)a_{n+1} + na_n = 2n - 3 \ \forall \ n \geq 1$. Find a_n

9. Find the n^{th} term of the sequence $\{a_n\}$ such that
$a_1 = 1,\ a_{n+1} = na_n + n - 1\ (n = 1, 2, 3, \ldots)$.

10. Find the n^{th} term of the sequence $\{a_n\}$ such that
$a_1 = 1/2,\ (n-1)\, a_{n+1} = (n+1)a_n + 1 (n \geq 2)$.

11. Find the n^{th} term of the sequence $\{x_n\}$ such that

$$x_1 = 2,\ x_{n+1} = (n+1)^2 \left(\frac{2x_n}{n^2} - 1\right)\ (n = 1, 2, 3, \ldots)$$

12. Find the n^{th} term of the sequence $\{a_n\}$ which is defined by

$$a_1 = 0,\ a_n = \left(1 - \frac{1}{n}\right)^3 a_{n+1} + \frac{n-1}{n^2}\ (n = 1, 2, 3, \ldots).$$

4.4 First Order Non-linear

4.4.1 First Order Non-linear of the Form

$$a_n = \frac{\alpha a_{n-1}}{\beta a_{n-1} + \gamma}; \quad n \geq 2, a_1 > 1; \quad \alpha \cdot \beta \cdot \gamma \neq 0.$$

By taking reciprocal of both sides, we get

$$\frac{1}{a_n} = \frac{\beta}{\alpha} + \frac{\gamma}{\alpha a_{n-1}}$$

Let,
$$\frac{1}{a_n} = b_n$$

$$\Rightarrow b_n = c_1 b_{n-1} + c_2 \quad \left(\text{where, } c_1 = \frac{\gamma}{\alpha}, c_2 = \frac{\beta}{\alpha}\right)$$

Example 5 *Let $\{a_n\}$ be a sequence such that $a_1 = 1,\ a_2 = \frac{1}{4},\ a_{n+1} = \frac{(n-1)a_n}{n - a_n}$, for $n = 2, 3, \ldots$.*
Find a_n.

Solution:

$$\frac{1}{a_{n+1}} = \left(\frac{n}{n-1}\right)\frac{1}{a_n} - \frac{1}{n-1}$$

$$\Rightarrow \frac{1}{na_{n+1}} - \frac{1}{(n-1)a_n} = -\frac{1}{n(n-1)} = -\frac{1}{n(n-1)} = \frac{1}{n} - \frac{1}{n-1}$$

Plugging $n = 2, 3, \ldots, (n-1)$, in above equation and adding all we get,

$$\frac{1}{(n-1)a_n} - \frac{1}{a_2} = \frac{1}{n-1} - 1 = \frac{2-n}{n-1}$$

$$\Rightarrow \frac{1}{(n-1)a_n} = \frac{2-n}{n-1} + 4 = \frac{3n-2}{n-1}, n \geq 2$$

$$\Rightarrow a_n = \frac{1}{3n-2}; n \geq 2$$

We can see that $a_1 = 1 = \frac{1}{3-2}$

Hence, $a_n = \dfrac{1}{3n-2} \quad \forall\, n \geq 1$

4.4.2 First Order Non-linear of the Form

$$a_n = \frac{\alpha a_{n-1} + \beta}{\gamma a_{n-1} + \delta} \quad \left(\text{where } \alpha\beta\gamma\delta \neq 0,\, n \geq 2,\, \frac{\alpha}{\gamma} \neq \frac{\beta}{\delta};\, a_1 \neq \frac{\alpha a_1 + b}{ca_1 + d}\right)$$

We will transform this to previous form, let $a_n = b_n + x$

$$\Rightarrow b_n + x = \frac{\alpha b_{n-1} + \alpha x + \beta}{\gamma b_{n-1} + \gamma x + \delta}$$

$$\Rightarrow b_n = \frac{\alpha b_{n-1} + \alpha x + \beta}{\gamma b_{n-1} + \gamma x + \delta} - x$$

$$= \frac{(\alpha - x\gamma)b_{n-1} + (\alpha x + \beta) - x(\gamma x + \delta)}{\gamma b_{n-1} + \gamma x + \delta}$$

Now choose x such that,

$$\alpha x + \beta = x(\gamma x + \delta) \tag{1}$$

Solving, $\gamma x^2 + (\delta - \alpha)x - \beta = 0$, we get $x = x_1, x_2$.

Take any root, say 'x_1',

$$b_n = \frac{(\alpha - x_1\gamma)b_{n-1}}{\gamma b_{n-1} + \gamma x_1 + \delta}$$

$$\Rightarrow \frac{1}{b_n} = \frac{\gamma x_1 + \delta}{(\alpha - x_1\gamma)b_{n-1}} + \frac{\gamma}{(\alpha - x_1\gamma)}$$

Let,

$$\frac{1}{b_n} = f_n$$

$$\Rightarrow f_n = c_1 f_{n-1} + c_2$$

Where $c_1 = \dfrac{\gamma x_1 + \delta}{(\alpha - x_1\gamma)}$, $c_2 = \dfrac{\gamma}{(\alpha - x_1\gamma)}$.

Note: Observe that we can get equation (1) directly from recurrence by replacing a_i by x, $x = \dfrac{\alpha x + \beta}{\gamma x + \delta}$. The value of x satisfying the equation is called fixed point of the sequence.

Example 6 *Let $\{a_n\}$ be a sequence such that $a_1 = 1$, $a_{n+1}a_n = 4(a_{n+1} - 1)$ $\forall n \in \mathbb{N}$, find a_n.*

Solution:

$$a_{n+1} = \frac{4}{4 - a_n}$$

Let, $$a_{n+1} = b_{n+1} + \lambda$$

$$\Rightarrow b_{n+1} = \frac{4 - 4\lambda + \lambda^2 + \lambda b_n}{4 - \lambda - b_n}$$

Take $$\lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \quad \lambda = 2$$

$$\Rightarrow b_{n+1} = \frac{2b_n}{2-b_n}$$

$$\Rightarrow \frac{1}{b_{n+1}} = \frac{1}{b_n} - \frac{1}{2}$$

$\Rightarrow \frac{1}{b_n}$ in an arithmetic progression with common difference $= -\frac{1}{2}$, first term $= \frac{1}{b_1} = \frac{1}{a_n - 2} = -1$.

$$\Rightarrow \frac{1}{a_n - 2} = -1 + (n-1)\left(-\frac{1}{2}\right) = -\frac{n+1}{2}$$

$$\Rightarrow a_n - 2 = -\frac{2}{n+1} \Rightarrow a_n = \frac{2n}{n+1}.$$

Example 7 *Let $\{a_n\}$ be a sequence such that $a_1 = 2$, $a_{n+1} = \frac{3a_n + 4}{2a_n + 3}$, $n \ge 1$. Find a_n.*

Solution: Let $a_n = b_n + \lambda, \forall n \ge 1$

$$\Rightarrow b_{n+1} = \frac{3b_n + 3\lambda + 4}{2b_n + 2\lambda + 3} - \lambda$$

$$= \frac{(3-2\lambda)b_n - 2\lambda^2 + 4}{2b_n + 2\lambda + 3}$$

Get $-2\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm\sqrt{2}$

Take $\lambda = \sqrt{2}$

$$\Rightarrow b_{n+1} = \frac{(3-2\sqrt{2})b_n}{2b_n + (3+2\sqrt{2})}$$

$$\Rightarrow \frac{1}{b_{n+1}} = \alpha + \frac{\beta}{b_n} \qquad \left(\text{where } \alpha = \frac{2}{3-2\sqrt{2}},\ \beta = \frac{3+2\sqrt{2}}{3-2\sqrt{2}} = \frac{1}{(3-2\sqrt{2})^2}\right)$$

Let $\frac{1}{b_n} = c_n + \mu$,

So, $c_{n+1} + \mu = \alpha + \beta(c_n + \mu)$

Taking $\alpha + \mu\beta - \mu = 0 \quad \Rightarrow \quad \mu = \frac{\alpha}{1-\beta} = -\frac{1}{2\sqrt{2}}$

Hence, $c_{n+1} = \beta c_n$

$$\Rightarrow \quad c_n = \beta^{n-1} \cdot c_1$$

$$\Rightarrow \quad \frac{1}{b_n} + \frac{1}{2\sqrt{2}} = \beta^{n-1} \cdot c_1$$

$$= \beta^{n-1} \cdot \left(\frac{1}{a_1 - \sqrt{2}} + \frac{1}{2\sqrt{2}} \right)$$

$$= \frac{3 + 2\sqrt{2}}{2\sqrt{2}} \beta^{n-1}$$

$$= \frac{3 - 2\sqrt{2}}{2\sqrt{2}} \beta^{n}$$

$$\frac{1}{b_n} = \frac{(3 - \sqrt{2})\beta^n - 1}{2\sqrt{2}}$$

$$b_n = \frac{2\sqrt{2}}{(3 - \sqrt{2})\beta^n - 1}$$

$$\Rightarrow a_n = \sqrt{2}\left[\frac{(3 - 2\sqrt{2})\beta^n + 1}{(3 - 2\sqrt{2})\beta^n - 1} \right]$$

$$\Rightarrow a_n = \sqrt{2}\left[\frac{\dfrac{1}{(3 - 2\sqrt{2})^{2n-1}} + 1}{\dfrac{1}{(3 - 2\sqrt{2})^{2n-1}} - 1} \right] = \sqrt{2}\left[\frac{1 + (3 - 2\sqrt{2})^{2n-1}}{1 - (3 - 2\sqrt{2})^{2n-1}} \right]$$

$$= \sqrt{2}\left[\frac{1 + (\sqrt{2} - 1)^{4n-2}}{1 - (\sqrt{2} - 1)^{4n-2}} \right]$$

Aliter: After getting $\lambda = \pm\sqrt{2}$

Consider,

$$\frac{a_{n+1} - \sqrt{2}}{a_{n+1} + \sqrt{2}} = \frac{3a_n + 4 - \sqrt{2}(2a_n + 3)}{3a_n + 4 + \sqrt{2}(2a_n + 3)}$$

$$= \frac{(\sqrt{2} - 1)^2}{(\sqrt{2} + 1)^2} \left(\frac{a_n - \sqrt{2}}{a_n + \sqrt{2}} \right)$$

$$\Rightarrow \quad b_{n+1} = (\sqrt{2} - 1)^4 b_n \quad \left(\text{where } b_n = \frac{a_n - \sqrt{2}}{a_n + \sqrt{2}} \right)$$

$$\Rightarrow \quad b_1 = \frac{a_1 - \sqrt{2}}{a_1 + \sqrt{2}} = \frac{2 - \sqrt{2}}{2 + \sqrt{2}} = (\sqrt{2} - 1)^2$$

$$\Rightarrow \quad b_n = b_1 \cdot (\sqrt{2} - 1)^{4(n-1)} = (\sqrt{2} - 1)^{4n-2}$$

$$\Rightarrow \quad \frac{a_n - \sqrt{2}}{a_n + \sqrt{2}} = \frac{(\sqrt{2} - 1)^{4n-2}}{1}$$

$$\Rightarrow \quad a_n = \sqrt{2}\left(\frac{1 + (\sqrt{2} - 1)^{4n-2}}{1 - (\sqrt{2} - 1)^{4n-2}} \right), n = 1, 2, \ldots.$$

Example 8 *Let* $\{a_n\}$ *be a sequence such that* $a_1 = 1,\ a_{n+1} = \frac{1}{16}(1 + 4a_n + \sqrt{1+24a_n}),\ n \geq 1.$

Solution: Let us get rid of radical sign by assuming, $1 + 24a_n = b_n^2$ (with $b_n > 0$)

or $a_n = \frac{b_n^2 - 1}{24}$, also $b_1 = 5$.

$$\Rightarrow \frac{b_{n+1}^2 - 1}{24} = \frac{1}{16}\left(1 + 4 \cdot \frac{1}{24}\left(b_n^2 - 1\right) + b_n\right)$$

$$\Rightarrow 4b_{n+1}^2 - 4 = b_n^2 + 6b_n + 5$$

or $$(2b_{n+1})^2 = (b_n + 3)^2$$

$$\Rightarrow \quad 2b_{n+1} = b_n + 3, \quad n \geq 1 \quad (\text{as } b_n > 0)$$

Let, $$b_n = c_n + \lambda$$

$$\Rightarrow 2c_{n+1} = c_n + 3 - \lambda$$

set, $$\lambda = 3$$

$$\Rightarrow c_{n+1} = \frac{1}{2}c_n,\ n \geq 1$$

$$\Rightarrow c_n = \left(\frac{1}{2}\right)^{n-1} c_1$$

$$\Rightarrow b_n - 3 = \left(\frac{1}{2}\right)^{n-1} (b_1 - 3)$$

$$\Rightarrow b_n = \left(\frac{1}{2}\right)^{n-1} \cdot 2 + 3 \quad (\text{as } b_1 = 5)$$

$$\Rightarrow b_n = 3 + \frac{1}{2^{n-2}}$$

$$\Rightarrow b_n^2 = 9 + \frac{1}{2^{2n-4}} + \frac{6}{2^{n-2}}$$

$$\Rightarrow a_n = \frac{1}{24}\left(8 + \frac{1}{2^{2n-4}} + \frac{6}{2^{n-2}}\right)$$

or $$a_n = \frac{1 + 3 \cdot 2^{n-1} + 2^{2n-1}}{3 \cdot 2^{2n-1}}$$

Build-up Your Understanding 2

1. $a_n = \frac{3a_{n-1}}{2a_{n-1}+1}, n \geq 1, a_0 = \frac{1}{4}$ find a_n.
2. Find the n^{th} term of the sequence $\{a_n\}$ such that $a_1 = 1,\ a_{n+1} = \frac{a_n}{2a_n + 3}(n \geq 1)$.
3. Solve: $a_n = \frac{3a_{n-1}}{2a_{n-1}+1},\ a_0 = \frac{1}{4}$.
4. Solve: $a_n = \frac{3a_{n-1}+1}{a_{n-1}+3},\ a_0 = 5$.

5. Let $a_1 = 0,\ a_{n+1} = \dfrac{6a_n + 2}{4 - 13a_n}$. Find a_n.
6. $a_0 = 3,\ a^2_{n+1} = a_n,\ n \geq 1$.
7. Find the n^{th} term of the sequence $\{a_n\}$ such that $a_1 = 1,\ a_{n+1} = 2a_n^{\ 2}$ $(n = 1, 2, 3, \ldots)$.
8. Find the n^{th} term of the sequence $\{x_n\}$ such that $x_{n+1} = x_n(2 - x_n)$ $(n = 1, 2, 3, \ldots)$ in terms of x_1.
9. Find the n^{th} term of the sequence $\{a_n\}$ such that $\sum_{k=1}^{n} a_k = n^3 + 3n^2 + 2n$ and Calculate $\sum_{k=1}^{n} \dfrac{1}{a_k}$

4.5 Linear Homogeneous Recurrence Relation with Constant Coefficient of Order '2'

Consider the recurrence relation

$$a_n = pa_{n-1} + qa_{n-2}$$

where p and q are constant.

As we have seen, in first order homogeneous recurrence relation, solution are of the form x^n (usually). Let us plug this solution in second order with $x \neq 0$

$$\Rightarrow \quad x^n = px^{n-1} + qx^{n-2}$$
$$\Rightarrow \quad x^2 = px + q$$
$$\text{or} \quad x^2 - px - q = 0$$

This equation is called the characteristic equation of the recurrence and the quadratic appearing on the left hand side is called the characteristic polynomial.

After solving this quadratic we get two roots, $x = \alpha, \beta$

There are two cases:

Case 1: $\alpha \neq \beta$, in this case,

$$a_n = \lambda\alpha^n + \mu\beta^n$$

for value of λ, μ use initial conditions.

Case 2: $\alpha = \beta$, in this case,

$$a_n = (\lambda + \mu n)\alpha^n$$

Example 9 *Let $\{a_n\}$ be a sequence such that, $a_n = a_{n-1} + 2a_{n-2}; n \geq 2, a_0 = 1, a_1 = 3$, find a_n.*

Solution: Replace a_n by x^n, $x \neq 0$

$$\Rightarrow x^n = x^{n-1} + 2x^{n-2}$$
$$\Rightarrow x^2 - x - 2 = 0 \quad \Rightarrow \quad x = -1, 2$$
$$\Rightarrow a_n = \lambda(-1)^n + \mu 2^n$$

Now, $\quad a_0 = \lambda + \mu = 1$

also $\quad a_1 = -\lambda + 2\mu = 3$

$$\Rightarrow \mu = \frac{4}{3}, \lambda = -\frac{1}{3}$$

$$\Rightarrow a_n = \frac{1}{3}(2^{n+2} + (-1)^{n+1}).$$

Example 10 *Let* $\{a_n\}$ *be a sequence such that,* $a_n = a_{n-1} + a_{n-2} \quad \forall n \geq 3, a_1 = 1, a_2 = 1$. *Find* a_n.

Solution: Replace a_n by x^n, $x \neq 0$

$$\Rightarrow x^n = x^{n-1} + x^{n-2}$$

$$\Rightarrow x^2 - x - 1 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow a_n = \lambda\left(\frac{1+\sqrt{5}}{2}\right)^n + \mu\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\Rightarrow a_1 = \lambda\left(\frac{1+\sqrt{5}}{2}\right) + \mu\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Also, $$a_2 = \lambda\left(\frac{3+\sqrt{5}}{2}\right) + \mu\left(\frac{3-\sqrt{5}}{2}\right) = 1$$

$$\Rightarrow \lambda = \frac{1}{\sqrt{5}}, \mu = -\frac{1}{\sqrt{5}}$$

$$\Rightarrow a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], n = 1, 2, \ldots.$$

Example 11 *Let* $\{a_n\}$ *be a sequence such that,* $a_0 = 1$, $a_1 = 0$, $a_n = 2a_{n-1} - 2a_{n-2}$, *find* a_n.

Solution: Characteristic equation of the recurrence is,

$$x^2 - 2x + 2 = 0$$

$$\Rightarrow x = 1 \pm i = \sqrt{2}\left(\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}\right)$$

$$\Rightarrow a_n = \lambda\left[(\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n\right] + \mu\left[(\sqrt{2})^n\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^n\right]$$

$$= (\sqrt{2})^n\left[\lambda\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + \mu\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)\right] \text{(De Moivere's Theorem)}$$

$$= (\sqrt{2})^n\left[(\lambda + \mu)\cos\frac{n\pi}{4} + i(\lambda - \mu)\sin\frac{n\pi}{4}\right]$$

Now, $$a_0 = \lambda + \mu = 1$$

$$a_1 = \sqrt{2}\left(\frac{\lambda+\mu}{\sqrt{2}} + i\frac{\lambda-\mu}{\sqrt{2}}\right) = 0$$
$$\Rightarrow 1 + i(\lambda-\mu) = 0$$
$$\Rightarrow \lambda - \mu = i$$
$$\Rightarrow a_n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} - \sin\frac{n\pi}{4}\right).$$

Example 12 *Let* $\{a_n\}$ *be a sequence such that,* $a_0 = 2, a_1 = 25, a_n = 10a_{n-1} - 25a_{n-2}$ $\forall n \geq 2, n \in \mathbb{N}$, *find* a_n.

Solution: Characteristic equation of the recurrence is,

$$x^2 - 10x + 25 = 0$$
$$\Rightarrow x = 5, 5$$

As characteristic roots are equal,

$$a_n = (\lambda + \mu n)5^n$$

Now, $$a_0 = \lambda = 2$$

$$a_1 = (\lambda+\mu)5 = 25 \quad \Rightarrow \quad \mu = 3$$

Hence, $$a_n = (2+3n)5^n.$$

4.6 General Form of Linear Homogeneous Recurrence Relation with Constant Coefficients

Consider the relation $c_0a_n + c_1a_{n-1} + \cdots + c_ka_{n-k} = 0$

where c_i's are constants c_0, $c_k \neq 0$, this is called kth order recurrence relation.

By replacing the terms a_r by x^r, $r = n, n-1, \ldots, n-r$.

$$\Rightarrow c_0x^n + c_1x^{n-1} + \cdots + c_kx^{n-k} = 0$$

$$\Rightarrow c_0x^k + c_1x^{k-1} + \cdots + c_{k-1}x + c_k = 0 \tag{1}$$

This equation is called characteristic equation of the recurrence.

Case 1: $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are all distinct and roots of the equation then,

$$a_n = \lambda_1(\alpha_1)^n + \lambda_2(\alpha_2)^n + \cdots + \lambda_k(\alpha_k)^n$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are constants will be calculated using initial conditions.

Case 2: $\alpha_1, \alpha_2, \ldots, \alpha_p$ $(1 \leq p \leq k)$ are the distinct characteristic roots of (1) such that α_i is of multiplicity m_i, $i = 1, 2, \ldots, p$ then

$$\begin{aligned} a_n = &\left(\lambda_{11} + \lambda_{12}n + \lambda_{13}n^2 + \cdots + \lambda_{1m_1}n^{m_1-1}\right)\alpha_1^n \\ &+\left(\lambda_{21} + \lambda_{22}n + \lambda_{23}n^2 + \cdots + \lambda_{2m_2}n^{m_2-1}\right)\alpha_2^n \\ &+\cdots \\ &+\left(\lambda_{p1} + \lambda_{p2}n + \lambda_{p3}n^2 + \cdots + \lambda_{pm_p}n^{m_p-1}\right)\alpha_p^n \end{aligned}$$

where λ_{ij}'s are constants will be calculated using initial conditions.

Example 13 *Let* $\{a_n\}$ *be a sequence such that,*

$$a_n = 5a_{n-1} - 9a_{n-2} + 7a_{n-3} - 2a_{n-4},\ n \geq 4,\ a_0 = 3,\ a_1 = 8,\ a_2 = 17,\ a_3 = 32.$$

Find a_n.

Solution: Characteristic equation of the recurrence is,

$$x^n = 5x^{n-1} - 9x^{n-2} + 7x^{n-3} - 2x^{n-4},\ x \neq 0$$

$$\Rightarrow x^4 - 5x^3 + 9x^2 - 7x + 2 = 0$$

$$\Rightarrow (x-1)^3(x-2) = 0 \quad \Rightarrow \quad x = 1, 1, 1, 2.$$

$$\Rightarrow a_n = (\lambda_1 + \lambda_2 n + \lambda_3 n^2)(1)^n + \lambda_4 2^n$$

Now,

$$a_0 = \lambda_1 + \lambda_4 = 3$$
$$a_1 = \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 = 8$$
$$a_2 = \lambda_1 + 2\lambda_2 + 4\lambda_3 + 4\lambda_4 = 17$$
$$a_3 = \lambda_1 + 3\lambda_2 + 9\lambda_3 + 8\lambda_4 = 32$$
$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 1,\ \lambda_4 = 2$$
$$\Rightarrow a_n = (1 + 2n + n^2) + 2 \cdot 2^n$$
$$= (n+1)^2 + 2^{n+1}$$

Build-up Your Understanding 3

1. Let $x_0 = 1, x_1 = 1, x_{n+1} = x_n + 2x_{n-1}\ \ \forall n \geq 1$. Find x_n
2. Let $a_0 = 1, a_1 = 7, a_{n+1} = 2a_n + 3a_{n-1}$. Find a_n.
3. Let $a_1 = 1,\ a_2 = 3,\ a_{n+2} = 4a_{n+1} - 4a_n$.Find$a_n$.
4. $a_0 = 3,\ a_3 = 6,\ a_n = -6\ a_{n-1} - 9a_{n-2}$ find a_n
5. Let $a_n = 7a_{n-1} - 6a_{n-2},\ a_0 = 2,\ a_1 = 7$. Find a_n.
6. Solve the following recurrence relation.
   (a) $a_n = 5a_{n-1} - 6a_{n-2},\ n \geq 2,\ a_0 = 1,\ a_1 = 5$
   (b) $a_n = 6a_{n-1} - 9a_{n-2},\ n \geq 2,\ a_0 = 1,\ a_1 = 2$
7. $a_n = 3\ a_{n-1} - 4\ a_{n-3},\ a_0 = -4,\ a_1 = 2,\ a_2 = 6.$
8. Let $a_0 = a, a_1 = b, a_2 = 2b - a + 2,\ a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$. Find a_n.
9. Let $a_1 = a_2 = 1,\ a_n = \dfrac{a_{n-1}^2 + 2}{a_{n-2}}\ \ \forall n \geq 3$. Prove that $\forall\ n \in \mathbb{N}, a_n \in \mathbb{N}$.

4.7 General Method For Non-Homogeneous Linear Equation

Non-homogeneous linear equations are usually solved using the method of undetermined coefficients (basically guessing the solution of non-homogeneous part and checking with the recurrence).

We do this in three parts:

Part 1: Find the general solution $a_n^{(H)}$ of associated homogeneous equation.

Part 2: Find a single solution $a_n^{(P)}$ to the non-homogeneous equation. This solution is referred as particular solution.

Part 3: Now $a_n = a_n^{(H)} + a_n^{(P)}$. Now put initial condition to get constants in a_n^H

Example 14 *Find the general solution to the recurrence:*

$$a_n = a_{n-1} + 2a_{n-2} + n,\ n \geq 2,\ a_0 = \frac{7}{4},\ a_1 = \frac{5}{4}.$$

Solution: As non-homogeneous term is a polynomial of degree 1, we guess that particular solution will be of the form, $a_n^{(P)} = pn + q$

$$\begin{aligned}
&\Rightarrow pn + q = p(n-1) + q + 2p(n-2) + 2q + n \\
&\Rightarrow -2pn + (5p - 2q) = n \\
&\Rightarrow -2p = 1 \quad \text{and} \quad 5p - 2q = 0 \\
&\Rightarrow p = -\frac{1}{2},\ q = -\frac{5}{4}
\end{aligned}$$

Associated homogeneous equation is,

$$\begin{aligned}
&a_n^{(H)} = a_{n-1}^{(H)} + 2a_{n-2}^{(H)} \\
&\Rightarrow x^2 - x - 2 = 0 \quad \Rightarrow \quad x = 2, -1 \\
&\Rightarrow a_n^{(H)} = \lambda 2^n + \mu(-1)^n \\
&\Rightarrow a_n = a_n^{(H)} + a_n^{(P)} \\
&\qquad = \lambda 2^n + \mu(-1)^n - \frac{n}{2} - \frac{5}{4} \\
&a_0 = \lambda + \mu - \frac{5}{4} = \frac{7}{4} \quad \Rightarrow \quad \lambda + \mu = 3 \\
&a_1 = 2\lambda - \mu - \frac{1}{2} - \frac{5}{4} = \frac{5}{4} \quad \Rightarrow \quad 2\lambda - \mu = 3 \\
&\Rightarrow \lambda = 2,\ \mu = 1 \\
&\Rightarrow a_n = 2^{n+1} + (-1)^n - \frac{n}{2} - \frac{5}{4}.
\end{aligned}$$

Example 15 *Let $\{a_n\}$ be a sequence such that, $a_n = 2a_{n-1} + 4^{n-1}$, $a_0 = 0$, find a_n.*

Solution: Solution to homogeneous put,

$$\begin{aligned}
&a_n^{(H)} = 2a_{n-1}^{(H)} \\
&\Rightarrow a_n^{(H)} = \lambda 2^n
\end{aligned}$$

As non-homogeneous term is 4^{n-1}, let us guess that the particular solution,

$$\begin{aligned}
&a_n^{(P)} = a \cdot 4^n \\
&\Rightarrow a \cdot 4^n = 2 \cdot a \cdot 4^{n-1} + 4^{n-1} \\
&\Rightarrow 4a = 2a + 1 \quad \Rightarrow \quad a = \frac{1}{2} \\
&\Rightarrow a_n^{(P)} = \frac{1}{2} \cdot 4^n
\end{aligned}$$

Hence,

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$a_n = \lambda 2^n + \frac{1}{2} 4^n$$

Now,

$$a_0 = \lambda + \frac{1}{2} = 0$$

$$\Rightarrow \lambda = -\frac{1}{2}$$

$$\Rightarrow a_n = \frac{1}{2}(4^n - 2^n).$$

Example 16 *Let* $\{a_n\}$ *be a sequence such that,*
$a_n = 3a_{n-1} - 2a_{n-2} + 2^n$, $n \ge 2$, $a_0 = 3$, $a_1 = 8$, find a_n.

Solution: Solution to homogeneous part,

$$a_n^{(H)} = 3a_{n-1}^{(H)} - 2a_{n-2}^{(H)}$$

$$\Rightarrow x^2 - 3x + 2 = 0 \quad \Rightarrow 1, 2$$

$$\Rightarrow a_n^{(H)} = \lambda(1)^n + \mu 2^n = \lambda + \mu 2^n$$

Unfortunately corresponding non-homogeneous term is 2^n which is also appearing in homogeneous part. Let us guess particular solution.

$$a_n^{(P)} = \gamma n 2^n$$

$$\Rightarrow \gamma n 2^n = 3\gamma(n-1)2^{n-1} - 2\gamma(n-2)2^{n-2} + 2^n$$

$$\Rightarrow 4n\gamma = 6\gamma(n-1) - 2\gamma(n-2) + 4$$

$$\Rightarrow 2\gamma - 4 = 0$$

$$\Rightarrow \gamma = 2$$

$$\Rightarrow a_n^{(P)} = n \cdot 2^{n+1}$$

$$\Rightarrow a_n = a_n^{(H)} + a_n^{(P)} = \lambda + \mu 2^n + n2^{n+1}$$

Now, $a_0 = \lambda + \mu = 3$ and $a_1 = \lambda + 2\mu + 4 = 8$

$$\Rightarrow \lambda = 2,\ \mu = 1$$

$$\Rightarrow a_n = 2 + 2^n + n2^{n+1}.$$

4.7.1 A Special Case

If the recurrence relation is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} + \alpha_1^n f_1(n) + \alpha_2^n f_2(n) + \cdots + \alpha_k^n f_k(n)$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are constants and $f_1, f_2, f_3, \ldots, f_k$ are polynomials in 'n' of degree $p_1, p_2, p_3, \ldots, p_k$ respectively, then the characteristic equation of the recurrence will be

$$\left(x^r - c_1 x^{r-1} - c_2 x^{r-2} - \cdots - c_r\right)(x-\alpha_1)^{p_1+1}(x-\alpha_2)^{p_2+1} \cdots (x-\alpha_k)^{p_k+1} = 0$$

Here we added additional roots in the equation as our equation contains terms of the form $\alpha^n f(n)$ where α is a constant and $f(n)$ is a polynomial in 'n'.

The general form of the solution is the same as in article 4.6, But we need to know more than r initial terms as there are additional $p_1 + 1 + p_2 + 1 + \cdots + p_k + 1$ coefficients needed to be resolve. We can obtain these additional terms from the given recurrence.

Example 17 *Let $a_0 = 2$, $a_n = 3a_{n-1} - 4n$. Find a_n.*

Solution: Given recurrence can be rewritten as $a_n = 3a_{n-1} - (1)^n\, 4n$

$$\Rightarrow \alpha = 1 \text{ and } f(n) = 4n, \text{ which is of degree 1.}$$

Hence corresponding characteristic equation is,

$$(x-3)(x-1)^{1+1} = 0$$

$$\Rightarrow a_n = \lambda 3^n + (\mu + \gamma n)1^n \quad (1)$$

Now we need two more terms to resolve μ and γ

$$a_0 = 2 \Rightarrow a_1 = 2,\ a_2 = -2 \text{ (From the recurrence relation)}$$

Plugging $n = 0, 1, 2$ in equation (1), we get

$$\lambda + \mu = 2$$
$$3\lambda + \mu + \gamma = 2$$
$$9\lambda + \mu + 2\gamma = -2$$
$$\Rightarrow \lambda = -1,\ \mu = 3, \text{ and } \gamma = 2$$

Hence $a_n = -3^n + 2n + 3$

Example 18 *Let $a_n = 2a_{n-1} + n + 2^{n+1}$, $a_0 = 0$. Find a_n*

Solution: Given recurrence can be rewritten as $a_n = 2a_{n-1} + n(1)^n + 2^n(2)$

$\Rightarrow \alpha_1 = 1$, $f_1(n) = n$, which is of degree 1 and $\alpha_2 = 2$, $f_2(n) = 2$, , which is of zero degree.

Hence corresponding characteristic equation is,

$$(x-2)(x-1)^{1+1}(x-2)^{0+1} = (x-1)^2(x-2)^2 = 0$$

$$\Rightarrow a_n = (\lambda + \mu n)1^n + (\gamma + \delta n)2^n \quad (1)$$

Now we need three more terms. From the recurrence: $a_1 = 5$, $a_2 = 20$, $a_3 = 59$.

Plugging $n = 0, 1, 2, 3$ in equation (1), we get

$$\lambda + \gamma = 0$$
$$\lambda + \mu + 2\gamma + 2\delta = 5$$
$$\lambda + 2\mu + 4\gamma + 8\delta = 20$$
$$\lambda + 3\mu + 8\gamma + 24\delta = 59$$
$$\Rightarrow \lambda = -2,\ \mu = -1,\ \gamma = 2, \text{ and } \delta = 2$$

Hence $$a_n = -n - 2 + (n+1)2^{n+1}$$

Build-up Your Understanding 4

1. Find the n^{th} term of the sequence $\{b_n\}$ such that $b_1 = 2$, $b_{n+1} = 2b_n + n$ $(n = 1, 2, 3, \ldots)$.
2. Given the sequence $\{a_n\}$ which is defined by
   $a_1 = 1$, $a_{n+1} = 2a_n + 2^n$ $(n = 1, 2, 3, \ldots)$. Find the n^{th} term a_n and sum $\sum_{k=1}^{n} a_k$
3. Let $a_0 = 2$, $a_n = 2a_{n-1} + 3^{n-1}$, $n \geq 1$. Find a_n.
4. Define the sequence $\{a_n\}$ such that $a_1 = -4$, $a_{n+1} = 2a_n + 2^{n+3}\, n - 13 \cdot 2^{n+1}$ $(n = 1, 2, 3, \ldots)$. Find the value of n for which a_n is minimized.
5. Find the n^{th} term of the sequence $\{a_n\}$ such that $a_1 = 1$, $a_{n+1} = 2a_n - n^2 + 2n$ $(n = 1, 2, 3, \ldots)$.
6. Let $a_0 = 1$, $a_n = 3a_{n-1} - 2n^2 + 6n - 3$, $n \geq 1$. Find a_n.
7. Let $a_1 = 8, a_n = 3a_{n-1} - 4n + 3.2^n$. Find a_n
8. Let $a_0 = 2$, $a_n = 9a_{n-1} - 56n + 6^n$, $n \geq 1$. Find a_n.
9. Find the n^{th} term of the sequence $\{a_n\}$ such that $a_1 = 1$, $a_2 = 3$, $a_{n+1} - 3a_n + 2a_{n-1} = 2^n$ $(n \geq 2)$.
10. Find the n^{th} term of the sequence $\{a_n\}$ such that
    $$a_1 = \frac{1}{2},\ a_2 = \frac{1}{3},\ a_{n+2} = \frac{a_n a_{n+1}}{2a_n - a_{n+1} + 2a_n a_{n+1}}.$$
11. Let $x_0 = 1$, $x_1 = 1$, $x_{n+2} = \dfrac{1 + x_{n+1}}{x_n}$ $\forall\, n = 0, 1, 2, \ldots$. Find x_{2017}.

Solved Problems

Problem 1 *Let $a_1 = 1$, $a_2 = e$, $a_{n+2} = a_n^{-2}\, a^3_{n+1}$, $n \geq 1$. Find a_n*

Solution: $\ln a_{n+2} = -2 \ln a_n + 3 \ln a_{n+1}$

Let $\ln a_n = b_n$, $b_1 = 0$, $b_2 = 1$

$$\Rightarrow \quad b_{n+2} = -2b_n + 3b_{n+1}$$

Its characteristic equation,

$$x^2 - 3x + 2 = 0$$
$$\Rightarrow x = 1, 2$$
$$\Rightarrow b_n = \lambda(1)^n + \mu_2^n$$
$$\Rightarrow 0 = \lambda + 2\mu \text{ and } 1 = \lambda + 4\mu$$
$$\Rightarrow \mu = \frac{1}{2},\ \lambda = -1$$
$$\Rightarrow b_n = 2^{n-1} - 1$$
$$\Rightarrow a_n = e^{2^{n-1}-1}$$

Problem 2 *Let $a_n = 7a_{n/2} - 6a_{n/4}$, $a_1 = 2$, $a_2 = 7$. Find a_n.*

Solution: Take $n = 2^m$

$$a_{2^m} = 7a_{2^{m-1}} - 6a_{2^{m-2}}$$

Let
$$a_{2^m} = b_m$$
$$\Rightarrow b_m = 7b_{m-1} - 6b_{m-2},\ b_0 = 2,\ b_1 = 7$$

Characteristic equation is

$$x^2 - 7x + 6 = 0 \Rightarrow x = 6, 1$$

$$\Rightarrow b_m = \lambda\, 6^m + \mu 1^m$$

For $m = 0,\ 2 = \lambda + \mu$

For $m = 1,\ 7 = 6\lambda + \mu$

$$\Rightarrow \lambda = 1, \mu = 1$$

$$\Rightarrow b_m = 6^m + 1$$

$$\Rightarrow a_{2^m} = 6^m + 1$$

$$\Rightarrow a_n = 6^{\log_2 n} + 1$$

$$\Rightarrow a_n = n^{\log_2 6} + 1$$

Problem 3 *Let* $a_1 = 1,\ a_{n+1} = 2a_n + \sqrt{3a_n^2 - 2}\ \forall n \geq 1$. *Prove that* $a_n \in \mathbb{N}$.

Solution: $a_{n+1}^2 - 4a_{n+1}\, a_n + 4a_n^2 = 3a_n^2 - 2$

$$\Rightarrow a_{n+1}^2 - 4a_{n+1}a_n + a_n^2 + 2 = 0$$

$$\Rightarrow a_n^2 - 4a_n a_{n-1} + a_{n-1}^2 + 2 = 0 \text{ or } a_{n-1}^2 - 4a_n a_{n-1} + a_n^2 + 2 = 0$$

$$\Rightarrow a_{n+1}, a_{n-1} \text{ are the roots of } p(x) = x^2 - 4a_n\, x + a_n{}^2 + 2 = 0$$

$$\Rightarrow a_{n+1} + a_{n-1} = 4a_n$$

$$\Rightarrow a_{n+1} = 4a_n - a_{n-1} \text{ and by induction we are done}$$

Problem 4 *Let* $a_n - 2a_{n-1} + a_{n-2} = \binom{n+4}{4}\ \forall\, n \geq 2,\ a_0 = 0,\ a_1 = 5$. *Find* a_n

Solution: Let $a_n - a_{n-1} = b_n;\ b_1 = 5$

$$\Rightarrow b_n - b_{n-1} = \binom{n+4}{4}\ \forall\, n \geq 2$$

Plugging $n = 2, 3, \ldots$, and adding all, we get,

$$b_n - b_1 = \sum_{r=2}^{n} \binom{r+4}{4} = \binom{n+5}{5} - \binom{4}{4} - \binom{5}{4}$$

$\left(\textit{Note: } \sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}\text{, known as hockey stick identity} \right)$

$$\Rightarrow b_n - b_1 = \binom{n+5}{5} - 6$$

$$\Rightarrow a_n - a_{n-1} = \binom{n+5}{5} - 1$$

Again plugging $n = 1, 2, 3, \ldots, n$ and adding all, we get,

$$a_n = \sum_{r=1}^{n} \binom{r+5}{5} - n$$

$$= \binom{n+6}{6} - \binom{5}{5} - n$$

Problem 5 *Let* $\sum_{m=1}^{n}\binom{n}{k}a_k = \frac{n}{n+1}, n = 1, 2, \ldots,$ *find* a_n

Solution: Let $f(x) = \sum_{m=1}^{n}\binom{n}{k}x^k = (x+1)^n - 1$

$$\int_0^x f(t)dt = \sum_{m=1}^{n}\binom{n}{k}\frac{x^{k+1}}{k+1} = \frac{(x+1)^{n+1}-1}{n+1} - x$$

Put $x = -1$, we get

$$\sum_{m=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k+1} = -\frac{1}{n+1} + 1 = \frac{n}{n+1}$$

And since there is obviously a unique sequence a_k matching the requirements, we get $\Rightarrow a_n = \frac{(-1)^{n+1}}{n+1}$.

Problem 6 *Let* $a_0 = 1, a_1 = 2$ *and* $a_n = 4a_{n-1} - a_{n-2} \forall n \geq 2$. *Find an odd prime factor of* a_{2015}.

[Putnam, 2015]

Solution: Characteristic equation

$$x^2 - 4x + 1 = 0 \quad \Rightarrow \quad x = 2 \pm \sqrt{3}$$

$$\Rightarrow \quad a_n = \lambda(2+\sqrt{3})^n + \mu(2-\sqrt{3})^n$$

$$\Rightarrow \quad \lambda + \mu = 1, \lambda(2+\sqrt{3}) + \mu(2-\sqrt{3}) = 2$$

$$\Rightarrow \quad \lambda = \mu = \frac{1}{2}$$

$$\Rightarrow \quad a_n = \frac{1}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n\right] \tag{1}$$

For the sake of notational ease, let $2+\sqrt{3} = \alpha$ and $2-\sqrt{3} = \beta$, then $a_n = \frac{1}{2}(\alpha^n + \beta^n)$

Claim: If k is an odd positive integer and $a_n \neq 0$ then $a_n \mid a_{kn}$

Proof: $\frac{a_{kn}}{a_n} = \frac{\alpha^{kn} + \beta^{kn}}{\alpha^n + \beta^n} = \frac{(\alpha^n)^k + (\beta^n)^k}{\alpha^n + \beta^n}$

$$= \alpha^{(k-1)n} - \alpha^{(k-2)n}\beta^n + \cdots - \alpha^n\beta^{(k-2)n} + \beta^{(k-1)n} \tag{1}$$

As $\alpha \cdot \beta = 1$ and $\alpha^m + \beta^m \in \mathbb{Z}\ \forall\ m$, RHS of (1) is an integer $\Rightarrow a_n \mid a_{kn}$

Now $2015 = 403 \times 5$

$$\Rightarrow \quad a_5 \mid a_{5.403}, \text{ i.e., } \Rightarrow \quad a_5 \mid a_{2015}$$

Here $$a_5 = \frac{(2+\sqrt{3})^5 + (2-\sqrt{3})^5}{2}$$
$$= 362 = 2 \times 181$$

Hence one possible answer is 181.

Problem 7 $a_0 = 0, a_1 = 1, a_n = 2a_{n-1} + a_{n-2}, n \geq 2$. *Prove that* $2^k \mid a_n$ *if and only if* $2^k \mid n$.

[IMO Shortlisted Problem, 1988]

Solution: By the binomial theorem, if $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$, then $(1-\sqrt{2})^n = A_n - B_n\sqrt{2}$. Multiplying these 2 equations, we get $A_n^2 - 2B_n^2 = (-1)^n$.
This implies A_n is always odd. Using characteristic equation method to solve the given recurrence relations on a_n, we find that $a_n = B_n$.

Now write $n = 2^k m$, where m is odd.

We have $k = 0$ (*i.e.,* n is odd) if and only if $2B_n^2 = A_n^2 + 1 \equiv 2 \pmod 4$, (*i.e.*, B_n is odd). Next suppose case k is true.

Since $(1+\sqrt{2})^{2n} = (A_n + B_n\sqrt{2})^2 = A_{2n} + B_{2n}\sqrt{2}$, so $B_{2n} = 2A_nB_n$.

Then it follows case k implies case $k+1$.

Aliter: From given recurrence we can easily get,

$$a_n = \frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right) = \binom{n}{1} + 2\binom{n}{3} + 2^2\binom{n}{5} + \cdots$$

Let $n = 2^k$m with m being odd; then for $r > 0$ the summand

$2^r\binom{n}{2r+1} = 2^r\dfrac{n}{2r+1}\binom{n-1}{2r} = 2^{r+k}\dfrac{m}{2r+1}\binom{n-1}{2r}$ is divisible by 2^{r+k} (As $2r+1$ is odd)

Hence, $a_n = n + \sum\limits_{r>0} 2^r\binom{n}{2r+1} = 2^k m + 2^{k+1}s$, for some integer s.

$\Rightarrow a_n$ is exactly divisible by 2^k.

Problem 8 *Let* $a_0 = 0$, $a_1 = 1$, $a_{n+2} = a\,a_{n+1} + ba_n$ *where gcd(a, b) = 1. Let c be a given positive integer, m is the least positive integer such that* $c \mid a_m$, *and n is an arbitrary positive integer such that* $c \mid a_n$. *Prove that* $m \mid n$.

Solution: Let us first prove that consecutive terms are pair wise coprime

Given $a_0, a_1, a_2, a_3, a_4, \ldots$ is $0, 1, a, a^2+b, \ldots$

First 4 terms are co-prime pairwise

Let $(a_{k+1}, a_k) = 1$ and suppose $p \mid a_{k+2}$, and $p \mid a_{k+1}$, where p is prime number.

As $a_{k+2} = a\,a_{k+1} + ba_k$

$\Rightarrow p \mid ba_k \Rightarrow p \mid b$ (As p does not divides a_k)

Also $a_{k+1} = aa_k + ba_{k-1}$

$\Rightarrow p \mid a \cdot a_k \Rightarrow p \mid a$, which is a contradiction.

Hence any two consecutive terms are pair wise co-prime.

Now Let a_m be the first term divisible by 'c', *i.e.*, m is minimal such number.

Consider the sequence

$0, 1, a, a^2+b, \ldots, x, a_m, a_m + bx, a^2 a_m + ab\,x + ba_m, \ldots$

By taking mod c of the sequence we get,

$0, 1, a, \ldots, x, 0, b\,x, ab\,x, \ldots.$

$\Rightarrow a_{m+k} \equiv bx\,a_k \pmod c$

Let $c \mid bx\,a_k$ but c does not divides a_k.

Now, gcd $(a_m, a_m + bx) = 1$ and $c \mid a_m \Rightarrow c$ does not divides bx.

Hence, if $c \mid a_{m+k} \Rightarrow c \mid a_k$

$\Rightarrow k$ must be a multiple of m, since otherwise we can continue retrieving values m from k until reaching a term divisible by C, with an index strictly between 0 and m, contradicting the minimality of m.

Problem 9 *Let* $a_1 = 1$, $a_n = \sum_{k=1}^{n-1}(n-k)a_k, \forall n \geq 2$. *Find* a_n

Solution: $a_1 = 1 \Rightarrow a_2 = 1, a_3 = 3, a_4 = 8$ and so on.

Then
$$a_{n+1} = \sum_{k=1}^{n}(n+1-k)a_k \tag{1}$$

Also
$$a_n = \sum_{k=1}^{n-1}(n-k)a_k \tag{2}$$

$$\Rightarrow \quad a_{n+1} - a_n = \sum_{k=1}^{n} a_k \qquad \text{(From (1) – (2))} \tag{3}$$

$$\Rightarrow \quad a_{n+2} - a_{n+1} = \sum_{k=1}^{n+1} a_k \tag{4}$$

$$\Rightarrow \quad (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = a_{n+1} \quad \text{(From (4) – (3))}$$
$$\Rightarrow \quad a_{n+2} = 3a_{n+1} - a_n \;\forall\; n \geq 2$$

Characteristic equation is

$$x^2 - 3x + 1 = 0 \quad \Rightarrow \quad x = \frac{3 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad a_n = \lambda\left(\frac{3+\sqrt{5}}{2}\right)^n + \mu\left(\frac{3-\sqrt{5}}{2}\right)^n$$

$$a_2 = 1 = \lambda\left(\frac{3+\sqrt{5}}{2}\right) + \mu\left(\frac{3+\sqrt{5}}{2}\right)$$

$$a_3 = 3 = \lambda\left(\frac{7+3\sqrt{5}}{2}\right) + \mu\left(\frac{7-3\sqrt{5}}{2}\right)$$

$$\lambda = \frac{2}{\sqrt{5}(3+\sqrt{5})} \quad \text{and} \quad \mu = -\frac{2}{\sqrt{5}(3-\sqrt{5})}$$

$$\Rightarrow \quad a_n = \frac{(3+\sqrt{5})^{n-1} - (3-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} \quad \forall n \geq 2$$

Problem 10 *For real numbers* $a_1, a_2, a_3, \ldots,$ *if* $a_{n-1} + a_{n+1} \geq 2a_n$ *for* $n = 2, 3, \ldots,$ *then prove that*
$A_{n-1} + A_{n+1} \geq 2A_n$ for $n = 2, 3, \ldots,$ where A_n is the average of $a_1, a_2, \ldots, a_n$.

Solution: $A_{n-1} + A_{n+1} \geq 2A_n$

$$\Leftrightarrow \frac{\sum_{r=1}^{n-1} a_r}{n-1} + \frac{\sum_{r=1}^{n+1} a_r}{n+1} - \frac{2\sum_{r=1}^{n} a_r}{n} \geq 0$$

$$\Leftrightarrow \left[\frac{1}{n-1}+\frac{1}{n+1}-\frac{2}{n}\right]\sum_{r=1}^{n-1} a_r + a_n\left[\frac{1}{n+1}-\frac{2}{n}\right]+\frac{a_{n+1}}{n+1}\geq 0$$

$$\Leftrightarrow \left[\frac{2}{(n-1)n(n+1)}\right]\sum_{r=1}^{n-1} a_r - a_n\left[\frac{n+2}{n(n+1)}\right]+\frac{a_{n+1}}{n+1}\geq 0$$

$$\Leftrightarrow \sum_{r=1}^{n-1} a_r - a_n\frac{(n+2)(n-1)}{2}+a_{n+1}\frac{n(n-1)}{2}\geq 0$$

$$\Leftrightarrow \sum_{r=2}^{n} a_{r-1} - \sum_{r=2}^{n}\left(a_r\frac{(r+2)(r-1)}{2}-a_{r-1}\frac{(r+1)(r-2)}{2}\right)$$

$$+\sum_{r=2}^{n}\left(a_{r+1}\frac{r(r-1)}{2}-a_r\frac{(r-1)(r-2)}{2}\right)\geq 0$$

$$\Leftrightarrow \sum_{r=2}^{n}\left(a_{r-1}\left(1+\frac{(r+1)(r-2)}{2}\right)-a_r\left(\frac{(r+2)(r-1)}{2}+\frac{(r-1)(r-2)}{2}\right)+a_{r+1}\left(\frac{r(r-1)}{2}\right)\right)$$

$$\geq 0$$

$$\Leftrightarrow \sum_{r=2}^{n}\left(a_{r-1}\left(\frac{r(r-1)}{2}\right)-2a_r\left(\frac{r(r-1)}{2}\right)+a_{r+1}\left(\frac{r(r-1)}{2}\right)\right)\geq 0$$

$$\Leftrightarrow \sum_{r=2}^{n}\frac{r(r-1)}{2}(a_{r-1}-2a_r+a_{r+1})\geq 0$$

Which is true as $a_{r-1}+a_{r+1}\geq 2a_r$ for $r = 2, 3,\ldots$

Problem 11 *The first term x_1 of a sequence is 2014. Each subsequent term of the sequence is defined in term of the previous term. The iterative formula is $x_{n+1}=\frac{(\sqrt{2}+1)x_n-1}{(\sqrt{2}+1)+x_n}$. Find the 2015th term, i.e., x_{2015}.*

[BMO, 2015]

Solution: $x_{n+1}=\dfrac{x_n-(\sqrt{2}-1)}{1+(\sqrt{2}-1)x_n}$

Let $x_n = \tan a_n$ also $\sqrt{2}-1=\tan\frac{\pi}{8}$

$$\Rightarrow\quad x_{n+1}=\frac{\tan a_n-\tan\frac{\pi}{8}}{1+\tan\frac{\pi}{8}\tan a_n}$$

$$\Rightarrow\quad x_{n+1}=\tan\left(a_n-\frac{\pi}{8}\right)$$

$$\Rightarrow\quad x_{2015}=\tan\left(a_1-2014.\frac{\pi}{8}\right)=\tan\left(a_1+\frac{\pi}{4}\right)=\frac{\tan a_1+1}{1-\tan a_1}$$

$$=\frac{x_1+1}{1-x_1}=-\frac{2015}{2013}$$

Note: For any k, $a_1 - k\frac{\pi}{8}\neq$ odd multiple of $\frac{\pi}{2}$; Even we can say $a_1\neq\frac{\pi}{8}$ (integer)

As $\tan a_1 = 2014 \notin \left\{\tan 0, \pm\tan\frac{\pi}{8}, \pm\tan\frac{2\pi}{8}, \pm\tan\frac{3\pi}{8}\right\}$

Problem 12 *It is given that the sequence* $(a_n)_{n=1}^{\infty}$ *with* $a_1 = a_2 = 2$ *is given by the recurrence relation* $\dfrac{2a_{n-1}a_n}{a_{n-1}a_{n+1} - a_n^2} = n^3 - n \;\forall\; n = 2, 3, 4...$

Find integer that is closest to the value of $\displaystyle\sum_{k=2}^{2011} \frac{a_{k+1}}{a_k}$

[Singapore MO, 2012]

Solution: $\dfrac{2a_{n-1}a_n}{a_{n-1}a_{n+1} - a_n^2} = n^3 - n$

$$\Rightarrow \quad \frac{a_{n-1}a_{n+1} - a_n^2}{a_{n-1}a_n} = \frac{2}{n^3 - n} = \frac{2}{n(n-1)(n+1)}$$

$$\Rightarrow \quad \frac{a_{n+1}}{a_n} - \frac{a_n}{a_{n-1}} = \frac{(n+1)-(n-1)}{n(n-1)(n+1)}$$

$$= \frac{1}{(n-1)n} - \frac{1}{n(n+1)}$$

Plugging $n = 2, 3, 4, \ldots, n$ and adding all, we get,

$$\frac{a_n}{a_{n-1}} - \frac{a_2}{a_1} = \frac{1}{2} - \frac{1}{(n+1)n}$$

$$\frac{a_{n+1}}{a_n} = \frac{3}{2} - \frac{1}{n} + \frac{1}{n+1}$$

Again plugging $n = 2, 3, \ldots, n$ and adding all, we get,

$$\sum_{k=2}^{n} \frac{a_{k+1}}{a_k} = \frac{3}{2} \times n - \frac{1}{2} + \frac{1}{n+1}$$

$$= \frac{3n-1}{2} + \frac{1}{n+1}$$

For $n = 2011$,

$$\sum_{k=2}^{2011} \frac{a_{k+1}}{a_k} = \frac{6033-1}{2} + \frac{1}{2012}$$

$$= 3016 + \frac{1}{2012}$$

$\Rightarrow$ Closest integer is 3016

Problem 13 *Let x and y be distinct complex number such that* $\dfrac{x^n - y^n}{x - y}$ *is an integer for some four consecutive positive integers n. Show that* $\dfrac{x^n - y^n}{x - y}$ *is an integer for all positive integers n.*

Solution: For non-negative integer n, let $t_n = \dfrac{x^n - y^n}{x - y}$. So $t_0 = 0$, $t_1 = 1$ and we have a recurrence relation $t_{n+2} + bt_{n+1} + ct_n = 0$, where $b = -(x + y)$, $c = xy$.
Suppose t_n is an integer for m, $m + 1$, $m + 2$, $m + 3$.

Since $c^n = (xy)^n = t^2_{n+2} - t_n t_{n+2}$ is an integer for $n = m$, $m + 1$, so c is rational. Since c^{m+1} is integer, c must, in fact, be an integer. Next

$$b = \frac{t_m t_{m+3} - t_{m+1} t_{m+2}}{c^m}$$

So b is rational.

Form the recurrence relation, it follows by induction that $t_n = f_{n-1}$ (b) for some polynomial f_{n-1} of degree $n - 1$ with integer coefficients. Not the coefficient of x^{n-1} in f_{n-1} is 1, *i.e.*, f_{n-1} is monic.

Since b is a root of the integer coefficient polynomial $f_m(z) - t_{m+1} = 0$, b must be an integer.

So the recurrence relation implies all t_n's are integers.

Problem 14 *Let* $a_1 = 1, a_2 = -1, a_n = -a_{n-1} - 2a_{n-2} \forall n \geq 3$. *Prove that* $2^{n+2} - 7a_n^2$ *is a perfect square.*

Solution: Let us generate enough data

$$a_3 = -a_2 - 2a_1 = 1 - 2(1) = -1$$

$$a_4 = -a_3 - 2a_2 = 1 + 2 = 3$$

Now Let

$$2^{n+2} - 7a_n^2 = b_n^2$$

$$\Rightarrow b_1^2 = 2^3 - 7a_1^2 = 8 - 7 = 1$$

$$b_2^2 = 2^4 - 7a_2^2 = 16 - 7 = 9$$

$$b_3^2 = 2^5 - 7a_3^2 = 32 - 7 = 25$$

$$b_4^2 = 2^6 - 7a_4^2 = 64 - 63 = 1$$

Let us define $b_1 = -1$, $b_2 = -3$, and $b_n = -b_{n-1} - 2b_{n-2} \ \forall \ n \geq 3$

Claim 1: $a_{n+1} = \dfrac{b_n}{2} - \dfrac{a_n}{2}$ and $b_{n+1} = -\dfrac{7a_n}{2} - \dfrac{b_n}{2}$

Proof: for $n = 1$, $a_2 = \dfrac{b_1}{2} - \dfrac{a_1}{2} = -\dfrac{1}{2} - \dfrac{1}{2} = -1$, which is true.

$b_2 = -\dfrac{7a_1}{2} - \dfrac{b_1}{2} = -\dfrac{7}{2} - \dfrac{1}{2} = -3$, which is true.

for $n = 2$, $a_3 = \dfrac{b_2}{2} - \dfrac{a_2}{2} = -\dfrac{3}{2} + \dfrac{1}{2} = -1$, which is true.

$b_3 = -\dfrac{7a_2}{2} - \dfrac{b_2}{2} = \dfrac{7}{2} + \dfrac{3}{2} = 5$, which is true.

Let for $n = k$, claim be true

For $n = k + 1$

$$a_{k+2} = -a_{k+1} - 2a_k$$

$$= -\frac{b_k}{2} + \frac{a_k}{2} - 2a_k$$

$$= -\frac{3a_k}{2} - \frac{b_k}{2} = \frac{1}{2}\left(-\frac{7a_k}{2} - \frac{b_k}{2}\right) - \frac{1}{2}\left(\frac{b_k}{2} - \frac{a_k}{2}\right)$$

$$= \frac{1}{2}a_{k+1} - \frac{1}{2}b_{k+1}$$

and $$b_{k+2} = -b_{k+1} - 2b_k = \frac{7a_k}{2} - \frac{3b_k}{2}$$

$$= \frac{7b_{k+1}}{2} - \frac{b_{k+1}}{2}$$

Hence by induction our claim is true!

Claim 2: $$2^{n+2} = 7an^2 + bn^2$$

for $n = 1$, $2^3 = 7{a_1}^2 + {c_1}^2 = 7 + 1 = 8$ true
Let for $n = k$, claim be true
For $n = k + 1$

$$2^{k+3} = 2(2^{k+2}) = 2(7{a_k}^2 + {b_k}^2)$$

$$= 7\left(\frac{b_k}{2} - \frac{a_k}{2}\right)^2 + \left(-7\frac{a_k}{2} - \frac{b_k}{2}\right)^2$$

$$= 7a_{k+1}^2 + b_{k+1}^2$$

Hence by induction our claim is true.
As $2^{n+2} = 7a_n^2 + b_n^2$
$\Rightarrow \quad 2^{n+2} - 7a_n^2 = b_n^2$
$\Rightarrow \quad 2^{n+2} - 7a_n^2$ is a perfect square.

Problem 15 *Let $\{a_n\}$, $\{b_n\}$, $n = 1, 2, 3, \ldots$, be two sequences of integers defined by $a_1 = 1$, $b_2 = 0$ and $n \geq 1$.*

$$a_{n+1} = 7a_n + 12b_n + 6$$
$$b_{n+1} = 4a_n + 7b_n + 3$$

Prove that a_n^2 is the difference of two consecutive cubes.

[Singapore MO, 2010]

Solution: Consider the equation, $x^2 - 3y^2 = 1$ (Pell's Equation)
Its fundamental solution is (2, 1) and all other solutions (x_k, y_k) will satisfy

$$x_{k+1} + y_{k+1}\sqrt{3} = (x_k + y_k\sqrt{3})(2 + \sqrt{3})$$

Or $$x_{k+1} + y_{k+1}\sqrt{3} = 2x_k + 3y_k + \sqrt{3}(2y_k + x_k)$$

$$\Rightarrow \quad x_{k+1} = 2x_k + 3y_k \tag{1}$$

And $$y_{k+1} = 2y_k + x_k \tag{2}$$

Note that x_k is even and y_k is odd only when k is odd. For odd $k = 2n - 1$,

Let $$x_{2k-1} = 2f_n, f_1 = 1$$

$$y_{2k-1} = 2g_n + 1, g_1 = 0$$

From (1) and (2)

$$x_{k+2} = 2x_{k+1} + 3y_{k+1}$$
$$= 2(2x_k + 3y_k) + 3(2y_k + x_k)$$

$$= 7x_k + 12y_k$$

and $$y_{k+2} = 2y_{k+1} + x_{k+1}$$
$$= 2(2y_k + x_k) + (2x_k + 3y_k)$$
$$= 4x_k + 7y_k$$
$$\Rightarrow 2f_{n+1} = 7 \cdot 2f_n + 12 \cdot (2g_n + 1)$$

and $$2g_{n+1} + 1 = 4(2f_n) + 7(2g_n + 1)$$
$$\Rightarrow f_{n+1} = 7f_n + 12g_n + 6$$

and $$g_{n+1} = 4f_n + 7g_n + 3$$

Thus f_n and g_n are exactly equal to a_n and b_n respectively.

Now $(2a_n)^2 - 3(2b_{n+1})^2 = 1$

$$\Rightarrow \quad a_n^2 = 3b_n^2 + 3b_n + 1 = (b_{n+1})^3 - b_n^3$$

Hence proved.

Check Your Understanding

1. Solve the following recurrence relation.
   (a) $a_n = 4a_{n-1} - 3 \cdot 2^n$, $n \ge 1$, $a_0 = 1$
   (b) $a_n = 3a_{n-1} + 2 - 2n^2$, $n \ge 1$, $a_0 = 3$
   (c) $a_n = 6a_{n-1} - 9a_{n-2} + 2^n$, $n \ge 2$, $a_0 = 1$, $a_1 = 4$
2. The function f is given by the table

X	1	2	3	4	5
$f(x)$	4	1	3	5	2

   If $a_0 = 4$ and $a_{n+1} = f(a_n)$ then find a_{2017}.
3. Let $a_{n+1} = \dfrac{a_n^2 + 1}{2a_n + 1}$. Prove that $a_n < -\dfrac{1}{2} \Leftrightarrow a_{n+1} < -\dfrac{1}{2}$
4. Let $a_n = n(a_1 + a_2 + \cdots + a_{n-1})$ $\forall n \ge 2$ $a_1 = 1$. Find a_n
5. Let a_n be a real sequence $a_1 = 1$ and $a_n = \dfrac{n+1}{n-1}$ $(a_1 + a_2 + a_3 + \cdots + a_{n-1})$, $n \ge 2$. Find a_{2017}.
6. Let $\{a_n\}$ be a sequence such that, $a_1 = \dfrac{1}{2}$, $a_1 + a_2 + \cdots + a_n = n^2 a_n$.

[CMO, 1985]

7. Find the n^{th} term of the sequence $\{a_n\}$ such that
$$a_1 = 1,\ na_n = (n-1)\sum_{k=1}^{n} a_k \quad (n = 2, 3, \ldots)$$
8. Let $a_1 = 0$, $a_{n+1} = -\dfrac{1}{2} + \dfrac{1}{n(n+1)}\displaystyle\sum_{k=1}^{n} ka_k$, $n \in \mathbb{N}$. Find a_n.
9. Let $a_1 = 1$, $a_n = a_{n-1} + 1 + \displaystyle\sum_{k=1}^{n-1} a_k$ $\forall\ n > 1$. Find a_n
10. Let $(n + 2)(n + 1)a_{n+2} + (n + 1)\, a_{n+1} - a_n = 0\ \forall\ n \ge 1$
    $a_1 = 2$, $a_2 = 1$, find a_n
11. Let $x_n = 2\, x_n^{\ 2} - 1$, $n \ge 0$, $-1 \le x_0 \le 1$,Find x_n
12. Let $n(n - 1)a_n = (n - 1)\,(n - 2)\, a_{n-1} - (n - 2)\,(n - 3)\, a_{n-2}$, $n \ge 2$
    $a_0 = \alpha$, $a_1 = \beta$. Find a_n

13. Let $n(n-1)a_n = (n-1)(n-2)a_{n-1} + (n-3)a_{n-2}$, $n \geq 2$; $a_0 = \alpha$, $a_1 = \beta$. Find a_n

14. Find the n^{th} term of the sequence $\{a_n\}$ such that

$\sum_{k=1}^{n} a_k = 3n^2 + 4n + 2 (n = 1, 2, 3, \ldots)$ and calculate $\sum_{k=1}^{n} a_k^2$.

15. Find the n^{th} term of the sequence $\{a_n\}$ such that

$a_1 = 0, a_2 = 1, (n-1)^2 a_n = \sum_{k=1}^{n} a_k \ (n \geq 1)$.

16. Let a_n be the n^{th} term of the arithmetic sequence with $a_1 = 7$, the common difference 2, and b_n be the n^{th} term of the geometric sequence with $b_1 = \frac{1}{3}$, the common ratio $\frac{1}{3}$. For the sequence $\{c_n\}$, if $\sum_{k=1}^{n} a_k b_k c_k = \frac{1}{3}(n+1)(n+2)(n+3)$ holds, then find c_n and evaluate $\sum_{n=1}^{\infty} \frac{1}{c_n}$.

17. Let $a_n = 2\dfrac{a_{n-1}^3}{a_{n-2}^2}, n \geq 2,\ a_0 = 2, a_1 \geq 2$. Find a_n.

18. Find the n^{th} term of the positive sequence $\{a_n\}$ such that
$a_1 = 1, a_2 = 10, a_n^{\ 2} a_{n-2} = a^3_{\ n-1}$ $(n = 1, 2, 3, \ldots)$.

19. Let $a_n = \frac{n}{2} a_{\frac{n}{2}} + 2$, $a_1 = 1$. Find a_n

20. Let $a_1 = 1$. $a_n = a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n,\ n > 1$. Find $a(n)$

Challenge Your Understanding

1. If $D_n = (n-1)(D_{n-1} + D_{n-2}), n \geq 3, D_1 = 0, D_2 = 1,$ then prove that,

$D_n = nD_{n-1} + (-1)^n; n \geq 2, D_1 = 0$

And hence or otherwise prove that $D_n = n!\sum_{r=1}^{n} \frac{(-1)^r}{r!}$.

2. Let $x_{n+1} = 2x_n - 5x_n^{\ 2}$. Find x_n in terms of x_0

3. Find the n^{th} term of the sequence $\{a_n\}$ such that

$a_1 = \frac{3}{2}$, $a_{n+1} = 2a_n(a_n + 1)$ $(n \geq 1)$.

4. The operation $\otimes$ which makes two non zero integers m, n correspond to the integers $m \otimes n$ satisfies the following three conditions.

(a) $0 \otimes n = n + 1$

(b) $m \otimes 0 = m + 1$

(c) $m \otimes n = (m-1) \otimes (m \otimes (n-1)),\ (m \geq 1, n \geq 1)$.

Evaluate the following $1 \otimes n,\ 2 \otimes n,\ 3 \otimes n$

5. Let $a_0 = 0,\ a_1 = 1,\ (n^2 - n)a_n - (n-2)^2 a_{n-2} = 0\ \forall\, n \geq 2$. Find a_n.

6. Let $\{a_n\}$ be the sequence defined as follows $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 4a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$

(a) Prove that $a_n^2 - a_{n-1}a_{n+1} = 1\ \forall\, n \geq 1$.

(b) Evaluate $\sum_{k=1}^{\infty} \arctan\left(\frac{1}{4a_k^2}\right)$.

7. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$ and $n \geq 3$. $T_n = (n + 4)\, T_{n-1} - 4n\, T_{n-2} + (4n - 8)\, T_{n-3}$. Find T_n?

8. Find the n^{th} term of the sequence $\{a_n\}$ such that

$$a_1 = a,\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right)(n \geq 1).$$

9. The sequence $a_0, a_1, a_2, \ldots$ Satisfies $a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})\ \forall\ m, n \in \mathbb{N}_0$ with $m \geq n$. If $a_1 = 1$ find a_{2017}.

10. Let $a_n = 5a_{n-1} + 29b_{n-1}$, $n \geq 2$, $b_n = a_{n-1} + 5b_{n-1}$, $n \geq 2$, $a_1 = 5$, and $b_1 = 1$. Find a_n and b_n.

11. Let $p_{n+1} = -p_n - 6q_n$, $n \geq 1$, $q_{n+1} = p_n + 4q_n$, $n \geq 1$, $p_1 = 4$, and $q_1 = -1$. Find p_n and q_n

12. Solve the system of recurrence relations $a_{n+1} = a_n - b_n$ and $b_{n+1} = a_n + 3b_n$

Given $a_0 = -1, b_0 = 5$.

13. The sequence $\{a_n\}$ is given by $a_0 = 3$, $a_n = 2 + a_0 a_1 \cdots a_{n-1}\quad \forall n \geq 1$.

(i) Prove that any two term of $\{a_n\}$ are relatively prime.

(ii) Find a_{2007}. **[Croatia MO, 2007]**

14. Let $x_1 = 1$, $x_n^2 + 1 = (n + 1)\, x_{n+1}^2\ \forall\ n \geq 1$. Find x_n

15. Let $P_0(x)$, $P_1(x)$, $P_2(x), \ldots$ are polynomial in 'x' such that $P_0(x) = 0$, $P_1(x) = x - 2017$ and $P_n(x) = (x - 2017)\, P_{n-1}(x) + (2018 - x)\, P_{n-2}(x)\ \forall\ n \geq 2$. Find $P_n(x)$

16. Consider $a_{n+2}\, a_n = a_{n+1}^2 + 2$, $n \geq 1$, $a_1 = a_2 = 1$. Prove the following:

(i) $a_n \in \mathbb{Z}$

(ii) a_n is an odd number $\forall n \in \mathbb{N}$

(iii) set $\{a_n, a_{n+1}, a_{n+2}\}$ is pairwise coprime $\forall n \in \mathbb{N}$

17. Let $a_1 = 1$, $a_2 = 7$ and $a_{n+2} = \dfrac{a_{n+1}^2 - 7}{a_n}\quad \forall\ n \geq 1$.

Prove that $9a_n\, a_{n+1} + 1$ is a perfect square $\forall\ n \in \mathbb{N}$.

18. The sequence $\{x_n\}$ is defined by $x_1 = a$, $x_2 = b$, $x_{n+2} = 2008x_{n+1} - x_n$. Prove that there exist a, b such that $1 + 2006x_{n+1}x_n$ is a perfect square for all $n \in \mathbb{N}$.

[Turkey MO, 2008]

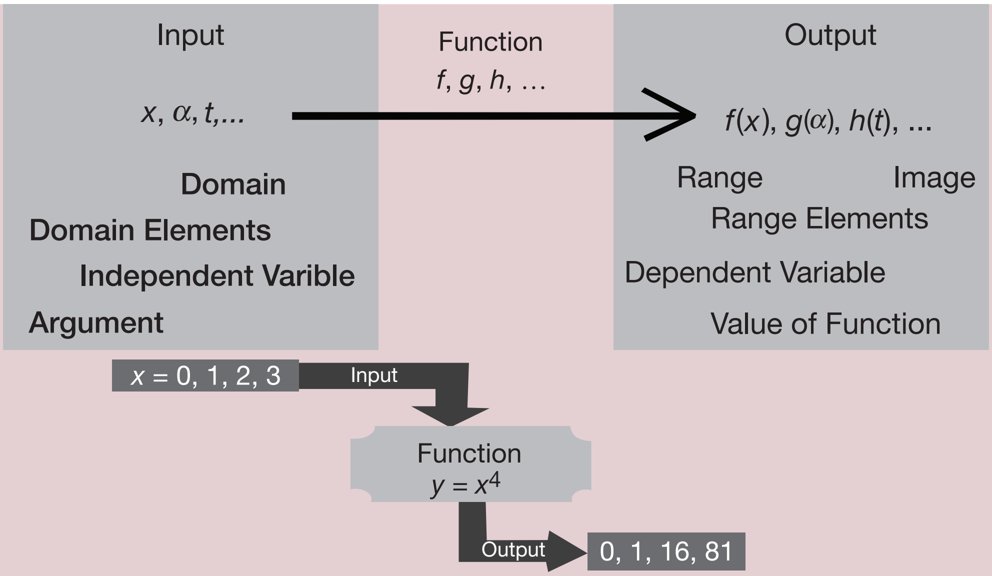
19. The sequence x_n is defined by $x_1 = 2$, $x_{n+1} = \dfrac{2 + x_n}{1 - 2x_n}$, $n = 1, 2, 3, \ldots$.

Prove that $x_n \neq \frac{1}{2}$ or 0 for all n and the terms of the sequence are all distinct.

20. The sequence $\{a_n\}$ of integers is defined by $-\frac{1}{2} \leq a_{n+1} - \dfrac{a_n^2}{a_{n-1}} \leq \frac{1}{2}$

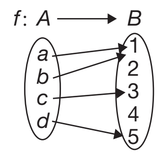
with $a_1 = 2$, $a_2 = 7$, prove that a_n is odd for all values of $n \geq 2$. **[BMO, 1988]**

Chapter 5



Functional Equations

5.1 Function



A function f is a rule 'f' that assigns to each element x of its domain of definition one definite value $f(x)$ belonging to its co-domain.

Formally,

A function f from A to B is a subset of Cartesian product $A \times B$ subject to the following condition:

Every element of A is the first component of one and only one ordered pair in the subset. In other words, for every $x \in A$, there is exactly one element y such that the ordered pair (x, y) is contained in the subset defining the function f.

The expression $f : A \to B$ means f is a function that has domain A and co-domain B or f is a function from A to B.

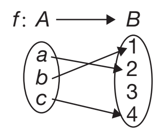
Usually (but not always) function is denoted with an expressions such as,

$f : A \to B$

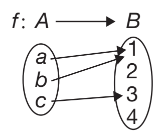
$f(x) = \text{Expression}$

where x is an argument of the function belongs to A and $f(x)$ is a value or f image of the function belongs to B.

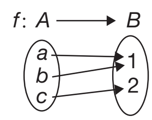
Collection of all f image, is called range of the function. It is always a subset of co-domain (*i.e.*, B here)

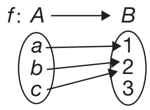

One to one

5.1.1 Some Properties of Function

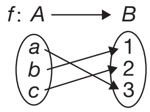

Many to one

1. **One to one function (Injective function):** If $f(a) = f(b) \Rightarrow a = b$, then f is called Injective or one to one function. In other words no value in co-domain may be taken by $f(x)$ more than once.
2. **Many to one function:** If for atleast one a, b such that $a \neq b$, $f(a) = f(b)$, then f is called many to one function.
3. **Onto function (Surjective function):** If range of the function is equal to co-domain of the function, then function is called onto. In other words for every $b \in$ Co-domain, there exist $a \in$ Domain such that $f(a) = b$.


Onto function



Into function



Bijective function

4. **Into function:** If range is a proper subset of co-domain then function is called into.
5. **Bijective function:** If f is injective as well as subjective, then f is called bijective function.
6. **Increasing function:** $f(x)$ is called increasing function (or non-decreasing function) over I, If
   $\forall\ a, b \in I, a < b \Rightarrow f(a) \le f(b)$.
7. **Strictly increasing function:** $f(x)$ is called strictly increasing function over I, if
   $\forall\ a, b \in I, a < b \Leftrightarrow f(a) < f(b)$
8. **Decreasing function:** $f(x)$ is called decreasing function (or non-increasing function) over I, If
   $\forall\ a, b \in I, a < b \Rightarrow f(a) \ge f(b)$.
9. **Strictly decreasing function:** $f(x)$ is called strictly decreasing function over I, If
   $\forall\ a, b \in I, a < b \Leftrightarrow f(a) > f(b)$
10. **Monotonic function:** If f is either increasing or decreasing then it is monotonic.
11. **Strictly monotonic function:** If f is either strictly increasing or strictly decreasing then it is called strictly monotonic.
12. **Even/odd function:** If Domain is symmetric about 'O (origin)' that is $x \in$ Domain $\Leftrightarrow -x \in D$ then we can define $f(x)$ even function, if $f(-x) = f(x)\ \forall\ x \in$ Domain and an odd function if $f(-x) = -f(x)\ \forall\ x \in$ Domain.
13. **Periodic function:** If $f(x + T) = f(x)\ \forall\ x \in$ Domain then f is called periodic, where T is a fixed positive real number independent of 'x'. Least positive T (if it exist) called fundamental period of f.
14. **Fixed point of function:** If $f(a) = a$ for some a belongs to domain then a is called a fixed point of the function.
15. **Identity function:** If $f(x) = x\ \forall\ x \in$ Domain, then f is called an Identity function.
16. **Self invertiable or involutory function:** If $f: A \to A$ has the property that $f(f(x)) = x$ for all $x \in A$, then f is called an involution on A or an involutory function. Involutory function are very special function. If $f: A$ to A is an involutory function then A can be partitioned as the union of sets A_i, such that each A_i has either one or two elements, and f swaps the two elements (if there are two) or maps the element to itself (if there is only one).

5.1.2 Continuity of a Function

Intuitively a continuous function is function whose graph does not 'breakup'. But one should only view this definition informal. Formally, $f(x)$ is continuous at $a \in A$. If $f(x)$ approaches $f(a)$ as x approaches a. In mathematical notation, this can be written as $\lim_{x\to a} f(x) = f(a)$. More intuitively, we can say that if we want to get all the $f(x)$ values to stay in some small neighbourhood around $f(x_0)$, we simply need to choose a small enough neighbourhood for the x values around x_0. If we can do that no matter how small the $f(x)$ neighbourhood is, then f is continuous at x_0. In mathematical notation:

f is a continuous at $x = a \Leftrightarrow$

$\forall\ \varepsilon > 0\ \exists\ a\ \delta > 0$ such that $\forall\ x \in D_f : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Read it "for all epsilon > 0 there exist a delta > 0 such that …"

Note: $f(x)$ is continuous over A, if it is continuous at every $a \in A$.

5.1.2.1 Intermediate Value Theorem

Let f be continuous over $[a, b]$. Then for every λ lying between $f(a)$ and $f(b)$ (including $f(a)$ and $f(b)$), there exist atleast one $c \in [a, b]$ such that $f(c) = \lambda$.

5.2 Functional Equation

A functional equation is an equation whose variables are ranging over functions and our aim is to find all possible functions satisfying the equation.

There is no fixed method to solve a functional equation few standard approaches as follows:

5.2.1 Substitution of Variable/Function

This is most common method for solving functional equations. By substitution we get simplified form or some time some additional information regarding equation. We replace old variable with new variable by keeping domain of old variable unchanged. See the following examples:

Charles Babbage

26 Dec 1791–18 Oct 1871
Nationality: British

Example 1 *Let* $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be such that* $f\left(1+\frac{1}{x}\right)=x^2+\frac{1}{x^2}\quad\forall x\in\mathbb{R}\setminus\{0\}$, *find* $f(x)$.

Solution: Let $y=1+\frac{1}{x}\Rightarrow x=\frac{1}{y-1}$

$$\Rightarrow\quad f(y)=\left(\frac{1}{y-1}\right)^2+(y-1)^2\quad\forall y\in\mathbb{R}-\{1\}.$$

Example 2 *Let p, q be fixed non-zero real numbers. Find all function* $f:\mathbb{R}\to\mathbb{R}$ *such that* $f\left(x-\frac{q}{p}\right)+2x\le\frac{p}{q}x^2+\frac{2q}{p}\le f\left(x+\frac{q}{p}\right)-2x\,\forall x\in\mathbb{R}$.

Solution: Substitute $x-\frac{q}{p}=y$ in left inequality, we get $f(y)\le\frac{p}{q}y^2+\frac{q}{p}$ (1)

Similarly substituting $x+\frac{q}{p}=y$ in right inequality, we get $f(y)\ge\frac{p}{q}y^2+\frac{q}{p}$ (2)

From Inequations (1) and (2), we get

$$f(y)=\frac{p}{q}y^2+\frac{q}{p}\quad\forall y\in\mathbb{R}.$$

Example 3 $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *such that* $2f(x)+3f\left(\frac{1}{x}\right)=x\ \forall x\in\mathbb{R}\setminus\{0\}$, *find* $f(x)$.

Solution: Replace x by $\frac{1}{x}$, we get $2f\left(\frac{1}{x}\right)+3f(x)=\frac{1}{x}$

Now by eliminating $f\left(\frac{1}{x}\right)$ from the two equations, we get

$$(9-4)f(x)=\frac{3}{x}-2x$$

$$\Rightarrow f(x)=\frac{3-2x^2}{5x}.$$

Example 4 *Find all functions* $f:\mathbb{R}\to\mathbb{R}$ *such that* $x^2f(x)+f(1-x)=2x-x^4\ \forall x\in\mathbb{R}$.

Solution: Replace x by $(1-x)$, we get

$$(1-x)^2f(1-x)+f(x)=2(1-x)-(1-x)^4$$

Now eliminating $f(1-x)$ from the two equations, we get $f(x)=1-x^2$.

Example 5 $f:\mathbb{R}-\{0,1\}\to\mathbb{R},\ f(x)+f\left(\frac{x-1}{x}\right)=1+x$ *find* $f(x)$

Solution: Replacing x by $\frac{x-1}{x}$, we get

$$f\left(\frac{x-1}{x}\right)+f\left(\frac{\frac{x-1}{x}-1}{\frac{x-1}{x}}\right)=1+\frac{x-1}{x}$$

or $f\left(\frac{x-1}{x}\right)+f\left(\frac{1}{1-x}\right)=\frac{2x-1}{x}$ (1)

again replacing x by $\frac{1}{1-x}$ in parent equation, we get

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{\frac{1}{1-x}-1}{\frac{1}{1-x}}\right)=1+\frac{1}{1-x}=\frac{2-x}{1-x}$$

$f\left(\frac{1}{1-x}\right)+f(x)=\frac{2-x}{1-x}$ (2)

By adding parent equation + Eq. (2) and subtracting Eq. (1), we get

$$2f(x)=1+x+\frac{2-x}{1-x}-\frac{2x-1}{x}\quad\Rightarrow\quad f(x)=\frac{x^3-x^2-1}{2x(x-1)}$$

5.2.2 Isolation of Variables

We try to bring all functions of x to one side and all functions of y on other side. For some particular type of problems this works wonderfully. See the following examples:

Example 6: *Find* $f(x)$ *such that* $xf(y)=yf(x)\ \forall x, y\in\mathbb{R}-\{0\}$.

Solution: $x\,f(y)=y\,f(x)$

$$\Rightarrow \frac{f(x)}{x}=\frac{f(y)}{y}$$

as x, y are independent of each other

$$\Rightarrow \frac{f(x)}{x}=\text{Constant}=c$$

$\Rightarrow f(x)=cx.$

Example 7 *If* $(x-y)\,f(x+y)-(x+y)\,f(x-y)=4xy(x^2-y^2)\ \forall\ x, y\in\mathbb{R}$, *find* $f(x)$.

Solution: Given equation is equivalent to

$$\frac{f(x+y)}{x+y}-\frac{f(x-y)}{x-y}=4xy$$

$$=(x+y)^2+(x-y)^2$$

$$\Rightarrow \frac{f(x+y)}{x+y}-(x+y)^2=\frac{f(x-y)}{x-y}-(x-y)^2$$

$$\Rightarrow \frac{f(t)}{t}-t^2 \text{ is constant}$$

Let $\dfrac{f(x)}{x} - x^2 = c \Rightarrow f(x) = x^2 + cx.$

which satisfies the parent equation.

Build-up Your Understanding 1

1. Find $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$, such that
$$f\left(\frac{x}{x-1}\right) = 2f\left(\frac{x-1}{x}\right) \forall x \in \mathbb{R} \setminus \{0, 1\}.$$
2. Find $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$, such that
$$f(x) + f\left(\frac{1}{1-x}\right) = x \ \forall x \in \mathbb{R}\backslash\{0, 1\}.$$
3. $f(x^2 + x) + 2f(x^2 - 3x + 2) = 9x^2 - 15x \ \forall \ x \in \mathbb{R}$, find $f(2016)$.
4. Find $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) + xf(1 - x) = 1 + x \ \forall \ x \in \mathbb{R}$.
5. $f : \mathbb{R} \to \mathbb{R}$, $f(x + y) + f(x - y) = 2f(x) \cos y \ \forall \ x, y \in \mathbb{R}$, find all such functions.
6. Find all functions $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$, such that
$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)} \forall x \in \mathbb{R} \setminus \{0, 1\}.$$
7. Find all functions $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$, such that
$$f(x) + 2f\left(\frac{1}{x}\right) + 3f\left(\frac{x}{x-1}\right) = x.$$

5.2.3 Evaluation of Function at Some Point of Domain

We try to determine the unknown function at points 0, 1, –1, etc, which is mostly crucial to simplify the complex functional equation. Observe the following examples:

Example 8 *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*
$f(xf(x) + f(y)) = (f(x))^2 + y \ \forall \ x, y \in \mathbb{R}$.

Solution: Let $P(x, y) : f(xf(x) + f(y)) = (f(x))^2 + y$
$P(0, x) : f(f(x)) = (f(0))^2 + x$
Let $f(0) = a \Rightarrow f(f(x)) = x + a^2$ (1)
at $x = -a^2$, $f(f(-a^2)) = 0$
Let $f(-a^2) = b$
$\Rightarrow f(b) = 0$
$P(b, b) : f(bf(b) + f(b)) = (f(b))^2 + b$
$\Rightarrow f(b(0) + 0) = 0^2 + b$
$\Rightarrow f(0) = b$
Also $P(0, b) : f(0 \cdot f(0) + f(b)) = (f(0))^2 + b$
$f(0) = (f(0))^2 + b$
$\Rightarrow (f(0))^2 = 0$ (as $f(0) = b$)

$\Rightarrow f(0) = 0 \Rightarrow a = 0$

From Eq. (1), we get $f(f(x)) = x \ \forall \ x \in \mathbb{R}$ (2)
Also from $P(x, 0) : f(xf(x)) = (f(x))^2$ (3)
Replace x by $f(x)$ in Eq. (3)

We get, $f(f(x)\cdot f(x)) = (f(f(x)))^2$

$\Rightarrow f(f(x)\,x) = x^2$ (from Eq. (2)) (4)

From Eqs. (3) and (4), we get

$(f(x))^2 = x^2$

$\Rightarrow f(x) = x$ or $-x$

Now we will prove either $f(x) = x\ \forall\ x \in \mathbb{R}$ or $f(x) = -x\ \forall\ x \in \mathbb{R}$.

If possible let $f(x_1) = x_1$ and $f(x_2) = -x_2$, $x_1 \neq x_2$

$P(x_1, x_2) : f(x_1 f(x_1) + f(x_2)) = (f(x_1))^2 + x_2$

$f\left(x_1^2 - x_2\right) = x_1^2 + x_2$

$\Rightarrow \pm\left(x_1^2 - x_2\right) = x_1^2 + x_2$

+ve, $x_1{}^2 + x_2 = x_1{}^2 + x_2 \Rightarrow x_2 = 0$

−ve, $-x_1{}^2 + x_2 = x_1{}^2 + x_2 \Rightarrow x_1{}^2 = 0$

$\Rightarrow x_1 = 0$

Hence either $f(x) = x\ \forall\ x \in \mathbb{R}$

or, $f(x) = -x\ \forall\ x \in \mathbb{R}$.

Example 9 $f : \mathbb{N}_0 \to \mathbb{N}_0$ *such that* $f(x^2 + f(y)) = x\,f(x) + y\ \forall\ x, y \in \mathbb{N}_0$.

Solution: $P(x, y) : f(x^2 + f(y)) = x\,f(x) + y$

$P(0, x) : f(f(x)) = x\ \forall\ x \in \mathbb{N}_0$ (1)

$P(1, 0) : f(1 + f(0)) = f(1)$ (2)

$\Rightarrow f(f(1 + f(0))) = f(f(1))$ (taking f on both side of Eq. (2))

$\Rightarrow 1 + f(0) = 1$ (using Eq. (1))

$\Rightarrow f(0) = 0$

$P(1, f(x)) : f(1^2 + f(f(x))) = 1\cdot f(1) + f(x)$

$\Rightarrow f(1 + x) = a + f(x)$ (Let $f(1) = a$)

$f(x + 1) - f(x) = a$ (3)

Plugging $x = 0, 1, 2, \ldots, n - 1$ in Eq. (3) and adding all, we get

$f(n) = na \forall n \in \mathbb{N}_0$

Checking it in parent equation, we get

$a(x^2 + ay) = ax^2 + y$

$\Rightarrow a^2\, y = y \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$

But $a = -1$, not possible as co-domain $= \mathbb{N}_0$.

$\Rightarrow f(n) = n.$

Example 10 *Find all functions* $f : \mathbb{R} \to \mathbb{R}$, *such that* $f(f(x + y)) = f(x + y) + f(x)\cdot f(y) - x\,y\ \forall\ x, y \in \mathbb{R}$.

Solution: $P(x, y) : f(f(x + y)) = f(x + y) + f(x)\cdot f(y) - xy$

$P(x, 0) : f(f(x)) = f(x)\,(1 + f(0))$

Let $f(x) = t$

$\Rightarrow f(t) = (1 + f(0))t$ (1)

When $t \in$ image set of f

$\Rightarrow f(f(x + y)) = (1 + f(0))\,f(x + y)$

$\Rightarrow f(x + y) + f(x)\cdot f(y) - xy = (1 + f(0))\,f(x + y)$

$\Rightarrow f(x)\cdot f(y) - xy = f(0)\cdot f(x + y)$ (2)

Let $f(0) = a$, $x = -a$ and $y = a$ in Eq. (2)
$f(a) \cdot f(-a) + a^2 = a^2$
$\Rightarrow f(a) \cdot f(-a) = 0$
$\Rightarrow 0 \in I_m(f)$
From Eq. (1), we get
$f(0) = (1 + f(0)) \cdot 0 = 0$
Using this in Eq. (2), we get
$f(x) \cdot f(y) = xy$
$\Rightarrow (f(1))^2 = 1 \Rightarrow f(1) = \pm 1$
$\Rightarrow f(x) = x$ or $-x$
But $f(x) = x$ only satisfy the parent equation.

5.2.4 Application of Properties of the Function

Sometime investigating for injectivity or surjectivity of function involved in the equation is very useful in order to determine it. Sometime identifying function as monotonous reduces the complexity of the problem at great length. See the following examples:

Example 11 *Let $f: \mathbb{Z} \to \mathbb{Z}$ satisfies*
$f(f(n) + 2) = n \ \forall\ n \in \mathbb{Z}, f(1) = 0$ *find* $f(n)$.

Solution: Let $f(n) + 2 = g(n)$
$\Rightarrow f(g(n)) = n$
as $f \circ g$ is one to one and onto function, g is one to one and f must be onto. As $g(n) = f(n) + 2 \Rightarrow f$ is one to one function and $g(n)$ is onto also $\Rightarrow f$ and g are inverse of each other.
As $f(1) = 0 \Rightarrow g(0) = 1 \Rightarrow f(0) + 2 = g(0) = 1$
$\Rightarrow f(0) = -1$
from $f(n) + 2 = g(n)$, we get
$f(f(n)) + 2 = g(f(n)) = n$
$\Rightarrow n = f(f(n)) + 2$
Replacing n by $f(n + 2)$, we get
$f(n + 2) = f(f(f(n + 2))) + 2$
$= f(n + 2 - 2) + 2 \quad (\text{as } f(f(n)) = n - 2)$
$\Rightarrow f(n + 2) = f(n) + 2$
$\Rightarrow f(n + 2) - f(n) = 2$
using telescoping sum we get
$\Rightarrow f(n) = n - 1 \quad (\text{as } f(0) = -1, f(1) = 0)$

Example 12 *Find all functions $f: \mathbb{R} \to \mathbb{R}$ that have the following two properties:*
$f(f(x)) = x \ \forall\ x \in \mathbb{R}$ *and* $x \geq y$ *then* $f(x) \geq f(y)$.

Solution: Fix any number $x \in \mathbb{R}$ and Let $y = f(x)$.
From first property $f(y) = x$
Let $x \neq y, \Rightarrow x < y$ or $x > y$

Case 1: $x < y \Rightarrow f(x) \leq f(y)$
$\Rightarrow y \leq x$ contradiction

Case 2: $y < x \Rightarrow f(y) \leq f(x)$
$\Rightarrow x \leq y$ contradiction
Hence $x = y \Rightarrow f(x) = x \ \forall\ x \in \mathbb{R}$.

Example 13 *Prove that there is no function* $f: \mathbb{N}_0 \to \mathbb{N}_0$ *such that* $f(f(n)) = n + 1987$ **[IMO, 1987]**

Solution: f must be injective (if it exists)

Let $x \neq y, f(x) = f(y)$

$\Rightarrow f(f(x)) = f(f(y))$

$\Rightarrow x + 1987 = y + 1997$

$\Rightarrow x = y$ Contradiction.

$\Rightarrow f$ must be injective.

Let $f(n)$ misses exactly k distinct values $C_1, C_2, \ldots, C_k$ in $\mathbb{N}_0$, *i.e.*, $f(n) \neq C_1, C_2, \ldots, C_k$ $\forall\, n \in \mathbb{N}_0$, then $f(f(n))$ misses the $2k$ distinct values $C_1, C_2, \ldots, C_k$ and $f(C_1), f(C_2), \ldots, f(C_k)$ in $\mathbb{N}_0$ (No two $f(C_i)$ is equal as f is one to one function). Let $y \in \mathbb{N}_0$ and $y \neq C_1, C_2, \ldots, C_k, f(C_1), f(C_2), \ldots, f(C_k)$, then there exist $x \in \mathbb{N}_0$ such that $f(x) = y$. Since $y \neq f(C_j), x \neq C_j$, so there is $n \in \mathbb{N}_0$ such that $f(n) = x$, then $f(f(n)) = y$.

This implies $f(f(n))$ misses only the $2k$ values $C_1, C_2, \ldots, C_k, f(C_1), f(C_2), \ldots, f(C_k)$ and no others since $n + 1987$ misses the 1987 values $0, 1, \ldots, 1986$ and $2k \neq 1987$ this is a contradiction.

5.2.5 Application of Mathematical Induction

Many functional equation on natural number or on integer can be solved using induction, sometimes it is also applicable in case of rational numbers. See the following examples:

Example 14 *Let* $f: \mathbb{N} \to \mathbb{N}$ *be a function such that* $f(n + 1) > f(f(n))\ \forall n \in \mathbb{N}$. *Prove that* $f(n) = n\ \forall\ n \in \mathbb{N}$.

Solution: Our claim is $f(1) < f(2) < f(3) < \ldots$. This follows if we can show that, for every $n > 1, f(n)$ is the unique smallest element of $\{f(n), f(n + 1)\ , f(n + 1), \ldots\}$.

Let us apply introduction on n.

Firstly for $m \geq 2, f(m) \geq f(f(m - 1))$. Since $f(m - 1) \in \{1, 2, 3, \ldots\}$, this mean that $f(m)$ cannot be the smallest of $\{f(1), f(2), f(3), \ldots\}$.

Since $\{f(1), f(2), \ldots\}$ is bounded below by 1, it follows that $f(1)$ must be the unique smallest element of $\{f(1), f(2), f(3), \ldots\}$.

Now suppose that $f(n)$ is the smallest of $\{f(n), f(n + 1), \ldots\}$. Let $m > n + 1$. By the induction hypothesis, $f(m - 1) > f(n)$. Since $f(n) > f(n - 1) > \ldots > f(1) \geq 1$, we have $f(n) \geq n$ and so $f(m - 1) \geq n + 1$, so $f(m - 1) \in \{n + 1, n + 2, \ldots\}$.

But $f(m) > f(f(m - 1))$, so $f(m)$ is not smallest in $\{f(n + 1), f(n + 2), \ldots\}$. Since $\{f(n + 1), f(n + 2), \ldots\}$ is bounded below, it follows that $f(n + 1)$ is the unique smallest element of $\{f(n + 1), f(n + 2), \ldots\}$.

Now since, $1 \leq f(1) < f(2) < f(3) < \ldots$, clearly we have $f(n) \geq n\ \forall\ n \in \mathbb{N}$. But if $f(n) > n$ for some n, then $f(f(n)) > f(n + 1)$ a contradiction. Hence $f(n) = n\ \forall\ \mathbb{N}$.

Example 15 *Let* $f: \mathbb{Q} \to \mathbb{Q}$, *such that* $f(1) = 2$ *and* $f(xy) = f(x) \cdot f(y) - f(x + y) + 1$, *find* $f(x)$.

Solution: Putting $y = 1$, then

$f(x) = f(x) \cdot f(1) - f(x + 1) + 1$

$= 2f(x) - f(x + 1) + 1$

$\Rightarrow f(x + 1) = f(x) + 1$

Therefore by applying condition $f(1) = 2$ and by mathematical induction, for all integer n, we have $f(x) = x + 1$.

For any rational number, let $x = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$, putting $x = \frac{m}{n}$, $y = n$ then

$$f\left(\frac{m}{n}\cdot n\right) = f\left(\frac{m}{n}\right)\cdot f(n) - f\left(\frac{m}{n}+n\right)+1$$

$$f(m) = f\left(\frac{m}{n}\right)(n+1) - f\left(\frac{m}{n}+n\right)+1$$

$$m+1 = f\left(\frac{m}{n}\right)(n+1) - f\left(\frac{m}{n}\right)-n+1 \qquad (\text{as } f(x+1) = f(x)+1\ \forall\, x \in \mathbb{Q})$$

$$\Rightarrow nf\left(\frac{m}{n}\right) = n+m$$

or $f\left(\frac{m}{n}\right) = 1+\frac{m}{n}$

$\Rightarrow f(x) = x + 1\ \forall\, x \in \mathbb{Q}$.

Build-up Your Understanding 2

1. The function f is defined for all real numbers and satisfies $f(x) \leq x$ and $f(x + y) \leq f(x) + f(y)$ for all real x, y. Prove that $f(x) = x$ for every real number x.
2. Let R denote the real numbers and $f : \mathbb{R} \to [-1, 1]$ satisfy
$$f\left(x+\frac{13}{42}\right)+f(x) = f\left(x+\frac{1}{6}\right)+f\left(x+\frac{1}{7}\right)$$
for every $x \in \mathbb{R}$. Show that f is a periodic function, *i.e.*, there is a non-zero real number T such that $f(x + T) = f(x)$ for every $x \in \mathbb{R}$. **[IMO Shortlisted Problem, 1996]**
3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$f(f(x + y)) = f(x + y) + f(x)\, f(y) - xy$ for all $x, y \in \mathbb{R}$.
4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$f((x + y\, f(x)) = f(x) + xf(y)$ for all x, y in $\mathbb{R}$.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(f(x)) + f(x) + x = 0\ \forall\, x \in \mathbb{R}$. Find all such $f(x)$.

5.2.6 Method of Undetermined Coefficients

It is mostly used when we know that given function is a polynomial then we assume a polynomial with unknown coefficients and using given functional equation we try to get the coefficients. See the following example:

Example 16 *Let f be a polynomial and* $f(x)\cdot f\left(\frac{1}{x}\right) = f(x)+f\left(\frac{1}{x}\right)$ $\forall x \in \mathbb{R} - \{0\}$. *Find 'f'.*

Solution: Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $a_n \neq 0$

Now using given equation we get

$$(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)\left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}\right)$$

$$= (a_0 + a_1x + \cdots + a_nx^n) + \left(a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n}\right)$$

Multiply x^n on both side and clearing the denominators, we get

$(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)(a_0x^n + a_1x^{n-1} + \cdots + a_n)$

$= (a_0x^n + a_1x^{n+1} + \cdots + a_nx^{2n}) + (a_0x^n + a_1x^{n-1} + \cdots + a_n)$

Comparing $[x^{2n}]$ on both side, we get $a_0a_n = a_n \Rightarrow a_0 = 1$ (as $a_n \neq 0$)

Comparing $[x^{2n-1}]$, we get

$a_na_1 + a_{n-1}a_0 = a_{n-1}$

$\Rightarrow a_na_1 + a_{n-1} = a_{n-1}$

$\Rightarrow a_na_1 = 0$

$\Rightarrow a_1 = 0$

Similarly $a_2 = a_3 = a_4 = \ldots = a_{n-1} = 0$

Comparing $[x^n]$, we get

$a_n^2 + a_{n-1}^2 + \cdots + a_0^2 = 2a_0$

$a_n^2 = 1$

$\Rightarrow a_n = \pm 1$

$\Rightarrow f(x) = 1 \pm x^n$

which satisfy the given functional equation.

5.2.7 Using Recurrence Relation

When functional equation involves relation between $f(n)$, $f(f(n))$, $f(f(f(n)))$, etc., then we can use this method effectively. See the following examples:

Example 17 *$f: \mathbb{N}_0 \to \mathbb{R}$ such that $f(n + m) + f(n - m) = f(an)$ $\forall n \geq m$ where a be a positive integer, find f.*

Solution: Plug $m = 0$, $2f(n) = f(an)$

for $n = 0$, $2f(0) = f(0) \Rightarrow f(0) = 0$

Plug $m = 1$, $f(n + 1) + f(n - 1) = f(an) = 2f(n)$ (1)

Let $f(n)$ be a_n $\forall\ n \geq 0$

Then from Eq. (1), we get

$a_{n+1} - 2a_{n+1} + a_{n-1} = 0$

Its characteristic equation

$x^{n+1} - 2x^{n+1} + x^{n-1} = 0, x \neq 0$

$\Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x - 1)^2 = 0 \Rightarrow n = 1, 1$

$a_n = \alpha n + \beta$

Now $a_0 = f(0) = 0 \Rightarrow \beta = 0$

$\Rightarrow a_n = \alpha n$

or $f(n) = \alpha n$

Checking it with parent equation, we get

$\alpha(n + m) + \alpha(n - m) = a\alpha n$

$\Rightarrow \alpha(2 - a)n = 0 \Rightarrow \alpha(2 - a) = 0$

for $a \neq 2$, $\alpha = 0 \Rightarrow f(n) = 0$; for $a = 2$, $f(n) = \alpha n$.

Example 18 *If $f: \mathbb{N} \to \mathbb{N}$ such that $f(f(f(n))) + f(f(n)) + n = 3f(n)$ $\forall n \in \mathbb{N}$, then find f.*

Solution: Replace n by $f(n)$ successively in parent functional equation k times we get

$$\underbrace{fof\ldots of(n)}_{k+3\text{ times}}+\underbrace{fofof\ldots of(n)}_{k+2\text{ times}}+\underbrace{fofofof\ldots of(n)}_{k\text{ times}}=3\,\underbrace{fofo\ldots of(n)}_{k+1\text{ times}} \qquad (1)$$

Let $a_0 = n$ for some fix n and $a_{k+1} = f(a_k)$ $\forall k \geq 0$

$\Rightarrow$ From Eq. (1) we get

$a_{k+3} + a_{k+2} - 3a_{k+1} + a_k = 0$

Its characteristic equation is

$x^{k+3} + x^{k+2} - 3x^{k+1} + x^k = 0, x \neq 0$

or $x^3 + x^2 - 3x + 1 = 0$

$\Rightarrow (x-1)(x^2 + 2x - 1) = 0$

$\Rightarrow x = 1, -1 \pm \sqrt{2}$

$\Rightarrow a_k = c_0 + c_1(-1+\sqrt{2})^k + c_2(-1-\sqrt{2})^k \ \forall\ k \geq 0.$

Observe that $|-1-\sqrt{2}| > 1 > |-1+\sqrt{2}|$

for $c_2 > 0$, $a_{2k+1} \to -\infty$ which is a contradiction

for $c_2 < 0$, $a_{2k+1} \to \infty$ which is again a contradiction as n is fix.

$\Rightarrow c_2 = 0$

$\Rightarrow a_k = c_0 + c_1(\sqrt{2}-1)^k$

Now $a_0 = n \in \mathbb{N}$

$a_1 = f(a_0) = f(n) \in \mathbb{N}$

$a_0 = c_0 + c_1(\sqrt{2}-1)^0$

$a_1 = c_0 + c_1(\sqrt{2}-1)$

$\Rightarrow a_1 - a_0 = c_1(\sqrt{2}-2)$

If $c_1 \neq 0$ then $\sqrt{2} - 2 = \dfrac{a_1 - a_0}{c_1} \in \mathbb{Q}$

which is contradiction

$\Rightarrow c_1 = 0$

$\Rightarrow a_k = c_0$

$\Rightarrow a_1 = a_0$

$\Rightarrow f(n) = n.$

Build-up Your Understanding 3

1. Consider the function $f: [0, 1] \to [0, 1]$ defined by $f(x) = 4x(1-x)$. How many distinct roots does the equation $f^{1992}(x) = x$ have? [where $f^n(x) = f(f^{n-1}(x))$]
2. Prove that there exists a unique function f from the set $\mathbb{R}^+$ of positive real numbers to $\mathbb{R}^+$ such that $f(f(x)) = 6x - f(x)$ and $f(x) > 0$ for all $x > 0$. **[Putnam, 1988]**
3. Let $f(x) = x^2 - 2$ with $x \in [-2, 2]$. Show that the equation $f^n(x) = x$ has 2^n real roots. [where $f^n(x) = f(f^{n-1}(x))$.]
4. Let $\{a_n\}$ be the sequence of real numbers defined by $a_1 = t$ and $a_{n+1} = 4a_n(1-a_n)$, $n \geq 1$. For how many distinct values of t do we have $a_{1998} = 0$?
5. Given the expression

$$P_n(x) = \frac{1}{2^n}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right],$$

prove that $P_n(x)$ satisfies the identity

$$P_n(x) - xP_{n-1}(x) + \frac{1}{4}P_{n-2}(x) = 0,$$

and that $P_n(x)$ is a polynomial in x of degree n.

6. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $3f(2x+1) = f(x) + 5x$.

7. Find all increasing bijections f of $\mathbb{R}$ onto itself that satisfy $f(x) + f^{-1}(x) = 2x$, where f^{-1} is the inverse of f.

8. Find all function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ that satisfy

$$f(x) + f\left(\frac{1}{x}\right) = 1$$

and $f(1+2x) = \dfrac{f(x)}{2}$ for all x in the domain of f.

Augustin-Louis Cauchy

21 Aug 1789–23 May 1857
Nationality: French

5.2.8 Cauchy's Functional Equation

The equation $f : \mathbb{R} \to \mathbb{R}, f(x+y) = f(x) + f(y)\ \forall\ x \in \mathbb{R}$ is called Cauchy's functional equation (or additive function). Observe the Cauchy's step by step method to solve the following functional equation.

Example 19 *$f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x+y) = f(x) + f(y)\ \forall\ x, y \in \mathbb{Q}$, find f.*

Solution: $f(x+y) = f(x) + f(y)$

$\Rightarrow f(x_1 + x_2 + \cdots + x_n) = f(x_1) + f(x_2) + \cdots + f(x_n)$

Let $x_1 = x_2 = x_3 = \cdots = x_n = x$

$\Rightarrow f(nx) = nf(x), n \in \mathbb{N}, x \in \mathbb{Q}$ (1)

also from $x = y = 0$ in parent equation, we get $f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$

Plug $y = -x$ in parent equation, we get

$f(0) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$

$\Rightarrow f$ is odd

Replace x by $-x$ in Eq. (1)

$f(-nx) = nf(-x)$

$= -nf(x)$

Let $-n = k \in \mathbb{Z}^-$

$\Rightarrow f(kx) = k\,f(x)\ \forall\ x \in \mathbb{Q}, \forall\ k \in \mathbb{Z}^-$ (2)

From Eqs. (1) and (2), we get

$f(nx) = nf(x)\ \forall\ x \in \mathbb{Q}, \forall\ n \in \mathbb{Z}$ (3)

Now take $x = \dfrac{p}{q}; p, q \in \mathbb{Z}, q \neq 0$ and $n = q$

$$\Rightarrow f\left(q \cdot \frac{p}{q}\right) = q \cdot f\left(\frac{p}{q}\right)$$

$$\Rightarrow f(p \cdot 1) = q \cdot f\left(\frac{p}{q}\right)$$

$$\Rightarrow pf(1) = qf\left(\frac{p}{q}\right) \quad \text{(From Eq. (3))}$$

$$\Rightarrow f\left(\frac{p}{q}\right) = \frac{p}{q} \cdot f(1) \qquad (1)$$

$\Rightarrow f(x) = ax\ \forall\ x \in \mathbb{Q}$
where $a = f(1)$.

Example 20 *$f : \mathbb{R} \to \mathbb{R}$, such that $f(x + y) = f(x) + f(y)\ \forall\ x, y \in \mathbb{R}$ and f is bounded above on an interval then prove that $f(x) = ax\ \forall\ x \in \mathbb{R}$ where $a \in \mathbb{R}$.*

Solution: In previous example we already proved for additive function
$f(x) = ax\ \forall\ x \in \mathbb{Q}$.
Consider: $g(x) = f(x) - ax$
Now $g : \mathbb{R} \to \mathbb{R}$ is an additive function
$\Rightarrow g(r) = 0\ \forall\ r \in \mathbb{Q}$.
Let f be bounded above on (a, b)
$\Rightarrow g$ will be bounded above on (a, b).
Let $g(x) < M\ \forall\ x \in (a, b)$
Let x' be any arbitrary read number
Consider interval $(a - x', b - x')$
This interval contains, a rational number r.
As $r \in (a - x', b - x') \Rightarrow r + x' \in (a, b)$
Now $g(r + x') = g(r) + g(x')$ (as g is an additive function)
$\Rightarrow g(x') = g(r + x')$ (as $g(r) = 0$)
$\Rightarrow g(x') = g(r + x') < M$
$\Rightarrow g(x') < M$
Hence $g(x) < M\ \forall\ x \in \mathbb{R}$

Now $g(x') = g\left(\frac{1}{n} \cdot nx'\right) = \frac{1}{n} g(nx')$ (as g is an additive function)

$$\Rightarrow g(x') = \frac{1}{n} g(nx') \le \frac{M}{n}$$

Also $g(x') = g\left(-\frac{1}{n}(-nx')\right) = -\frac{1}{n} g(-nx') > -\frac{M}{n}$

Hence for $\forall n \in \mathbb{N}$

$$-\frac{M}{n} \le g(x') \le \frac{M}{n}$$

as $n \to \infty$, we get $g(x') = 0$
$\Rightarrow g(x) = 0\ \forall\ x \in \mathbb{R}$ (as x' is an arbitrary real number)
$\Rightarrow f(x) - ax = 0$
or $f(x) = ax\ \forall\ x \in \mathbb{R}$.

Notes: All the following statements are equivalent:

1. f is bounded above (or bounded below) over an interval and f is additive function.
2. f is increasing (or decreasing) and f is additive function.
3. f is continuous at a point and f is additive function.

We can easily prove that (2) $\to$ (1) and (3) $\to$ (1)
and from first we already got $f(x) = ax\ \forall\ x \in \mathbb{R}$.

Proof of (2) → (1): It is obvious as f is increasing over an interval then f is bounded above over some interval. Similarly for decreasing function bounded below over some interval.

Proof of (3) → (1): Let f be continuous at $a \in \mathbb{R}$.
Then there is $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$
$\Rightarrow \varepsilon - f(a) < f(x) < \varepsilon + f(a)$
$\Rightarrow f$ is bounded over $(a - \delta, a + \delta)$.

5.2.8.1 *Equations Reducible to Cauchy's Equations*

Example 21 $f: (0, \infty) \to \mathbb{R}$, $f(xy) = f(x) + f(y)\ \forall x \in (0, \infty)$ *and f is bounded over some interval, find f.*

Solution: Let $f(x) = g(\ln x)$
$\Rightarrow g(\ln xy) = g(\ln x) + g(\ln x)$
$\Rightarrow g(\ln x + \ln y) = g(\ln x) + g(\ln y)$
or $g(u + v) = g(u) + g(v)$ (where $\ln x = u$, $\ln y = v$)
and also g is bounded above as f bounded above.
$\Rightarrow g(t) = at$
$\Rightarrow f(x) = g(\ln x) = a \cdot \ln x$.

Example 22 $f: \mathbb{R} \to \mathbb{R}$, $f(x + y) = f(x) \cdot f(y)\ \forall\ x, y \in \mathbb{R}$ *and f is bounded below by a positive real number, find f.*

Solution: If there exist some x_0 such that $f(x_0) = 0$, then replace x by $x - x_0$ and $y = x_0$, we get
$f(x - x_0 + x_0) = f(x - x_0) \cdot (x_0)$
$\Rightarrow f(x) = 0\ \forall\ x \in \mathbb{R}$
For other solutions, let $f(x) \neq 0\ \forall\ x \in \mathbb{R}$

replace x by $\dfrac{x}{2}$ and y by $\dfrac{x}{2}$, we get

$$f\left(\frac{x}{2} + \frac{x}{2}\right) = f\left(\frac{x}{2}\right) \cdot \left(\frac{x}{2}\right)$$

$$\Rightarrow f(x) = \left(f\left(\frac{x}{2}\right)\right)^2 > 0$$

$\Rightarrow f(x) > 0\ \forall\ x \in \mathbb{R}$
Also plug $y = 0$ in parent equation

$\Rightarrow f(x + 0) = f(x) \cdot f(0) \Rightarrow f(0) = 1$

Let $g(x) = \ln f(x)$
$\Rightarrow \ln f(x + y) = \ln f(x) + \ln f(y)$
$\Rightarrow g(x + y) = g(x) + g(y)$
g is an additive function and bounded below also as f is bounded below by positive real number.
$\Rightarrow g(x) = ax$
$\Rightarrow \ln (f(x)) = ax$
$\Rightarrow f(x) = e^{ax}$
$\Rightarrow f(x) = b^x\ \forall\ x \in \mathbb{R}$.

Example 23 *Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(xy) = f(x) \cdot f(y)\ \forall\ x \in \mathbb{R}$, find all such 'f'.*

Solution: Such type of function called multiplicative function.

Plug $x = y = 0 \Rightarrow f(0) = f(0)^2$

$\Rightarrow f(0) = 0, 1$

Case 1: $f(0) = 1$

Plug $y = 0 \Rightarrow f(0) = f(x) \cdot f(0)$

$\Rightarrow f(x) = 1\ \forall\ x \in \mathbb{R}$

This is a solution of the equation.

Case 2: $f(0) = 0$

Plug $x = y = 1 \Rightarrow f(1) = (f(1))^2$

$\Rightarrow f(1) = 0$ or 1

Sub-case 1: $f(1) = 0$

Plug $y = 1, f(x) = f(x) \cdot f(1) = 0$

$\Rightarrow f(x) = 0\ \forall\ x \in \mathbb{R}$

This is a solution of the equation.

Sub-case 2: $f(0) = 0$ and $f(1) = 1$

Now $f(x) \neq 0\ \forall\ x \in \mathbb{R}\backslash\{0\}$

Otherwise if at some $x_0 \neq 0, f(x_0) = 0$

then replace x by $\dfrac{x}{x_0}$ and y by x_0

$$\Rightarrow f\left(\frac{x}{x_0} \cdot x_0\right) = f\left(\frac{x}{x_0}\right) \cdot f(x_0) = 0$$

$\Rightarrow f(x) = 0 \Rightarrow f(1) = 0$

Which is contradiction

Now plug $x = y = -1$

$\Rightarrow f((-1)x(-1)) = f(-1) \cdot f(-1)$

$\Rightarrow 1 = (f(-1))^2$

$\Rightarrow f(-1) = \pm 1$

Let $f(-1) = 1$, then $\forall\ x \in \mathbb{R} - \{0\}$,

$f(x) = f(|x| \operatorname{sgn} x)$

$= f(|x|) f(\operatorname{sgn} x)$

$= f(|x|) \cdot 1$

$\Rightarrow f(x) = f(|x|)\ \forall\ x \in \mathbb{R}\backslash\{0\}$

Now it sufficient to solve f for positive real x.

Let $f(xy) = f(x) \cdot f(y), x > 0, y > 0$

Set $x = e^u, y = e^v$ and $f(e^u) = g(u) \neq 0$

We get $g(u + v) = g(u) \cdot g(v)$ (1)

$$\text{Now } g(u) = g\left(\frac{u}{2} + \frac{u}{2}\right) = g\left(\frac{u}{2}\right) \cdot g\left(\frac{u}{2}\right)$$

$$= \left(g\left(\frac{u}{2}\right)\right)^2 > 0$$

Take log on base e of Eq. (1)

$\Rightarrow \ln g(u + v) = \ln g(u) + \ln g(v)$

Let $\ln g(u) = h(u)$

$\Rightarrow h(u + v) = h(u) + h(v)$
$\Rightarrow h$ is additive and continuous as it is given that f is continuous
$\Rightarrow h(x) = ax$
$\ln g(x) = ax \Rightarrow g(x) = e^{ax}$
$\Rightarrow f(e^x) = e^{ax}$
$\Rightarrow f(e^{\ln t}) = e^{a\ln t} = e^{\ln t^a}$
$\Rightarrow f(t) = t^a$
$\Rightarrow f(x) = x^a$
$\Rightarrow f(x) = |x|^a \ \forall x \in \mathbb{R}$
This is a solution of the given equation.
Let $f(-1) = -1$
Then $\forall\, x \in \mathbb{R} \setminus \{0\}$,
$f(x) = f(|x| \operatorname{sgn} x)$
$= f(|x|) \cdot f(\operatorname{sgn} x)$
$f(x) = \operatorname{sgn}(x) \cdot f(|x|) \ \forall\, x \in \mathbb{R}$
Solving this similar to previous case, we get
$f(x) = \operatorname{sgn}(x) \cdot |x|^a$
This is also a solution.
Hence, $f(x) = 0 \ \forall\, x \in \mathbb{R}$
or $f(x) = 1 \ \forall\, x \in \mathbb{R}$
or $f(x) = |x|^a \ \forall x \in \mathbb{R}$
or $f(x) = \operatorname{sgn}(x) \cdot |x|^a \ \forall x \in \mathbb{R}$
is complete set of solution of multiplicative and continuous function.

Example 24 *$f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x) \cdot f(y)$, find f.*

Solution: From $f(xy) = f(x) \cdot f(y)$
We get for positive real x

$$f(x) = f(\sqrt{x} \cdot \sqrt{x}) = f(\sqrt{x}) \cdot f(\sqrt{x})$$

$$= (f(\sqrt{x}))^2 \geq 0$$

$\Rightarrow f(x)$ is bounded below
$f(x)$ is also additive, hence $f(x) = ax$.
From second equation, we get $axy = ax \cdot ay \Rightarrow a = 0$ or 1.

Example 25 *$f : \mathbb{R} \to \mathbb{R}$ such that $f(ax + by + c) = p\,f(x) + q\,f(y) + r \ \forall\, x \in \mathbb{R}$, where a, b, c, p, q, r are real number and $ab \neq 0$. Prove that $g(x + y) = g(x) + g(y)$, where $g(x) = f(x) - f(0)$.*

Solution: $P(x, y) : f(ax + by + c) = pf(x) + qf(y) + r \ \forall x \in \mathbb{R}$

$$P\left(-\frac{c}{a}, 0\right) : f(0) = pf\left(-\frac{c}{a}\right) + qf(0) + r \tag{1}$$

$$P\left(\frac{x-c}{a}, 0\right) : f(x) = pf\left(\frac{x-c}{a}\right) + qf(0) + r \tag{2}$$

$$P\left(-\frac{c}{a}, \frac{y}{b}\right) : f(y) = pf\left(-\frac{c}{a}\right) + qf\left(\frac{y}{b}\right) + r \tag{3}$$

$$P\left(\frac{x-c}{a}, \frac{y}{b}\right) : f(x+y) = pf\left(\frac{x-c}{a}\right) + qf\left(\frac{y}{b}\right) + r \tag{4}$$

By Eqs. (4) − (3) − (2) + (1), we get
$f(x+y) - f(x) - f(y) + f(0) = 0$
$\Rightarrow f(x+y) - f(0) = (f(x) - f(0)) + (f(x) - f(0))$
$\Rightarrow g(x+y) = g(x) + g(y)$.

Build-up Your Understanding 4

1. Find the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional relation
   $f(x+y) = a^y f(x) + a^x f(y),\ \forall x, y \in \mathbb{R}$,
   where a is a positive constant.
2. Find the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation
   $$f\left(\sqrt{\frac{x^2+y^2}{2}}\right) = \sqrt{\frac{f(x)^2 + f(y)^2}{2}}, \quad \forall x, y \in \mathbb{R}.$$
3. Find the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation
   $f(x+y) = f(x) + f(y) + f(x)f(y),\ \forall x, y \in \mathbb{R}$.
4. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ that satisfy
   $f(xy) = x f(y) + y f(x)$.
5. If $a > 0$ find all continuous functions f for which
   $f(x+y) = a^{xy} f(x) f(y)$.
6. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
   $$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}.$$
7. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(1) = 1$, $f(a+b) = f(a) + f(b)$ for all a, b, $\in \mathbb{R}$ and
   $f(x) f\left(\frac{1}{x}\right) = 1$ for $x \neq 0$. Show that $f(x) = x$ for all x.

5.2.9 Using Fixed Points

This method is seldom used in very tough problems. Observe the following Examples:

Example 26 *Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that that $f(x f(y)) = y f(x)\ \forall\ x$, $y \in \mathbb{R}^+$ and as $x \to \infty$, $f(x) \to 0$.* **[IMO, 1983]**

Solution: $P(x, y) : f(x f(y)) = y f(x)$
$P(1, 1) : f(f(1)) = f(1)$ (1)
$P(1, f(1)) : f(f(f(1))) = (f(1))^2$
$\Rightarrow f(1) = (f(1))^2$ (using Eq. (1))
$\Rightarrow f(1) = 1$ (as $f \in \mathbb{R}^+$)
$\Rightarrow$ 1 is a fixed point
$P(x, x) : f(x \cdot f(x)) = x \cdot f(x)$ (2)
$\Rightarrow x \cdot f(x)$ is a fixed point $\forall x \in \mathbb{R}^+$
Let $x > 1$ is a fixed point
From Eq. (2), we get $f(x \cdot x) = x \cdot x$
or $f(x^2) = x^2$
$\Rightarrow x^2$ is a fixed point
$\Rightarrow x^{2^m}$ is a fixed point $\forall\ m \in \mathbb{N}$
Now $f(x^{2^m}) = x^{2^m}$

$$\lim_{m\to\infty} f(x^{2^m}) = \lim_{m\to\infty} x^{2^m} = \infty$$

which is a contradiction to $\lim_{x\to\infty} f(x) = 0$

$\Rightarrow$ fixed point x cannot be greater than 1.

Let $x \in (0, 1)$ be a fixed point, then

$$1 = f(1) = f\left(\frac{1}{x}\cdot x\right) = f\left(\frac{1}{x}\cdot f(x)\right) = x\cdot f\left(\frac{1}{x}\right)$$

$$\Rightarrow 1 = x\cdot f\left(\frac{1}{x}\right)$$

or $f\left(\frac{1}{x}\right) = \frac{1}{x} \Rightarrow \frac{1}{x}$ is a fixed point

as $x \in (0, 1)$, $\frac{1}{x} \in (1, \infty)$ which is a contradiction

Hence 1 is the only fixed point, which implies

$x\cdot f(x) \equiv 1$

$$\Rightarrow f(x) = \frac{1}{x}.$$

Example 27 *Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(m + f(n)) = f(f(m)) + f(n)\ \forall\ m, n \in \mathbb{N}_0$.* **[IMO, 1996]**

Solution: $P(m, n) : f(m + f(n)) = f(f(m)) + f(n)$

$P(0, 0) : f(f(0)) = f(f(0)) + f(0)$

$\Rightarrow f(0) = 0$

$\Rightarrow$ 0 is a fixed point

$P(0, n) : f(f(n)) = f(f(0)) + f(n)$

$= f(n)$

$\Rightarrow f(n)$ is a fixed point of $f\ \forall\ n \in \mathbb{N}_0$

$\Rightarrow f(m + f(n)) = f(m) + f(n)$

Let us prove if z is a fix point then kz is a fixed point $\forall\ k \in \mathbb{N}_0$

for $k = 0, 1$ it is true

Let mw be a fixed point. $\forall\ m \le k$

$$P(w, mw) : f(w + f(mw)) = f(w) + f(mw)$$
$$f(w + mw) = w + mw$$
$$f((m + 1)w) = (m + 1)w$$

Hence mw is a fixed point $\Rightarrow (m + 1)w$ is a fixed point.

If $w = 1$, then $f(nw) = nw \Rightarrow f(n) = n$ is a solution.

If 0 is the only fixed point of f, then $f(n) = 0\ \forall n \in \mathbb{N}_0$ (since $f(n)$ is a fixed point $\forall n \in \mathbb{N}_0$)

Otherwise f has a least fixed point $z \ge 2$.

Now we will prove that the only fixed points are kz, $k \in \mathbb{N}_0$.

Let x be a fixed point and $x = kz + r$, $0 \le r < z$,

We have $x = f(x) = f(r + kz) = f(r + f(kz))$

$= f(f(r)) + f(kz)$ (From Parent equation)

$= f(r) + kz$ (as $f(r)$ is a fixed point)

$\Rightarrow f(r) = x - kz$

$= r$

$\Rightarrow r$ is a fixed point but z is the least positive fixed point, hence $r = 0$.
$\Rightarrow x = kz$.
Now the identify $f(f(n)) = f(n)$
$f(n)$ in a fixed point and also all fixed point must be multiple of z.
Hence $f(n) = c_n z$, for some $c_n \in \mathbb{N}_0$, where $c_0 = 0$.
For $n \in \mathbb{N}_0$, we have

$$\begin{aligned} n &= kz + r,\ 0 \le r < z \\ f(n) &= f(kz + r) = f(f(kz) + r) \\ &= f(f(r)) + f(kz) \\ &= f(r) + f(kz) \\ &= c_r z + kz \\ &= (c_r + k) z \\ f(n) &= \left(c_r + \left\lfloor \frac{n}{z} \right\rfloor\right) z \end{aligned}$$

which is a solution of the equation.

Build-up Your Understanding 5

1. Find all polynomials $P(x)$ such that
   $P(F(x)) = F(P(x)), P(0) = 0,$
   where F is some function defined on $\mathbb{R}$ and that satisfies $F(x) > x, \forall x \ge 0$.
2. Let S be the set of real numbers strictly greater than -1. Find all functions $f : S \to S$ satisfying the two conditions
   (a) $f(x + f(y) + x f(y)) = y + f(x) + y f(x), \forall x, y \in S$;
   (b) $f(x)/x$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$. **[IMO, 1994]**
3. Let $\mathbb{R}$ denote the real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2$ or show no such function can exist.
4. Let $g(x)$ be a quadratic function such that the equation $g(g(x)) = x$ has at least three different real roots. Then there is no function $f : \mathbb{R} \to \mathbb{R}$ such that
   $$f(f(x)) = g(x)$$
   for all $x \in \mathbb{R}$.

Solved Problems

Problem 1 *Let f be a function on the positive integers, i.e., $f : \mathbb{N} \to \mathbb{Z}$ with the following properties:*
(i) $f(2) = 2$
(ii) $f(m \times n) = f(m) f(n)$ *for all positive integers m and n,*
(iii) $f(m) > f(n)$ *for $m > n$.*
Find $f(1998)$.

Solution: $2 = f(2) = f(1 \times 2) = f(1) \times f(2) = f(1) \times 2$

$$\therefore\ f(1) = \frac{2}{2} = 1$$

Now, $f(4) > f(3) > f(2) = 2$
and $f(4) = f(2) \times f(2) = 2 \times 2 = 4$

and so, $4 > f(3) > 2$, and $f(3)$ is an integer, hence $f(3) = 3$

and $\quad f(6) > f(5) > f(4)$

$\Rightarrow f(2) \times f(3) > f(5) > 4$

$\Rightarrow 2 \times 3 > f(5) > 4$

$\Rightarrow f(5) = 5$

So, we guess that $f(n) = n$. Let us prove it.

We will use mathematical induction for proving.

$f(n) = n$ is true for $n = 1, 2$.

Let us assume that the result is true for all $m < n$, and then we shall prove it for n, where $n > 2$.

If n is even, then let $n = 2m$

$f(n) = f(2m) = f(2) \times f(m) = 2 \times m = 2m = n$.

If n is odd and $n = 2m + 1$, then $n > 2m$

$2m < 2m + 1 < 2m + 2$

$\Rightarrow f(2m) < f(2m + 1) + f(2m + 2)$

$\Rightarrow f(2).f(m) < f(2m + 1) < f(2)\cdot f(m + 1)$

$\Rightarrow 2m < f(2m + 1) < 2m + 2$

There is exactly one integer $2m + 1$ between $2m$ and $2m + 2$ and hence,

$f(n) = f(2m + 1) = (2m + 1) = n$

Thus, $f(n) = n$ for all $n \in N$

Hence, $f(1998) = 1998$

Problem 2 *Let f be a function from the set of positive integers to the set of real numbers. If:* $\mathbb{N} \to \mathbb{R}$ *such that*

(i) $f(1) = 1$

(ii) $f(1) + 2f(2) + 3f(3) + \cdots + nf(n) = n(n + 1)\,f(n)$.

Find $f(1997)$.

Solution: $f(1) = 1$

$f(1) + 2f(2) = 2(2 + 1)\,f(2)$

$\Rightarrow 4f(2) = 1, \quad \Rightarrow \quad f(2) = \frac{1}{4}$.

Again, $f(1) + 2f(2) + 3f(3) = (3 \times 4)\,f(3)$

$\Rightarrow 9f(3) = 1 + \frac{1}{2} = \frac{3}{2}$

$\Rightarrow f(3) = \frac{1}{6}$.

The above calculation suggests that $f(n)$ may be $\frac{1}{2n}$ for $n > 1$.

Let us verify if it is so.

For $n = 2, \; f(2) = \frac{1}{2 \times 2} = \frac{1}{4}$ is true.

$n = 3, \; f(3) = \frac{1}{3 \times 2} = \frac{1}{6}$ is also true.

So, let us assume that $f(n) = \frac{1}{2n}$.

Now, we should show that $f(n+1) = \frac{1}{2(n+1)}$.

(Here we use the principle of mathematical induction.)

By the hypothesis (ii), we have

$$f(1) + 2f(2) + \cdots + nf(n) = n(n+1)f(n)$$

$$f(1) + 2f(2) + \cdots + nf(n) + (n+1)f(n+1) = (n+1)(n+2)f(n+1)$$

$$\Rightarrow 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n-1\,\text{times}} + (n+1)f(n+1)$$

$$= (n+1)(n+2)f(n+1)$$

$$\Rightarrow 1 + (n-1)\frac{1}{2} = (n+1)f(n+1)(n+2-1)$$

$$= (n+1)^2 \times f(n+1)$$

$$\Rightarrow \quad f(n+1) = \frac{1+(n-1)\frac{1}{2}}{(n+1)^2} = \frac{n+1}{2(n+1)^2} = \frac{1}{2(n+1)}.$$

Thus by the principle of mathematical induction, we have proved that $f(n) = \dfrac{1}{2n}$ for $n > 1$

$$\therefore \quad f(1997) = \frac{1}{2\times 1997} = \frac{1}{3994}.$$

Problem 3 *Find all functions $f : \mathbb{Q} \to \mathbb{Q}$, for which*
$f(xy) = f(x)f(y) - f(x+y) + 1 \ \forall\ x, y \in \mathbb{Q}$.

Solution: Let $P(x, y) : f(xy) = f(x)f(y) - f(x+y) + 1$

$P(0, 0) : f(0) = (f(0)^2) - f(0) + 1$

$\Rightarrow (f(0))^2 - 2f(0) + 1 = 0$

$\Rightarrow (f(0) - 1)^2 = 0$

$$\Rightarrow \ f(0) = 1 \tag{1}$$

$P(1, -1) : f(-1) = f(1) \cdot f(-1) - f(0) + 1$

$\Rightarrow f(-1) = f(1) \cdot f(-1) \quad (\text{as } f(0) = 1)$

$\Rightarrow (f(1) - 1) \cdot f(-1) = 0$

$\Rightarrow f(1) = 1$ or $f(-1) = 0$

Case 1: $f(-1) = 0$

$P(x, yz) : f(xyz) = f(x) \cdot f(yz) - f(x + yz) + 1$

$= f(x)\,[f(y) \cdot f(z) - f(y+z) + 1] - f(x + yz) + 1$

$$\Rightarrow f(xyz) - f(x) \cdot f(y) \cdot f(z) = -f(x) \cdot f(y+z) + f(x) - f(x+yz) + 1 \tag{1}$$

In Eq. (1), LHS is symmetric in x, y, z. But RHS is not so. Interchanging z and x, we get

$$f(zyx) - f(z) \cdot f(y) \cdot f(x).= -f(z) \cdot f(y+x) + f(z) - f(z+yx) + 1 \tag{2}$$

From Eqs. (1) and (2), we get

$$-f(x) \cdot f(y+z) + f(x) - f(x+yz) + 1 = -f(z) \cdot f(x+y) + f(z) - f(z+xy) + 1 \tag{3}$$

for $z = -1$

$-f(x)f(y-1) + f(x) - f(x-y) = -f(-1+xy) \quad (\text{as } f(-1) = 0)$

$$\text{or } f(x) \cdot (f(y-1) - 1) + f(x-y) = f(xy - 1) \tag{4}$$

Plugging $x = 1$, and $y = 2$ in Eq. (4), we get

$f(1) \cdot (f(1) - 1) = f(1)$

$\Rightarrow f(1) \cdot (f(1) - 2) = 0$

$\Rightarrow f(1) = 0$ or 2

Sub-case 1: $f(1) = 0$

Plugging $x = 1$, and $y = x + 1$ in Eq. (4), we get

$$f(1) \cdot (f(x+1-1)-1) + f(1-x-1) = f(1 \cdot (x+1)-1) \tag{5}$$

$\Rightarrow f(-x) = f(x)$

$P(x, -y) : f(-xy) = f(x) \cdot f(-y) - f(x-y) + 1$

or $f(xy) = f(x) \cdot f(y) - f(x-y) + 1$

Comparing it with parent equation, we get

$f(x+y) = f(x-y) \ \forall\, x, y \in \mathbb{Q}$

Replacing x by $\dfrac{x}{2}$ and y by $\dfrac{x}{2}$, we get

$$f\left(\frac{x}{2}+\frac{x}{2}\right) = f\left(\frac{x}{2}-\frac{x}{2}\right)$$

$$f(x) = f(0)$$

$\Rightarrow f(x) = \text{Constant}$

$\Rightarrow f(x) = f(1) = 0$

$\Rightarrow f(x) = 0$, but it does not satisfy the parent equation.

Sub-case 2: $f(1) = 2$ using this in Eq. (5), we get

$2(f(x) - 1) + f(-x) = f(x)$

$\Rightarrow f(x) + f(-x) = 2$

or $1 - f(x) = -(1 - f(-x))$

Let $g(x) = 1 - f(x) \Rightarrow g(x) = -g(-x)$

$\Rightarrow g$ is an odd function.

Now from parent equation, we get

$$Q(x, y) : g(xy) = g(x) + g(y) - g(x) \cdot g(y) - g(x+y) \tag{6}$$

$$Q(x, -y) : -g(xy) = g(x) - g(y) + g(x) \cdot g(y) - g(x-y) \tag{7}$$

From Eq. (6) + Eq. (7), we get

$0 = 2g(x) - g(x+y) - g(x-y)$

or $g(x+y) + g(x-y) = 2g(x)$

for $y = x$, $g(2x) + g(0) = 2g(x)$

$\Rightarrow g(2x) = 2g(x) \quad (\text{as } g(0) = 0)$

$\Rightarrow g(x+y) + g(x-y) = g(2x)$

Let $x + y = u$, $x - y = v$, $\Rightarrow 2x = u + v$

$\Rightarrow g(u) + g(v) = g(u+v)$

Which is a Cauchy's equation with domain $\mathbb{Q}$, so $g(x) = kx$ for some fix 'k'.

Using this in Eq. (6), we get

$kxy = kx + ky - k^2 \cdot xy - k(x+y)$

$\Rightarrow kxy = k^2 \cdot xy$

$\Rightarrow k = 0$ or $k = -1$

$k = 0$ is not possible $\Rightarrow k = -1$

$\Rightarrow g(x) = -x \Rightarrow f(x) = 1 + x \ \forall\, x \in \mathbb{Q}$.

Problem 4 *Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy $f(m + f(n)) = n + f(m)$.*

Solution: Let $P(m, n) : f(m + f(n)) = f(m) + n$

$$P(0, n) : f(f(n)) = f(0) + n$$

$$P(0, 0) : f(0 + f(0)) = 0 + f(0)$$

$$\Rightarrow f(f(0)) = f(0)$$

$$P(0, f(0)) : f(0 + f(f(0))) = f(0) + f(0)$$

$$f(f(f(0))) = 2f(0)$$
$$\Rightarrow f(0) = 2f(0)$$
$$\Rightarrow f(0) = 0$$
$$\Rightarrow f(f(n)) = n$$
$$P(m, f(n)) : f(m + f(f(n))) = f(m) + f(n)$$
$$\Rightarrow f(m + n) = f(m) + f(n)$$

It is an additive function

$\Rightarrow f(n) = an$ for some integer a

But $\quad f(f(n)) = n$

$$\Rightarrow f(an) = n$$
$$\Rightarrow a(an) = n \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$$

Hence, $f(n) = n$ or $f(n) = -n$ both satisfy the parent equation.

Problem 5 *The function f is defined on the positive integers and satisfies $f(2) = 1$, $f(2n) = f(n)$, $f(2n + 1) = f(2n) + 1$. Find the maximum value of $f(n)$ for $n \in \{1, 2, 3, .., 2002\}$.* **[Spain MO, 2002]**

Solution: $f(n)$ is obviously number of 1's in the binary expansion of n. we will prove it by induction on n.

Let $g(n) =$ The number of 1's in the binary representation of n.

Claim: $f(n) = g(n)\ \forall\ n \in \mathbb{N}$

For $n = 1$, $f(1) = f(2 \cdot 1) = f(2) = 1$

$g(1) = 1$

$\Rightarrow f(1) = g(1)$

Let for some $k \geq 1$, $f(n) = g(n)\ \forall\ n < k$

If k is even, then $k = 2l$, $(l < k)$ and $f(k) = f(2l) = f(l)$.

Also the binary representation of k is obtained from that of l by adding a 0 to the end.

So $g(k) = g(l)$.

The inductive hypothesis ensures that $f(l) = g(l)$

$\Rightarrow f(k) = g(k)$ for $k =$ Even

For $k =$ Odd, $k = 2l + 1 \Rightarrow f(k) = f(2l + 1) = f(2l) + 1 = f(l) + 1$.

Also the binary representation of k is obtained from that of l by adding a 1 at the end, therefore $g(k) = g(l) + 1$

so $f(k) = g(k)$ for $k =$ Odd

Now maximum value of $f(n)$ is $f(1023) = 9$.

Problem 6 *Prove that there exists a unique function $f : (0, \infty) \to (0, \infty)$, such that $f(f(x)) + f(x) = 6x\ \forall\ x \in (0, \infty)$.* **[Putnam, 1988]**

Solution: Let $a_0 = x$, $a_{k+1} = f(a_k)$, $k \geq 0$.

From given equation, we get $a_{k+2} + a_{k+1} = 6a_k$, $k \geq 0$

Corresponding characteristic equation

$x^{k+2} - x^{k+1} - 6x^k = 0$, $x \neq 0$

or $x^2 - x - 6 = 0$

$x = 2, -3$

$\Rightarrow a_k = \alpha \cdot 2^k + \beta \cdot (-3)^k$

$$\Rightarrow a_k = \left(\frac{3a_0 + a_1}{5}\right)2^k + \left(\frac{2a_0 - a_1}{5}\right)(-3)^k$$

Also $\lim\limits_{k\to\infty} \dfrac{3^k}{2^k} = \infty$

$\Rightarrow \lim\limits_{k\to\infty} a_{2k} = -\infty$ for $\beta < 0$

$\Rightarrow \lim\limits_{k\to\infty} a_{2k+1} = -\infty$ for $\beta > 0$

$\Rightarrow \beta = 0 \Rightarrow a_1 = 2a_0$

Hence, $f(x) = 2x\ \forall\ x \in (0, \infty)$.

Problem 7 *Find all functions* $f : \mathbb{Z} \to \mathbb{Z}$ *that satisfy* $f(m+n) + f(mn) = f(m) \cdot f(n) + 1$.

Solution: Let $P(m, n) : f(m+n) + f(mn) = f(m) f(n) + 1$

$P(0, 0) : f(0) + f(0) = (f(0))^2 + 1$

$\Rightarrow (f(0) - 1)^2 = 0 \Rightarrow f(0) = 1$

$P(-1, 1) : f(0) + f(-1) = f(-1) \cdot f(1) + 1$

$\Rightarrow f(-1) = 0$ or $f(1) = 1$ (as $f(0) = 1$)

For $f(1) = 1$,

Put $(m, 1)$: $f(m+1) + f(m) = f(m) \cdot f(1) + 1$

$\Rightarrow f(m+1) = 1$

$\Rightarrow f(n) = 1\ \forall n \in \mathbb{Z}$

For $f(-1) = 0$

$P(-1, -1) : f(-2) + f(1) = f(-1) \cdot f(-1) + 1$

$f(-2) + f(1) = 1$

$P(-2, 1) : f(-1) + f(-2) = f(-2) \cdot f(1) + 1$

$\Rightarrow f(-2) = f(-2) \cdot f(1) + 1$

$\Rightarrow f(-2)(1 - f(1)) = 1$

$\Rightarrow (1 - f(1))^2 = 1$ (As $f(-2) = 1 - f(1)$)

$\Rightarrow 1 - f(1) = \pm 1$

$\Rightarrow f(1) = 0$ or 2

For $f(-1) = 0$ and $f(1) = 0$

$P(m, 1) : f(m+1) + f(m) = 1$

$\Rightarrow f(m+1) = 1 - f(m)$

Claim: $f(2m) = 1, f(2m+1) = 0$

Proof: For $m = 0, f(0) = 1$

for $m = 1, f(1) = 0$

Let for $n = k$, claim be true:

then $f(k+1) = 1 - f(k)$

$$= \begin{cases} 1-1, & k = \text{Even} \\ 1-0, & k = \text{Odd} \end{cases}$$

$$= \begin{cases} 0, & k+1 = \text{Odd} \\ 1, & k+1 = \text{Even} \end{cases}$$

Similarly, $f(k-1) = 1 - f(k)$

$$= \begin{cases} 1-1, & k = \text{Even} \\ 1-0, & k = \text{Odd} \end{cases}$$

$$= \begin{cases} 0, & k-1 = \text{Odd} \\ 1, & k-1 = \text{Even} \end{cases}$$

For $f(-1) = 0, f(1) = 2$

$P(n, 1) : f(n+1) + f(n) = f(1) \cdot f(n) + 1$

$\Rightarrow f(n+1) = f(n) + 1$

$f(n+1) - f(n) = 1 \Rightarrow f(n)$ are in AP

with common difference = 1 as $f(0) = 1$

$\Rightarrow f(n) = n+1 \; \forall n \in \mathbb{Z}$.

Problem 8 *Find all functions $f : \mathbb{R} \to \mathbb{R}$, that satisfy $f(x^2 + y \cdot f(x)) = x \cdot f(x+y)$.*

Solution: $P(x, y) : f(x^2 + y \cdot f(x)) = x \cdot f(x+y)$

$$P(0,0) : f(0) = 0 \tag{1}$$

$P(x,0) : f(x^2) = x \cdot f(x)$

$P(x,-x) : f(x^2 - x \cdot f(x)) = x \cdot f(0) = 0$

$f(x^2 - x \cdot f(x)) = 0$

If possible let $x^2 - x f(x) \neq 0$ for some $x_0 \neq 0$ (otherwise $x^2 - x f(x) = 0 \Rightarrow f(x) = x$).
Also assume $x_0^2 - x_0 f(x_0) = a$

$\Rightarrow f(a) = 0$

$P(a, y) : f(a^2) = a \cdot f(a+y)$

$\Rightarrow af(a) = a \cdot f(a+y)$

$\Rightarrow a \cdot f(a+y) = 0$

$\Rightarrow a = 0$ or $f(a+y) = 0$

Case 1: $f(a+y) = 0$, replace y by $x - a$, we get

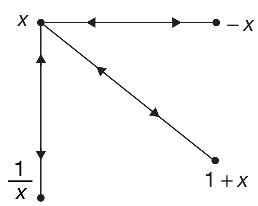
$\Rightarrow f(x) = 0$

Case 2: $a = 0, x^2 - x f(x) = 0 \Rightarrow f(x) = x$ for $x \neq 0$

$\Rightarrow f(x) = x \; \forall \, x \in \mathbb{R}$ (as $f(0) = 0$)

so $f(x) = 0 \; \forall \, x \in \mathbb{R}$ or $f(x) = x \; \forall \, x \in \mathbb{R}$.

Problem 9 *Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(-x) = -f(x)$, $f(x+1) = f(x) + 1 \; \forall \, x \in \mathbb{R}$ and $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2} \; \forall \, x \in \mathbb{R} \setminus \{0\}$.*



Solution: See the adjacent graph. It is a connected graph. From any node we can reach any other node. Let us find a cycle!

Observe the cycle,

$$x \to 1+x \to \frac{1}{1+x} \to -\frac{1}{1+x} \to 1 - \frac{1}{1+x} = \frac{x}{x+1} \to \frac{x+1}{x} = 1 + \frac{1}{x} \to \frac{1}{x} \to x$$

Now, let $f(x) = y$

$\Rightarrow f(1+x) = f(x) + 1 = y + 1$

$$\Rightarrow f\left(\frac{1}{1+x}\right) = \frac{f(1+x)}{(1+x)^2} = \frac{y+1}{x^2+2x+1}$$

$$\Rightarrow f\left(-\frac{1}{1+x}\right) = -\frac{y+1}{(x+1)^2}$$

$$\Rightarrow f\left(1-\frac{1}{1+x}\right) = -\frac{y+1}{(x+1)^2}+1$$

$$= \frac{x^2+2x-y}{(x+1)^2}$$

$$\Rightarrow f\left(\frac{x+1}{x}\right) = \frac{x^2+2x-y}{\cancel{(x+1)^2}\cdot\dfrac{x^2}{\cancel{(x+1)^2}}}$$

$$\Rightarrow f\left(\frac{1}{x}\right) = \frac{x^2+2x-y}{x^2} - 1 = \frac{2x-y}{x^2}$$

$$\Rightarrow f(x) = \frac{2x-y}{x^2\cdot\dfrac{1}{x^2}} = 2x-y$$

$\Rightarrow y = 2x - y \Rightarrow 2y = 2x \Rightarrow y = x \Rightarrow f(x) = x \ \forall\ x \neq 0, -1.$

Also from $f(-x) = -f(x)$ we get $f(-0) = -f(0) \Rightarrow f(0) = 0$ and $f(0+1) = f(0) + 1 = 1 \Rightarrow$ $f(-1) = -1$ so $f(x) = x \ \forall\ x \in \mathbb{R}$.

Check Your Understanding

1. Given a constant c, $|c| \neq 1$, find all function of f, such that $f(x) + cf(2-x) = (x-1)^3$ for all x.
2. Let $f_1(x) = \dfrac{1}{1-x}$ and $f_n(x) = f_1(f_{n-1}(x))$ for $n = 1, 2, 3, \ldots$; Evaluate $f_{2012}(2012)$ and $f_{2013}(2013)$.
3. For any positive integer n, let $f(n)$ be defined as $\dfrac{4n+\sqrt{4n^2-1}}{\sqrt{2n+1}+\sqrt{2n-1}}$.

   Show that the value of $f(1) + f(2) + f(3) + \cdots + f(40)$ is rational.
4. Let $f(n)$ be a function defined on the non-negative integers given the following facts:
   (i) $f(0) = f(1) = 0$
   (ii) $f(2) = 1$
   (iii) For $n > 2$, $f(n)$ gives the smallest positive integer, which does not divide n.
   Let $g(n) = f(f(f(n)))$. Find the value of $S_{2012} = g(1) + g(2) + g(3) + \cdots + g(2012)$.
5. If f denotes the function which gives $\cos 17x$ in terms of $\cos x$, that is $\cos 17x = f(\cos x)$, then, prove that it is the same function 'f' which gives $\sin 17x$ in terms of $\sin x$, Generalize this result.
6. A real valued function f is defined for positive integers and a positive integer a satisfies
   $f(a) = f(1995), f(a+1) = f(1996),$
   $f(a+2) = f(1997), f(n+a) = \dfrac{f(n)-1}{f(n)+1}$ for every integer n.

Prove that:
(a) $f(n+4a) = f(n)$ for any positive integer n.
(b) Determine the smallest possible value of a.

7. Let $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}$, evaluate:

$$f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)$$

8. Let $f(1) = 1, f(1) + \cdots + f(n) = n^2 \cdot f(n)$ for all $n \in \mathbb{N}$. What is $f(n)$?

9. Let x be the set of positive integers greater than or equal to 8. Let $f: \mathbb{N} \to \mathbb{N}$ be a function, such that $f(x+y) = f(xy)$ for all $x \geq 4, y \geq 4$. If $f(8) = 9$, determine $f(9)$.

10. The function defined on the set of ordered pairs of positive integers, has the following properties:
(a) $f(x, x) = x, \forall\, x$
(b) $f(x, y) = f(y, x)\ \forall\, x, y$
(c) $(x+y)\, f(x, y) = y f(x, x+y)\ \forall\, x, y$
Prove that $f(52, 14) = 364$.

11. Given $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(0) \neq f(-1)$ also $f(m+n) + f(mn-1) = f(m)\, f(n) + 2$ for all integers m, n. Show that $f(5) = 26$.

12. Find all $f: (0, \infty) \to (0, \infty)$ such that

$$\frac{(f(x))^2 + (f(y))^2}{f(z^2) + f(t^2)} = \frac{x^2+y^2}{z^2+t^2}\ \forall\, x, y, z, t \in (0, \infty) \text{ with } xy = zt.$$ **[IMO, 2008]**

13. Find all functions $f: \mathbb{N} \to \mathbb{N}$ which satisfy $f(2) = 2, f(mn) = f(m) \cdot f(n)\ \forall\, m, n \in \mathbb{N}$, gcd $(m, n) = 1$ and $f(m) < f(n)$ whenever $m < n$.

14. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f((x-y)^2) = (f(x))^2 - 2x\, f(y) + y^2$.

15. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)\ \forall\, x, y \in \mathbb{R}$ and $f\left(\dfrac{1}{x}\right) = \dfrac{f(x)}{x^2}\ \forall x \neq 0$.

Challenge Your Understanding

1. Find all polynomials $P(x)$ such that $(x-16)\, P(2x) = 16(x-1)\, P(x)\ \forall\, x \in \mathbb{R}$.

2. $f, g, h: \mathbb{R} \to \mathbb{R}$ all are continuous functions such that $f(x+y) = g(x) + h(y), \forall\, x, y \in \mathbb{R}$ find f, g, h.

3. $f: \mathbb{R}^+ \to \mathbb{R}$ such that $f(x)$ is strictly increasing function, $f(x) > -\dfrac{1}{x}\ \forall\, x > 0$ and

$$f(x) \cdot f\left(f(x) + \frac{1}{x}\right) = 1\ \forall\, x > 0. \text{ Find } f.$$ **[Greece MO, 1997]**

4. $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(n) = f(n-1) + f(n-2), f(0) = 0, f(1) = 1$, find f.

5. Find all solutions of the following system of equations:

$$\frac{4x^2}{4x^2+1} = y,\quad \frac{4y^2}{4y^2+1} = z,\quad \frac{4z^2}{4z^2+1} = x$$ **[Canada MO, 1996]**

6. Find all polynomials $f(x)$, $g(x)$ and $h(x)$ such that

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1, & x < -1 \\ 3x+2, & -1 \leq x \leq 0 \\ -2x+2, & x > 0 \end{cases}$$ **[Putnum, 1999]**

7. Do there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ $\forall x \in \mathbb{R}$.
8. Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(f(n))) + f(f(n)) + f(n) = 3n$, find f.
9. $f : [0, \infty) \to \mathbb{R}$ such that $f(0) = 0$ and $f(x) = 1 + 5f\left(\left\lfloor \frac{x}{2} \right\rfloor\right) - 6f\left(\left\lfloor \frac{x}{4} \right\rfloor\right) \forall x > 0$.
   Find f.
10. $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2, f(2) = 1, f(3n) = 3f(n), f(3n + 1) = 3f(n) + 2,$ $f(3n + 2) = 3f(n) + 1$. Find number of integer $n \le 2006$ for which $f(n) = 2^n$.
11. $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f(f(n)) = 3n \ \forall \ n \in \mathbb{N}$. Determine $f(2016)$.
12. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy $f(f(n)) = n, f(f(n + 2) + 2) = n$ and $f(0) = 1$. **[Putnam, 1992]**
13. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
    $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1 \ \forall \ x, y \in \mathbb{R}$. **[IMO, 1999]**
14. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
    $(f(x) + f(z))\,(f(y) + f(t)) = f(xy - zt) + f(xt + yz) \ \forall \ x, y, z, t \in \mathbb{R}$. **[IMO, 2002]**
15. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $x, f(y)$ and $f(y + f(x) - 1)$ are sides of a triangle for all $x, y \in \mathbb{N}$. **[IMO, 2009]**

Chapter 6

In the margin of his copy of a book by Diophantus, Pierre de Fermat wrote:

"*Cubum autem in duos cubos, aut quadrate-quadratum in duos quadrate-quadratos, et generaliter nullamin infinitum ultra quadratum potestatem in duos eiusdem nominis fas est divider cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.*"

"But to divide a cube into two cubes, or a doublesquare into two doublesquare and generally no power up to infinity from beyond the square into two of the same name, is not permissible. Of which thing I have of course uncovered a wonderful proof. The smallness of the margin would not be able to contain it."

[Known as Fermat's Last Theorem, the proof of which remained elusive for 358 years and in 1994, proven by Adrew Wiles, a British mathematician.]

Pierre de Fermat

(Between 31 Oct to 6 Dec 1607–12 Jan 1665), Nationality: French

Number Theory

6.1 Divisibility of Integers

An integer $a \neq 0$ divides b, if there exists an integer x such that $b = ax$, and thus, we write as $a|b$ (read a divides b). This can also be stated as b is divisible by a or a is a divisor of b or b is a multiple of a. If a does not divide b we write as $a \nmid b$.

6.1.1 Properties of Divisibility

1. $a|b$ and $b|c \Rightarrow a|c$
2. $a|b$, $a|c \Rightarrow a|(b+c)$, and $a|(b-c)$
3. $a|b$, $a|(b+c) \Rightarrow a|c$
4. $a|b$, $a|(b-c) \Rightarrow a|c$
5. $a|b$ and $a|c \Rightarrow a|(kb \pm lc)$ for all $k, l \in \mathbb{Z}$
6. $a|b$ and $b|a \Rightarrow a = \pm b$
7. $a|b \Rightarrow b = 0$ or $|a| \leq |b|$. In particular if $a|b$ where $a > 0$, $b > 0$, then $a < b$
8. $a|b \Rightarrow a|bc$ for any integer c
9. $a|b$ iff $ma|mb$ where $m \neq 0$

Notes:

1. $(x+y)|(x^{2n+1}+y^{2n+1}) \forall n \in \mathbb{N}_0$

   Proof:
   For $n = 0$ it is obvious, for $n \geq 1$, we have
   $(x^{2n+1}+y^{2n+1}) = (x+y)(x^{2n}-x^{2n-1}y+x^{2n-2}y^2-\ldots+y^{2n})$

2. $(x-y)|(x^n-y^n) \forall n \in \mathbb{N}$

   Proof:
   For $n = 1$ it is obvious, for $n \geq 2$, we have
   $x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots+y^{n-1})$

Example 1 *The equation $x^2 + px + q = 0$ has rational roots, where p and q are integers. Prove that the roots are integers.*

Solution: $x = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$,

since the roots are rational, $p^2 - 4q$ is a perfect square.

If p is even, p^2 and $4q$ are even and hence, $p^2 - 4q$ is an even integer and hence, $-p \pm \sqrt{p^2 - 4q}$ is an even integer and hence, $\dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$ is an integer.

If p is odd, $(p^2 - 4q)$ is odd and $-p \pm \sqrt{p^2 - 4q}$ is an even integer and hence, $\dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$ is an integer and hence, the result.

Example 2 *Find the number of positive integer n for which (i) $n \le 1991$ (ii) 6 is a factor of $n^2 + 3n + 2$.*

Solution: $6 \mid (n^2 + 3n + 2)$

$\Rightarrow \quad 6 \mid (n + 1)(n + 2)$

$\Rightarrow \quad 2 \mid (n + 1)(n + 2)$ and also $3 \mid (n + 1)(n + 2)$

But the product of two consecutive integers is even, and $3 \mid (n + 1)(n + 2)$ only when n is not a multiple of 3, *i.e.*, $n \ne 3, 6, \ldots, 1989$.

So, the number of $n \le 1991$ satisfying the conditions is 1991 – (the number of multiples of 3, up to 1991)

$$= 1991 - \left\lfloor \frac{1991}{3} \right\rfloor = 1991 - 663 = 1328.$$

Example 3 *Find all six-digit numbers $(a_1a_2a_3a_4a_5a_6)_{10}$ formed by using the digits 1, 2, 3, 4, 5, 6 once each such that the number $(a_1a_2\ldots a_k)_{10}$ is divisible by k for $1 \le k \le 6$.* **[RMO, 1994]**

Solution: $(a_1a_2a_3a_4a_5)_{10}$ is divisible by 5 and hence, $a_5 = 5$. a_1a_2, $a_1a_2a_3a_4$, and $a_1a_2a_3a_4a_5a_6$ are to be divisible by 2, 4 and 6, respectively. a_2, a_4 and a_6 should be even numbers.

So, $a_1 = 1$ and $a_3 = 3$ or $a_1 = 3$ and $a_3 = 1$.

Case 1: If $a_1 = 1$, a_2 can be 2, 4 or 6 and $a_1a_2a_3 = 123$, 143 or 163 but 143, 163 are not divisible by 3, so $a_1a_2a_3$ should be 123. For a_4, we have either 4 or 6 but for $a_4 = 4$, 1234 is not divisible by 4 and hence, $a_4 = 6$ and hence, the six-digit number, when $a_1 = 1$, is 123654.

Case 2: If $a_1 = 3$. a_2 can be 2 or 6 or 4 but then, $a_1a_2a_3 = 321$ is divisible by 3 and 361 or 341 is not divisible by 3.

So, a_2 cannot be 6 or 4.

Now, $a_1a_2a_3a_4 = (321a_4)_{10}$ and a_4 can be 4 or 6. For $a_4 = 4$, 3214 is not divisible by 4 and hence, $a_4 = 6$ and $a_6 = 4$.

Hence, the number is 321654.

Thus, there are exactly 2 numbers 123654 and 321654 satisfying the conditions.

Example 4 *Let T be the set of all triplets (a, b, c) of integers such that* $1 \le a \le b \le c \le 6$. *For each triplet (a, b, c) in T, take the number* $a \times b \times c$ *and add all these numbers corresponding to all the triplets in T. Prove that this sum is divisible by* 7.

Solution: If (a, b, c) is a valid triplet then $(7 - c, 7 - b, 7 - a)$ is also a valid triplet as $1 \le (7 - c) \le (7 - b) \le (7 - a) \le 6$.

Note that $(7 - b) \ne b$, etc.

Let $S = \sum_{1 \le a \le b \le c \le 6} (abc)$, then by the above

$$S = \sum_{1 \le a \le b \le c \le 6} (7-a)(7-b)(7-c)$$

$$2S = \sum_{1 \le a \le b \le c \le 6} [(a \cdot b \cdot c) + (7-a)(7-b)(7-c)]$$

$$= \sum_{1 \le a \le b \le c \le 6} [7^3 - 7^2(a+b+c) + 7(ab+bc+ca)]$$

In the RHS, every term is divisible by 7, *i.e.*, $7 | 2S$, and hence, $7 | S$.

Example 5 *Show that* $1^{1997} + 2^{1997} + \cdots + 1996^{1997}$ *is divisible by* 1997.

Solution: We shall make groups of the terms of the expression as follows:

$(1^{1997} + 1996^{1997}) + (2^{1997} + 1995^{1997}) + \cdots + (998^{1997} + 999^{1997})$.

Here each bracket is of the form $(a_i^{2n+1} + b_i^{2n+1})$ is divisible by $(a_i + b_i)$.

But $(a_i + b_i) = 1997$ for all i.

$\therefore$ Each bracket and hence, their sum is divisible by 1997.

Example 6 *Prove that for any natural number, n,* $E = 2903^n - 803^n - 464^n + 261^n$ *is divisible by* 1897.

Solution: $1897 = 7 \times 271$

Now, $(2903^n - 803^n) - (464^n - 261^n)$

As $(2903 - 803) | (2903^n - 803^n)$ and $(464 - 261) | (464^n - 261^n)$

i.e., $2100 | (2903^n - 803^n)$ and $203 | (464^n - 261^n)$

$\Rightarrow 7 | (2903^n - 803^n)$ and $7 | (464^n - 261^n)$ ($\because 2100 = 7 \times 300$ and $203 = 7 \times 29$)

Hence, $7 | E$

Again, $2903^n - 803^n - 464^n + 261^n = (2903^n - 464^n) - (803^n - 261^n)$

$2903 - 464 = 2439 | (2903^n - 464^n)$

and $(803 - 261) = 542 | (803^n - 261^n)$

i.e., $2439 = 271 \times 9 | (2903^n - 464^n)$ and $542 = 271 \times 2 | (803^n - 261^n)$

So, $271 | (2903^n - 464^n)$ and $271 | (803^n - 261^n)$

and hence, $271 | E$.

Thus, the given expression is divisible by the prime numbers 7 and 271 and hence, is divisible by $271 \times 7 = 1897$.

Euclid of Alexandria

Mid 4th century BCE to
Mid 3rd century BCE
Nationality: Greek

6.2 Euclids Division Lemma

If a and b are any two integers, $a \neq 0$, then there exist unique integers q and r such that $b = aq + r$, $0 \leq r < |a|$

b, a, q and r are called dividend, divisor, quotient and remainder respectively.

Example 7 *When the numbers 19779 and 17997 are divided by a certain three-digit number, they leave the same remainder. Find this largest such divisor and the remainder. How many such divisors are there?*

Solution: Let the divisor be d and the remainder be r.

Then by Euclidean Algorithm, we find

$$19779 = dq_1 + r \tag{1}$$

and

$$17997 = dq_2 + r \tag{2}$$

By subtracting Eq. (2) from Eq. (1), we get

$$1782 = d(q_1 - q_2)$$

$\therefore$ d is a three-digit divisor of 1782.

Therefore, possible values of d are 891, 594, 297 and 198, 162.

Hence, the largest three-digit divisor is 891 and the remainder is 177.

Build-up Your Understanding 1

1. Prove that $(a - c)|(ab + cd)$ if and only if $(a - c)|(ad + bc)$.
2. Prove that $6|(a + b + c)$ if and only if $6|(a^3 + b^3 + c^3)$.
3. Prove that $641|(2^{32} + 1)$.
4. Find all natural numbers n, such that, $\dfrac{(n+1)^2}{n+7}$ is an integer. Find, then, corresponding values of the expression also.
5. Prove that, for any natural number n, $1^n + 8^n - 3^n - 6^n$ is divisible by 10.
6. Prove that $1^k + 2^k + 3^k + \cdots + n^k$ is divisible by $1 + 2 + 3 + \cdots + n$, where n is an integer and k is odd.
7. Prove that for any natural number n, the result of $1^{1987} + 2^{1987} + \cdots + n^{1987}$ cannot be divided by $(n + 2)$ without a remainder.
8. If a, m, n are positive integers with $a > 1$ and $(a^m + 1)|(a^n + 1)$, them $m|n$.
9. Let a, b be positive integers with $b > 2$. Show that $(2^b - 1) \nmid (2^a + 1)$.
10. Let a, b, c, d be integers such that $ad - bc > 1$. Prove that there is at least one among a, b, c, d which is not divisible by $ad - bc$.

6.3 Greatest Common Divisor (GCD)

The greatest common divisor of any two integers a, b (at least one of them non-zero), is the greatest among the integral common divisors of a and b.

The greatest common divisor is denoted as GCD and represented as (a, b).

If $(a, b) = 1$, then we say that a and b are relatively prime integers or co-prime integers.

6.3.1 Properties of GCD

1. $(a, b) \geq 1$
2. $(a, b) = (|a|, |b|)$
3. $(a, 0) = |a|, a \neq 0$
4. $(a, b) = (a + kb, b) \quad \forall\, k \in \mathbb{Z}$
5. $(a, b) = (b, a)$
6. If $(a, b) = g$ and d is a common divisor of a and b, then $d|g$.
7. For any non-zero $m \in \mathbb{Z}$, $(ma, mb) = |m|(a, b)$.
8. If $d|a$ and $d|b$ and $d > 0$, then $\left(\frac{a}{d}, \frac{b}{d}\right) = \left(\frac{1}{d}\right)(a, b)$.
9. If $(a, b) = g$, then $\left(\frac{a}{g}, \frac{b}{g}\right) = 1$.
10. If $(a, b) = 1$ and $(a, c) = 1$, then $(a, bc) = 1$.
11. If $a|bc$ and $(a, b) = 1$, then $a|c$

    If $(a, b) \neq 1$, then we cannot conclude that $a|c$.

    For example, $a = 6$, $b = 21$, $c = 10$

    $6|21 \times 10$, but $(6, 21) = 3$ and $(6, 10) = 2$ and 6 divides neither 21 nor 10.
12. If $a, b \in \mathbb{N}$, $(a, b) = 1$ and $a \times b = c^k$, $k, c \in \mathbb{N}$, then each of a and b is a perfect kth power.
13. If $(a, b) = g$, then there exist two integers x and y such that $g = xa + yb$.

    Note: In general $xa + yb$ is a multiple of $g \;\forall\, x, y \in \mathbb{Z}$
14. $(a, b) = 1 \Leftrightarrow am + bn = 1$ for some $m, n \in \mathbb{Z}$. This is known as **Bézout's identity**.

    The Euclidean algorithm can be used to find the GCD of two integers as well as representing the GCD as linear combination of numbers.

    Consider two numbers 18, 28.

$$28 = 1 \cdot 18 + 10$$
$$18 = 1 \cdot 10 + 8$$
$$10 = 1 \cdot 8 + 2$$
$$8 = 4 \cdot 2 + 0$$

$(18, 28) = 2$ (retracing the steps)

$$\begin{aligned}(18, 28) = 2 &= 10 - 1 \cdot 8 \\ &= 10 - (18 - 1 \cdot 10) \\ &= 2 \cdot 10 - 1 \cdot 18 = 2(28 - 1 \cdot 18) - 1 \cdot 18 \\ &= 2 \cdot 28 - 3 \cdot 18 = 2 \cdot 28 + (-3) \cdot 18\end{aligned}$$

Note: The representation in property (13) is not unique. In fact we can represent (a, b) as $xa + yb$ in infinite number of ways, where $x, y \in \mathbb{Z}$.

$$\begin{aligned}(18, 28) &= 2 \cdot 28 + (-3)\, 18 \\ &= 2 \cdot 28 + 252k + (-3)18 - 252k \\ &= (2 + 9k)\, 28 + (-3 - 14k)18\end{aligned}$$

where k is any integer.

Étienne Bézout

31 Mar 1730–27 Sep 1783
Nationality: French

6.3.2 Least Common Multiple

Least common multiple of two integers a, b is the smallest positive integer divisible by both a and b and it is denoted by $[a, b]$.

In the above example, 252 is the least common multiple of 18 and 28.

$$252 = 9 \times 28 \text{ and } 252 = 14 \times 18$$

Note: $[a, b](a, b) = ab$

Example 8 *If a and b are relatively prime, show that* $(a + b)$ *and* $(a - b)$ *are either relatively prime or their gcd is* 2.

Solution: If d is the gcd of $(a + b)$ and $(a - b)$ then $d|(a + b)$ and $d|(a - b)$ and therefore, $d|(a + b) \pm (a - b)$

$\Rightarrow \quad d|2a$ and $d|2b$

$\Rightarrow \quad d|(2a, 2b)$

But $(a, b) = 1$

$\therefore \quad (2a, 2b) = 2$

$\therefore \quad d|2.$

Hence, d is either 1 or 2.

Example 9 *If* $(a, b) = 1$, *then* $(a \pm b, b) = 1$ *and* $(a, a \pm b) = 1$.

Solution: If $(a \pm b, b)$ is d, then $d|(a \pm b)$, and $d|b$ and this implies $d|a$

$\Rightarrow d|(a, b) = 1 \Rightarrow d|1 \Rightarrow d = 1$.

Again $(\mathrm{a}, a \pm b) = d$, then $d|a$ and $d|(a \pm b)$ and this implies $d|b$.

So, $d|a$ and $d|b$ implies $d|(a, b) \Rightarrow d|1 \Rightarrow d = 1$.

Example 10 *Prove that the fraction* $\dfrac{21m+4}{14m+3}$ *is irreducible for every natural number m.*

Solution: Assuming the contrary, if p is a number which divides both $21m + 4$ and also $14m + 3$, then p should divide,

$$3(14n + 3) - 2(21m + 4) = 1.$$

Thus, $p = 1$.

Therefore, the gcd of $(14m + 3)$ and $(21m + 4)$ is 1.

So, $\dfrac{21m+4}{14m+3}$ is irreducible,

Example 11 *Prove that the expressions* $3x + 11y$ *and* $29x + 23y$ *are divisible by* 125 *for the same set of positive integral values of x, y. Find at least two such pairs* (x, y).

Solution: Since $3(3x + 11y) + 4(29x + 23y) = 125(x + y)$

Now, 3 and 125 are relatively prime and so are 4 and 125.

Thus, if one of the expressions is divisible by 125, then the other expression should also be divisible by 125. Here we have used the following properly:

For $a|b$ and $a|c \Rightarrow a|(ka + lb)$ conversely $a|(ka + lb)$ and $a|ka$, then $a|lb$ and if $(a, l) = 1$, then $a|b$.

To find the values of x and y for which both the expressions are divisible by 125,

$$3x + 11y = 125n_1 \tag{1}$$
$$29x + 23y = 125n_2 \tag{2}$$

Solving Eqs. (1) and (2) for x and y, we get

$$\left.\begin{aligned} x &= \frac{11n_2 - 23n_1}{2} \\ y &= \frac{29n_1 - 3n_2}{2} \end{aligned}\right\}\text{for all } n_1, n_2 \in \mathbb{Z} \text{ and having same parity (i.e., both even or both odd).}$$

Example 12 *If* $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ *where a, b, c are positive integers with no common factor, prove that a + b is a square.* **[RMO, 1992]**

Solution: Let the gcd of a, b be k, then $a = kp$ and $b = kq$, and p, q are relatively prime.

$$\therefore \quad \frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$
$$\Rightarrow \quad c(a + b) = ab$$
$$\Rightarrow \quad ck(p + q) = k^2pq$$
$$\Rightarrow \quad c(p + q) = kpq. \tag{1}$$

Since, k is the GCD of a, b and a, b, c have no common factor $(c, k) = 1$.

So, $c \mid pq$ (2)

As $(p, q) = 1$; p, q are prime to $(p + q)$ and hence, $(p + q)$ is prime to pq and hence, $pq \mid c$. (3)

From Eqs. (2) and (3), we have

$$c = pq \tag{4}$$

From Eqs. (1) and (4), we have, $p + q = k$

So, $(a + b) = k(p + q) = k \times k = k^2$ and hence, the result.

Build-up Your Understanding 2

1. If $a = qb + r$ where a, q, b and r are integers, then prove that $(a, b) = (b, r)$.
2. If a, b are integers both greater than zero and d is their gcd, then, prove that $d = ax + by$ for some $x, y \in \mathbb{Z}$.
3. Prove that $\frac{12n+1}{30n+2}$ is irreducible for every positive integer n.
4. Prove that the expression $\frac{63n+14}{42n+9}$ is irreducible for every positive integer n.
5. Show that $\gcd(n! + 1, (n + 1)! + 1) = 1$ for any $n \in \mathbb{N}$.
6. Prove that the expression $2x + 3y$ and $9x + 5y$ are divisible by 17 for the same set of integral values of x and y.
7. If x, y are integers and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides $xy - 12x + 15y$. **[RMO, 2005]**
8. Find the least possible value of $a + b$, where a, b are positive integers such that 11 divides $a + 13b$ and 13 divides $a + 11b$. **[RMO, 2006]**
9. Show that if 13 divides $n^2 + 3n + 51$ then 169 divides $21n^2 + 89n + 44$. **[RMO, 2012]**

10. If $\gcd(a, b) = 1$, then prove that $(a^2 + b^2, ab) = 1$ and also prove that $\gcd(a + b, a^2 - ab + b^2) = 1$ or 3.

11. If $a, b \in \mathbb{N}$ and $ab|(a^2 + b^2)$, then prove that $a = b$.

12. Let a, b, c be positive integers such that a divides b^2, b divides c^2, c divides a^2. Prove that abc divides $(a + b + c)^7$. **[RMO, 2002]**

13. If $\gcd(a, b, c) = 1$ and $c = \dfrac{ab}{a-b}$, then prove that $a - b$ is a perfect square.

14. Let m, n be positive integers, such that, $3m + n = 3\ \mathrm{lcm}[m, n] + \gcd(m, n)$; prove that, n divides m.

15. Let a_1, b_1, c_1 be natural numbers. We define $a_2 = \gcd(b_1, c_1)$, $b_2 = \gcd(c_1, a_1)$, $c_2 = \gcd(a_1, b_1)$ and $a_3 = \mathrm{lcm}(b_2, c_2)$, $b_3 = \mathrm{lcm}(c_2, a_2)$, $c_3 = \mathrm{lcm}(a_2, b_2)$. Show that $\gcd(b_3, c_3) = a_2$. **[RMO, 2013]**

16. Find the minimum possible least common multiple (lcm) of twenty (not necessarily distinct) natural numbers whose sum is 801. **[RMO, 1998]**

17. Let $m, n, l \in \mathbb{N}$ and $\mathrm{lcm}[m + l, m] = \mathrm{lcm}[n + l, n]$, then prove that $m = n$.

18. Find the set of all ordered pairs of integers (a, b) such that, of $\gcd(a, b) = 1$ and $\dfrac{a}{b} + \dfrac{14b}{25a}$ is an integer.

19. Let $\dfrac{a}{b} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots + \dfrac{1}{1319}$ such that $\gcd(a, b) = 1$. Show that $1979 \mid a$. **[IMO, 1979]**

20. Let $\dfrac{a}{b} = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{2002}$ such that $\gcd(a, b) = 1$. Show that $2003 \mid a$.

21. Let $\dfrac{a}{b} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots + \dfrac{1}{67}$ such that $\gcd(a, b) = 1$. Show that $101 | a$.

22. Let $m, n \in \mathbb{N}$ and n be an odd number then prove that $\gcd(2^n - 1, 2^m + 1) = 1$.

23. For each positive integer n, define $a_n = 20 + n^2$ and $d_n = \gcd(a_n, a_{n+1})$. Find the set of all values that are taken by d_n and show by examples that each of these values are attained. **[RMO, 1997]**

24. Let $P(x) = x^3 + ax^2 + b$ and $Q(x) = x^3 + bx + a$, where a, b are non-zero real numbers. Suppose that the roots of the equation $P(x) = 0$ are the reciprocals of the roots of the equation $Q(x) = 0$. Prove that a and b are integers. Find the greatest common divisor of $P(2013! + 1)$ and $Q(2013! + 1)$. **[RMO, 2013]**

25. If $(a, b) = 1$ and $x^a = y^b$ for some natural numbers a, b, x, y all greater than 1 then show that $x = n^b$ and $y = n^a$ for some $n > 1$.

26. Prove that $\gcd(k^a - 1, k^b - 1) = k^{\gcd(a,b)} - 1$ where $k > 1$; $k, a, b, \in \mathbb{N}$

6.4 Primes

An integer $p > 1$ is called a prime number if it has exactly two distinct divisors namely 1 and p.

In other words, p is a prime, if there is no d, $1 < d < p$, such that $d|p$. A number more than 1 which is not prime is called a composite number. 1 is neither prime nor composite.

Some properties of a prime number p:

1. $p|ab \quad \Rightarrow \quad p|a$ or $p|b$
2. $p|a^n \quad \Rightarrow \quad p|a \quad \Rightarrow \quad p^n|a^n, n \in \mathbb{N}$

3. Every integer greater than 1 is divisible by at least one prime.
4. For $n > 1$ there is at least one prime p such that $n < p < 2n$. A slight generalization for $n > 3$, there always exists at least one prime p with $n < p < 2n - 2$. Another way let p_n be nth prime for $n \geq 1$ then $p_{n+1} < 2p_n$.
5. The number of primes less than or equal to a real number x is $\approx x/\ln x$.

6.4.1 Euclidean Theorem

The number of primes is infinite.

Proof:
Suppose on the contrary that there are only finitely many primes $p_1, p_2, \ldots, p_n$. Look at

$$p_1 \cdot p_2 \cdots p_n + 1$$

This number is not divisible by any of the primes $p_1, p_2, \ldots, p_n$, because it leaves a remainder of 1 when divided by any of them. But as every integer greater than 1 is divisible by a prime. This contradiction implies that there cannot be finitely many primes, *i.e.,* there are infinitely many.

Note: Given $k > 1$, we can find k consecutive composite numbers.

One such k consecutive composite numbers are
$(k + 1)! + 2, (k + 1)! + 3, (k + 1)! + 4, \ldots, (k + 1)! + (k + 1)$.

For $k > 1$, these numbers are divisible by $2, 3, 4, \ldots, k + 1$, respectively.

Example 13 *Prove that if p and* $(8p - 1)$ *are prime then* $(8p + 1)$ *is a composite number.*

Solution: If $3|p$ then $p = 3 \Rightarrow 8p + 1 = 24 + 1 = 25 \Rightarrow 8p + 1$ is a composite number otherwise consider $(8p - 1)$, $8p$ and $(8p + 1)$. These are three consecutive numbers, where $(8p - 1)$ is a prime number $> 3 \Rightarrow 3 \nmid (8p - 1)$.

Since $3 \nmid 8$ and $3 \nmid p$, hence, $3 \nmid 8p$.

So, $3 | (8p + 1)$ as among three consecutive integers, one must be a multiple of 3 and $8p + 1 > 3 \Rightarrow 8p + 1$ is a composite number.

Example 14 *Determine with proof all the arithmetic progression (AP) with integer terms, with the property that for each positive integer n, the sum of the first n terms is a perfect square.*

Solution: When $n = 1$, the first term itself is a perfect square. Let it be k^2.

The sum to n terms of the AP is

$$S_n = \frac{n}{2}[2a + (n-1)d], \quad \text{where } a = k^2.$$

Since S_n is a perfect square for every n, $2a + (n - 1)\,d > 0$, for every n and hence, $d > 0$.

If n is an odd prime, say p, then

$$S_p = \frac{p}{2}[2a + (p-1)d].$$

Since S_p is a perfect square $p / [2a + (p - 1)d]$, *i.e.,* $p | [(2a - d) + pd]$

But $p | pd$, so $p | (2a - d)$. This is possible for all prime p, if and only if, $2a - d = 0$ or $2a = d$, *i.e.,* $d = 2k^2$.

So the required AP is

$$k^2, 3k^2, 5k^2, \ldots, (2n - 1)k^2$$

where k is any natural number.

Example 15 *Prove that the polynomial* $f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$ *cannot be expressed as a product of two polynomials* $p(x)$ *and* $q(x)$ *with integral coefficients of degree less than* 4.

Solution: If possible, let us express

$$x^4 + 26x^3 + 52x^2 + 78x + 1989 = (x^2 + ax + b)(x^2 + cx + d),$$

where $a, b, c, d \in \mathbb{Z}$

By comparing coefficients of both sides, we get

$$a + c = 26 \quad (1)$$

$$ac + b + d = 52 \quad (2)$$

$$bc + ad = 78 \quad (3)$$

$$bd = 1989 = 13 \times 3^2 \times 17 \quad (4)$$

Now, we see that 13 is a divisor of 26, 52, 78, and 1989 and 13 is a prime number.

Thus, $13 \mid bd \quad \Rightarrow \quad$ 13 divides one of b or d, but not both.

If $13 \mid b$, say, and $13 \nmid d$ then from Eq. (3), $13 \mid a$.

Now, $13 \mid ac$, $13 \mid b$, and $13 \mid 52$.

$\therefore$ $13 \mid d$ from Eq. (2) is a contradiction.

So, if $13 \mid d$ and $13 \nmid b$,

Then, again, from Eq. (3), $13 \mid c \Rightarrow 13 \mid a$ (from Eq. 1)

Now, $b = 52 - ac - d$.

$13 \mid b$, but it is again a contradiction. So, there does not exist quadratic polynomials $p(x)$ and $q(x)$with integral coefficients, such that $f(x) = p(x) \times q(x)$.

Similarly, if $p(x)$ is a cubic polynomial and $q(x)$ is a linear one, then let

$$p(x) = x^3 + ax^2 + bx + c$$

$$q(x) = (x + d)$$

$$x^4 + 26x^3 + 52x^2 + 78x + 13 \times 3^2 \times 17 = (x^3 + ax^2 + bx + c)(x + d)$$

Again, comparing coefficients

$$a + d = 26 \quad (5)$$

$$ad + b = 52 \quad (6)$$

$$bd + c = 78 \quad (7)$$

$$cd = 13 \times 3^2 \times 17 \quad (8)$$

As before 13 divides exactly one of c and d.

If $13 \mid d$, and $13 \nmid c$, then by Eq. (7),

$c = 78 + bd \quad \Rightarrow \quad 13 \mid c$ is a contradiction.

So, let $13 \mid c$ and $13 \nmid d$

By Eq. (7), $13 \mid b$,

By Eq. (6) $ad = 52 - b$

$\Rightarrow \quad 13|ad \quad \Rightarrow \quad 13|a$ as $13\nmid d$

By Eq. (5), $d = 26 - a \quad \Rightarrow \quad 13\,|\,d$, (a contradiction).

Hence, there does not exist any polynomials $p(x)$ and $q(x)$as assumed, so is the result.

6.4.2 Sophie Germain Identity

Marie-Sophie Germain

1 Apr 1776–27 Jun 1831
Nationality: French

$$\begin{aligned} a^4 + 4b^4 &= (a^2)^2 + (2b^2)^2 + 2\cdot a^2\cdot 2b^2 - 2a^2\cdot 2b^2 \\ &= (a^2 + 2b^2)^2 - (2ab)^2 \\ &= (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab) \\ &= ((a+b)^2 + b^2)((a-b)^2 + b^2) \end{aligned}$$

This is very useful for proving whether a given number is a prime or composite.

Example 16 *Prove that $n^4 + 4$ is a composite number for all $n > 1$, $n \in \mathbb{N}$.*

Solution: Since $n^4 + 4 = (n^4 + 4n^2 + 4) - 4n^2$

$$\begin{aligned} &= (n^2 + 2)^2 - (2n)^2 \\ &= (n^2 + 2n + 2)(n^2 - 2n + 2). \\ &= [(n+1)^2 + 1]\,[(n-1)^2 + 1] \end{aligned}$$

For $n > 1$, $(n \pm 1)^2 + 1 > 1$ and hence, $n^4 + 4$ is a composite number for all $n > 1$, $n \in \mathbb{N}$.

Example 17 *Prove that $n^4 + 4^n$ is a composite number for all $n \in \mathbb{N}$, $n > 1$.*

[RMO, 1991]

Solution: If n is even, then both n^4 and 4^n are even and hence, $n^4 + 4^n$ is an even number and hence, it is, composite as $n^4 + 4^n$ is surely greater than 2.

If $n > 1$ is odd, then $n = 2k + 1$ where k is a natural number.

Now, $$\begin{aligned} n^4 + 4^n &= n^4 + 4^{2k+1} \\ &= n^4 + 4\cdot 4^{2k} \\ &= n^4 + 4(2^{4k}) \\ &= n^4 + 4(2^k)^4 \end{aligned}$$

Let $\quad a = 2^k$.

Then $a > 2$ as $k \geq 1$.

Then $$\begin{aligned} n^4 + 4^n &= n^4 + 4a^4 \\ &= n^4 + 4n^2a^2 + 4a^4 - 4n^2a^2 \\ &= (n^2 + 2a^2)^2 - (2na)^2 \\ &= (n^2 + 2a^2 + 2na)(n^2 + 2a^2 - 2na). \\ &= ((n+a)^2 + a^2)((n-a)^2 + a^2) \end{aligned}$$

$$(n \pm a)^2 + a^2 > a^2 > 2^2 = 4$$

$\therefore \quad n^4 + 4^n$ is composite number.

Build-up Your Understanding 3

1. Show that $4n^3+6n^2+4n+1$ is composite for $n = 1, 2, 3\ldots$
2. Prove that $512^3 + 675^3 + 720^3$ is not a prime number.
3. Prove that $5^{12} + 2^{10}$ is composite.
4. Show that $3^{2008} + 4^{2009}$ can be written as a product of two integers each of which is greater than 2009^{182}. **[RMO, 2009]**
5. Prove that if p and $p^2 + 2$ are primes, then $p^3 + 2$ is also a prime.
6. Prove that if $2n + 1$ and $3n + 1$ are squares, then $5n + 3$ is not prime where, $n \in \mathbb{N}$.
7. Find all distinct primes p, q such that $p^2 - 2q^2 = 1$.
8. Find all integers n such that $\left(\dfrac{n^3-1}{5}\right)$ is prime.
9. Find all numbers p such that all six numbers p, $p + 2$, $p + 6$, $p + 8$, $p + 12$, and $p + 14$ are primes.
10. Prove that $N = \dfrac{5^{125}-1}{5^{25}-1}$ is a composite number.
11. Find all primes p and q such that $p^2 + 7pq + q^2$ is a square of an integer. **[RMO, 2001]**
12. Find all triples (p, q, r) of primes such that $pq = r + 1$ and $2(p^2 + q^2) = r^2 + 1$. **[RMO, 2013]**
13. Prove that, if a, b are prime numbers ($a > b$), each containing at least two digits, then $(a^4 - b^4)$ is divisible by 240. Also prove that, 240 is the gcd of all the numbers which arise in this way.
14. Prove that there are infinitely many primes of the form $4n - 1$.
15. Prove that there are infinitely many primes of the form $6n - 1$.
16. If $ab = cd$, prove that $a^2 + b^2 + c^2 + d^2$ is composite.
17. Let $m, n \in \mathbb{N}$ such that $2m^2 + m = 2n^2 + n$, then prove that $m - n$ and $2m + 2n + 1$ are perfect squares.
18. Let $a, b, c, d \in \mathbb{N}$ and in strictly increasing order such that $b^2 - bd - d^2 = a^2 - ac - c^2$. Prove that $ab + cd$ is not a prime number.
19. Let $\langle p_1, p_2, p_3, \ldots, p_n, \ldots\rangle$ be a sequence of primes defined by $p_1 = 2$ and for $n \geq 1$, p_{n+1} is the largest prime factor of $p_1p_2\cdots p_n + 1$. (Thus $p_2 = 3$, $p_3 = 7$). Prove that $p_n \neq 5$ for any n. **[RMO, 2004]**
20. Let n be a positive integer and $p_1, p_2, \ldots, p_n$ be n prime numbers all larger than 5 such that 6 divides $p_1^2 + p_2^2 + \cdots + p_n^2$. Prove that 6 divides n. **[RMO, 1998]**
21. Prove that for $n \geq 5$, $p_{n+1}^3 < p_1p_2\cdots p_n$ where p_i is the ith prime.
22. (a) If n is not a prime, prove that $2^n - 1$ is not a prime.
    (b) Prove that if $a^n - 1$ is prime, then $a = 2$ and n must be a prime. The smallest p for which $2^p - 1$ is composite is $11(2^{11} = 2047 = 23 \times 89)$. Prime numbers of the form $2^p - 1$ are called MERSENNE primes and usually denoted by M_p.
    (c) Show that every prime divisor of $2^p - 1$ is of the form $2kp + 1$ for some $k \in \mathbb{N}$.
23. (a) If n has an odd divisor > 1, prove that $2^n + 1$ is not prime.
    (b) Prove that if $a^n + 1$ is prime and $a > 1$, then a must be even and $n = 2^k$ for some $k \in \mathbb{N}$. Numbers of the form $2^{2^n}+1$ are called FERMAT numbers, and usually denoted by F_n. The only Fermat numbers known to be prime correspond to $n < 4$.
    (c) Show that every prime divisor of $2^{2^n}+1$ is of the form $k2^{n+2}+1$ for some $k \in \mathbb{N}$.

Largest known Mersenne prime is $2^{74,207,281}-1$. It has 22,338,618 digits! As of Jan 2016, 49 Mersenne primes are known.

6.5 Fundamental Theorem of Arithmetic

Every integer greater than 1 can be expressed as a product of primes. The factorisation is unique but for the order of the factors.

Any number n can be written as

$$n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times p_3^{\alpha_3} \times \cdots \times p_m^{\alpha_m}$$

where $p_1, p_2, p_3, \ldots, p_m$ are distinct primes and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m$ are natural numbers.

Notes:

1. A number $n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times \cdots \times p_m^{\alpha_m}$ is a perfect square, if and only if each $\alpha_i (i = 1, 2, 3, \ldots, m)$ is an even number.
2. If $n = p_1 \times p_2 \times \cdots \times p_m$, then n is called a square-free number. That is if each α_i $(i = 1, 2, \ldots, m)$ is 1, then n is square-free integer.

6.6 Number of Positive Divisors of a Composite Number

If a composite number is

$$n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times \cdots \times p_m^{\alpha_m}$$

then the number of positive divisors of n is $\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_m + 1)$. This is read as 'tau of n'.

For example, if we take a number $24 = 2^3 \times 3^1$, the number of divisors of 24 is $\tau(24) = (3 + 1)(1 + 1) = 8$.

You can easily see that 1, 2, 3, 4, 6, 8, 12, 24 are the 8 divisors of 24.

Notes:

1. If n is a perfect square, $\tau(n)$ is odd as all the α_i are even.
2. If n is not *a* perfect square, $\tau(n)$ is even.
3. The number of ways of writing n as the product of two factors (order immaterial) is:

   if n is a perfect square, $\dfrac{\tau(n)+1}{2}$

   if n is not a perfect square, $\dfrac{\tau(n)}{2}$.
4. The number of ways, in which a composite number can be expressed as a product of two relative prime factors (order not considered), is 2^{m-1}, where m is the number of distinct prime.

   For example, $5^8 \times 3^7 \times 41^5$ can be resolved into product of two factors, in $2^{3-1} = 2^2 = 4$ ways so that the factors are co-prime numbers.

   Here they are

   $$5^8 \times (3^7 \times 41^5)$$
   $$3^7 \times (5^8 \times 41^5)$$
   $$41^5 \times (3^7 \times 5^8)$$

   and finally $1 \times (41^5 \times 3^7 \times 5^8)$.

Now $\sigma(n)$, (This is read as sigma of n) the sum of the positive divisors of n, is given by

$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \times \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \times \cdots \times \frac{p_m^{\alpha_m+1} - 1}{p_m - 1},$$

where $n = p_1^{\alpha_1} \times p_2^{\alpha_2} \times \cdots \times p_m^{\alpha_m}$.

For example,

$$\sigma(48) = \sigma(2^4 \times 3)$$
$$= \frac{2^5 - 1}{2 - 1} \times \frac{3^2 - 1}{3 - 1} = 31 \times 4 = 124$$

$\sigma_k(n)$, the sum of the kth power of the positive divisors of n

$$= \frac{p_1^{k(\alpha_1+1)} - 1}{p_1^k - 1} \times \frac{p_2^{k(\alpha_2+1)} - 1}{p_2^k - 1} \times \cdots \times \frac{p_m^{k(\alpha_m+1)} - 1}{p_m^k - 1}.$$

Example 18 *Find the smallest integer with exactly* 24 *divisors.*

Solution: If n is the required number and

$$n = p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}$$

then $\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$.

But 24 can be written as the product of 2 or 3 or 4 factors. Corresponding to each factorisation, we can get a smallest composite number.

24	2^{23}	148576
12×2	$2^{11} \times 3^1$	6144
6×4	$2^5 \times 3^3$	864
8×3	$2^7 \times 3^2$	1152
$6 \times 2 \times 2$	$2^5 \times 3^1 \times 5^1$	480
$4 \times 3 \times 2$	$2^3 \times 3^2 \times 5$	**360**
$3 \times 2 \times 2 \times 2$	$2^2 \times 3 \times 5 \times 7$	420

The smallest number having 24 divisors is 360.

Example 19 *Find the sum of the cubes of the divisors of* 12.

Solution: Since $12 = 2^2 \times 3$

$$\therefore \quad \sigma_3(12) = \frac{2^{3(2+1)} - 1}{2^3 - 1} \times \frac{3^{3(1+1)} - 1}{3^3 - 1}$$
$$= \frac{2^9 - 1}{7} \times \frac{3^6 - 1}{26}$$
$$= 73 \times 28 = 2044.$$

Example 20 *Show that* $\sigma(N) = 4N$ *when* $N = 30240$.

Solution: Since $N = 30240 = 2^5 \times 3^3 \times 5^1 \times 7^1$.

$$\text{So, } \sigma(N) = \frac{(2^6 - 1)}{2 - 1} \times \frac{(3^4 - 1)}{(3 - 1)} \times \frac{(5^2 - 1)}{(5 - 1)} \times \frac{(7^2 - 1)}{(7 - 1)}$$
$$= 63 \times 40 \times 6 \times 8$$
$$= 2^7 \times 3^3 \times 5 \times 7$$
$$= 2^2 \times 2^5 \times 3^3 \times 5^1 \times 7^1$$
$$= 4 \times N = 4N.$$

Example 21 *$N = P_1P_2P_3$ and P_1, P_2 and P_3 are distinct prime numbers. If $\sum_{d|N} d = 3N$ [or $\sigma(N) = 3N$], show that $\sum_{i=1}^{N} \frac{1}{d_i} = 3$.*

Solution: The divisors of N are

$$1, P_1, P_2, P_3, P_1P_2, P_1P_3, P_2P_3, P_1P_2P_3.$$

It is given that

$$1 + P_1 + P_2 + P_3 + P_1P_2 + P_1P_3 + P_2P_3 + P_1P_2P_3 = 3N.$$

Now

$$\sum_{i=1}^{N} \frac{1}{d_i} = \frac{1}{1} + \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} + \frac{1}{P_1P_2} + \frac{1}{P_1P_3} + \frac{1}{P_2P_3} + \frac{1}{P_1P_2P_3}$$
$$= \frac{P_1P_2P_3 + P_2P_3 + P_1P_3 + P_1P_2 + P_3 + P_2 + P_1 + 1}{P_1P_2P_3}.$$

But the numerator is the sum of the divisors of N,

$$i.e., \sum_{d|N} d = 3N = 3P_1P_2P_3 \text{ and hence, } \sum_{i=1}^{N} \frac{1}{d_1} = \frac{3P_1P_2P_3}{P_1P_2P_3} = 3.$$

Example 22 *Let $f(n)$ be sum of number of divisors of divisors of n. Prove that $f(18) = f(2) \cdot f(3^2)$.*

Solution: Divisors of 18 are 1, 2, 3, 6, 9, 18 and therefore,

$$f(18) = \sum_{q|18} \tau(q) = 1 + 2 + 2 + 4 + 3 + 6 = 18$$

$$f(2) = \sum_{q|2} \tau(q) = 1 + 2 = 3$$

$$f(3^2) = \sum_{q|3^2} \tau(q) = 1 + 2 + 3 = 6$$

$\therefore \quad f(2) \cdot f(3) = 3 \times 6 = 18 = f(18).$

Example 23 *Show that $f(p_1^{\alpha_1} \cdot p_2^{\alpha_2}) = f(p_1^{\alpha_1}) \cdot f(p_2^{\alpha_2})$,* where p_1 and p_2 are *distinct prime.*

Solution: The divisors of $p_1^{\alpha_1} \cdot p_2^{\alpha_2}$ of the form $p_1^{r} \cdot p_2^{s}$, where $0 \le r \le \alpha_1$ and $0 \le s \le \alpha_2$.

$$\text{Now, } f(p_1^{\alpha_1} \cdot p_2^{\alpha_2}) = \sum_{\substack{0 \le r \le \alpha_1 \\ 0 \le s \le \alpha_2}} \tau(p_1^{r} \cdot p_2^{s})$$
$$= \sum_{0 \le r \le \alpha_1} \sum_{0 \le s \le \alpha_2} (r+1)(s+1)$$
$$= \sum_{0 \le r \le g\alpha_1} (r+1) \left[\sum_{0 \le s \le \alpha_2} (s+1) \right]$$

$$= \sum_{0\le r\le \alpha_1} (r+1)\left(\frac{(\alpha_2+1)(\alpha_2+2)}{2}\right)$$

$$= \frac{(\alpha_2+1)(\alpha_2+2)}{2} \sum_{0\le r\le \alpha_1} (r+1)$$

$$= \frac{(\alpha_2+1)(\alpha_2+2)}{2}\,\frac{(\alpha_1+1)(\alpha_1+2)}{2}$$

$\because$ $$f(p_1^{\alpha_1}) = \sum_{0\le r\le \alpha_1} \tau(p_1^r) = \sum_{0\le r\le \alpha_1} (r+1)$$

$$= \frac{(\alpha_1+1)(\alpha_2+2)}{2}$$

Similarly, $f(p_2^{\alpha_2}) = \dfrac{(\alpha_2+1)(\alpha_2+2)}{2}$

$\therefore$ $f(p_1^{\alpha_1}\cdot p_2^{\alpha_2}) = f(p_1^{\alpha_1})\cdot f(p_2^{\alpha_2})$

where $p_1 \ne p_2$, *i.e.*, f is multiplicative.

Example 24 *Define* $F(n) = \sum_{d|n} \tau_3(d)$ *where* $\tau_3(d)$ = *cube of the number of divisors of d, i.e.,* $F(n)$ *is defined as the sum of the cubes of the number of divisors of the divisors of n. Prove that* $F(18) = F(3^2)\cdot F(2)$.

Solution: Consider $F(18)$.

Divisors of 18 are 1, 2, 3, 6, 9, 18.

Number of divisors of divisors of 18 are 1, 2, 2, 4, 3, 6.

So, $F(18) = 1^3 + 2^3 + 2^3 + 4^3 + 3^3 + 6^3 = 324$

Now, $18 = 2^1 \times 3^2$

$$F(2^1) = 1^3 + 2^3 = 9$$

$$F(3^2) = F(9) = 1^3 + 2^3 + 3^3 = 36$$

and $F(2)\times F(3^2) = 9\times 36 = 324 = F(18)$.

Thus, F is also multiplicative.

Example 25 *Show that* $F(p_1^{\alpha_1}\times p_2^{\alpha_2}) = F(p_1^{\alpha_1})\times(p_2^{\alpha_2})$.

Solution: Any divisor of $p_1^{\alpha_1}$ is p_1^r where $0 \le r \le \alpha_1$

$F(p_1^{\alpha_1}) = \sum_{r=0}^{\alpha_1} \tau_3(p_1^r) = \sum_{r=0}^{\alpha_1} (r+1)^3$ = sum of the cubes of the first α_1 + 1 natural numbers.

$$= \left[\frac{(\alpha_1+1)(\alpha_2+2)}{2}\right]^2.$$

Similarly, $F(p_2^{\alpha_2}) = \left[\dfrac{(\alpha_2+1)(\alpha_2+2)}{2}\right]^2$

$$
\begin{aligned}
F(p_1^{\alpha_1}\cdot p_2^{\alpha_2}) &= \sum_{\substack{0\le r\le\alpha_1\\0\le s\le\alpha_2}} \tau_3(p_1^r\cdot p_2^s)\\
&= \sum_{r=0}^{\alpha_1}\cdot\sum_{s=0}^{\alpha_2}(r+1)^3(s+1)^3\\
&= \sum_{r=0}^{\alpha_1}(r+1)^3\left(\sum_{s=0}^{\alpha_2}(s+1)^3\right)\\
&= \sum_{r=0}^{\alpha_1}(r+1)^3\cdot\left[\frac{(\alpha_2+1)(\alpha_2+2)}{2}\right]^2\\
&= F(p_2^{\alpha_2})\cdot\sum_{r=0}^{\alpha_1}(r+1)^3\\
&= F(p_2^{\alpha_2})\left[\frac{(\alpha_1+1)(\alpha_1+2)}{2}\right]^3\\
&= F(p_2^{\alpha_2})F(p_1^{\alpha_1}).
\end{aligned}
$$

Hence, proved.

Example 26 *Prove that* $F(p_1^{\alpha_1})=\{f(p_1^{\alpha_1})\}^2$, *where F and f are as defined in previous problems.*

Solution: Since

$$F(p_1^{\alpha_1})=1^3+2^3+3^3+\cdots+(\alpha_1+1)^3$$

$$
\begin{aligned}
[f(p_1^{\alpha_1})]^2 &= [1+2+3+\cdots+(\alpha_1+1)]^2\\
&= \left[\frac{(\alpha_1+1)(\alpha_1+2)}{2}\right]^3\\
&= 1^3+2^3+\cdots+(\alpha_1+1)^3\\
&= F(p_1^{\alpha_1}).
\end{aligned}
$$

Example 27 *Prove that sum of the cubes of the number of divisors of the divisors of a given number is equal to square of their sum.* [*For example, if* $N = 18$.] *The divisors of* 18 *are* 1, 2, 3, 6, 9, 18.

Number of divisors of divisors of 18 are 1, 2, 2, 4, 3, 6 *respectively*.

Sum of the cubes of these numbers

$$
\begin{aligned}
1^3+2^3+2^3+4^3+3^3+6^3 &= (1^3+2^3+3^3+4^3)+2^3+6^3\\
&= 100+224=324.
\end{aligned}
$$

Square of the sum of these divisors $= (1 + 2 + 2 + 4 + 3 + 6)^2$

$$= 18^2 = 324.$$

Solution: The solution is based on the result derived in previous problems.

We should show that $F(N) = f(N)^2$, where F and f are as defined in previous problems.

[This interesting property of numbers was originally given by Liouville, and Srinivasa Ramanujan, rediscovered it.]

If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, then

$F(n) = F(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_n^{\alpha_n})$ and $p_1, p_2, \ldots, p_n$ distinct prime numbers and we have proved earlier that F is multiplicative.

$$\therefore \quad F(n) = F(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_n^{\alpha_n})$$
$$= F(p_1^{\alpha_1}) \cdot F(p_2^{\alpha_2}) \ldots F(p_n^{\alpha_n}).$$

But $F(p_i^{\alpha_i}) = 1^3 + 2^3 + \cdots + \alpha_i^3$

$$= \left[\frac{(\alpha_i + 1)(\alpha_i + 2)}{2}\right]^2 \quad \text{for all } i \in \mathbb{N}$$

We have

$$F(n) = \left[\frac{(\alpha_1 + 1)(\alpha_1 + 2)}{2}\right]^2 \cdot \left[\frac{(\alpha_2 + 1)(\alpha_2 + 2)}{2}\right]^2 \cdots \left[\frac{(\alpha_n + 1)(\alpha_n + 2)}{2}\right]^2$$

$$= [(\alpha_1 + 1)(\alpha_1 + 2)(\alpha_2 + 1)(\alpha_2 + 2) \ldots (\alpha_n + 1) \times (\alpha_n + 2)]^2/(2^n)^2 \tag{1}$$

Now, $f(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_n^{\alpha_n})$

$$= f(p_1^{\alpha_1}) \cdot f(p_2^{\alpha_2}) \cdots f(p_n^{\alpha_n}) \quad [\because f \text{ is multiplicative}]$$

$$= \frac{(\alpha_1 + 1)(\alpha_1 + 2)}{2} \cdot \frac{(\alpha_2 + 1)(\alpha_2 + 2)}{2} \cdots \frac{(\alpha_n + 1)(\alpha_n + 2)}{2}$$

$$= (\alpha_1 + 1)(\alpha_1 + 2)(\alpha_2 + 1)(\alpha_2 + 2) \ldots \frac{(\alpha_n + 1)(\alpha_n + 2)}{2^n} \tag{2}$$

$\therefore$ From (1) and (2), we see that $F(n) = [f(n)]^2$.

6.6.1 Perfect Numbers

If the sum of the divisors of a number n, other than itself, is equal to n, then n is called a perfect number. For example, the first two perfect numbers are 6 and 28.

$$6 = 1 + 2 + 3$$

$$28 = 1 + 2 + 4 + 7 + 14$$

There are 49 perfect numbers known to date (January 2016) all even, and it is conjectured that there are no odd perfect numbers.

Example 28 *Show that $n = 2^{m-1}(2^m - 1)$ is a perfect number, if $(2^m - 1)$ is a prime number.*

Solution: Let $n = 2^{m-1} \times p$, where $p = 2^m - 1$ is a prime number.

The divisors of $2^{m-1} \times p$ are

$$1, 2, 2^2, 2^3, \ldots, 2^{m-1}, p, 2p, 2^2p, \ldots, 2^{m-2}p, 2^{m-1}p$$

Now, we should sum all these divisors except the last one, $2^{m-1}p$.

$$S = (1+2+2^2+\cdots+2^{m-1}) + p(1+2+2^2+\cdots+2^{m-2})$$
$$= \frac{1(2^m-1)}{2-1} + \frac{p[1(2^{m-1}-1)]}{2-1}$$
$$= 2^m - 1 + p(2^{m-1}-1)$$
$$= p + p(2^{m-1}-1) \quad [\because \quad p = 2^m - 1]$$
$$= p \cdot 2^{m-1}$$
$$= 2^{m-1}(2^m-1) = n.$$

Example 29 *$N = 2^{n-1}$ $(2^n - 1)$ and $(2^n - 1)$ is a prime number. $1 < d_1 < d_2 < \cdots < d_k = N$ are the divisors of N. Show that*

$$\frac{1}{1} + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = 2.$$

Solution: Let $2^n - 1 = q$.

We already saw that 1, d_1, d_2,..., d_k are 1, 2, 2^2, …, 2^{n-1}, q, $2q$, …, $2^{n-1}q$, respectively.

So, $S = \frac{1}{1} + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k}$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{q} \times \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right]$$

$$\therefore \quad S = \frac{2^n-1}{2^{n-1}} + \frac{1}{q}\frac{(2^n-1)}{2^{n-1}} = \frac{(2^n-1)q + (2^n-1)}{q2^{n-1}}$$
$$= \frac{(2^n-1)(q+1)}{q2^{n-1}} = \frac{(2^n-1)(2^n)}{(2^n-1)(2^{n-1})}$$
$$= \frac{2^n}{2^{n-1}} = 2.$$

Example 30 *If n_1 and n_2 are two numbers, such that the sum of all the divisors of n_1 other than n_1 is equal to n_2 and sum of all the divisors of n_2 other than n_2 is equal to n_1, then the pair (n_1, n_2) is called an amicable number pair.*

Given: $a = 3 \cdot 2^n - 1$,

$b = 3 \cdot 2^{n-1} - 1$

and $c = 9 \cdot 2^{2n-1} - 1$, $n > 1$

where a, b and c are all primes numbers, then show that $(2^n ab, 2^n c)$ *is an amicable pair.*

Solution: If $N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, then sum of the divisors: of N is given by the formula

$$\sum d(N) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \times \frac{p_2^{\alpha_2+1}-1}{p_2-1} \times \cdots \frac{p_n^{\alpha_n-1}-1}{p_n-1}$$

So, the sum of the divisors of $2^n ab$ is

$$(2^{n+1}-1) \times \frac{a^2-1}{a-1} \times \frac{b^2-1}{b-1}$$
$$= (2^{n+1} - 1)(a + 1)(b + 1)$$
$$= (2^{n+1} - 1)(9.2^{2n-1}).$$

But $2^n ab = 2^n[9\cdot 2^{2n-1} - 9\cdot 2^{n-1} + 1]$ (on simplification)
The sum of the divisors of $2^n ab$ other than $2^n a \cdot b$ is

$$
\begin{aligned}
& 9\cdot 2^{2n-1}(2^{n+1}-1) - 2^n(9.2^{2n-1} - 9.2^{n-1} + 1)\\
&= 9\cdot 2^{3n} - 9\cdot 2^{2n-1} - 9\cdot 2^{3n-1} + 9\cdot 2^{2n-1} - 2^n\\
&= 9\cdot 2^{3n-1}(2-1) - 2^n\\
&= 9\cdot 2^{3n-1} - 2^n\\
&= 2^n(9.2^{2n-1} - 1)\\
&= 2^n\cdot c
\end{aligned}
$$

Thus, the sum of the divisors of $2^n\cdot ab$ other than itself is $2^n c$. Now, sum of the divisors of $2^n c$ other than itself is

$$
\begin{aligned}
& \frac{2^{n+1}-1}{2-1}\times\frac{c^2-1}{c-1} - 2^n\cdot c\\
&= (2^{n+1}-1)(c+1) - 2^n\cdot c\\
&= (2^{n+1}-1)9\cdot 2^{2n-1} - 2^n(9\cdot 2^{2n-1} - 1)\\
&= 9\cdot 2^{3n} - 9\cdot 2^{2n-1} - 9\cdot 2^{3n-1} + 2^n\\
&= 9\cdot 2^{3n-1} - 9\cdot 2^{2n-1} + 2^n\\
&= 2^n[9\cdot 2^{2n-1} - 9\cdot 2^{n-1} + 1]\\
&= 2^n ab
\end{aligned}
$$

i.e., the sum of the divisors of $2^n c$ other than $2^n c$ is equal to $2^n ab$.

Build-up Your Understanding 4

1. Find the number of positive integers which divide 10^{999} but not 10^{998}. **[RMO, 1999]**
2. Find the number of rationals $\frac{m}{n}$ such that (i) $0 < \frac{m}{n} < 1$, (ii) gcd$(m, n) = 1$, (iii) $mn = 25!$. **[RMO, 1994]**
3. Determine largest 3-digit prime factor of $\binom{2000}{1000}$. **[RMO, 1992]**
4. Determine the smallest positive integer n, which has exactly 144 distinct divisors and there are 10 consecutive integers among these divisors.
5. Prove that every even perfect number is of the form $2^{p-1}(2^p-1)$, where 2^p-1 and p must be prime numbers.
6. Prove that every even perfect number ends in 6 or in 28.
7. Show that for any natural number $n \geq 1$, the sum $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2n+1}$ is never an integer.
8. Prove that the sum $\frac{1}{p}+\frac{1}{p+1}+\cdots+\frac{1}{p+n}$ is a fraction when reduced to simplest terms, has an even denominator.
9. $[a, b, c]$ and (a, b, c) denote the least common multiple (*lcm*) and the greatest common divisor (gcd). Show that $\frac{[a,b,c]^2}{[a,b][b,c][c,a]} = \frac{(a,b,c)^2}{(a,b)(b,c)(c,a)}$. **[USA MO, 1972]**

6.7 Modular Arithematic

The set of integers can be partitioned into n disjoint sets or module namely $S_0, S_1 \ldots, S_{n-1}$, where S_r = set of integers with r as remainder when divided by n, for $r = 0, 1, 2, \ldots, n-1$.

Any two numbers belonging to the same set or module S_r are said to be congruent modulo n.

Formally,

if a and b both leave the same remainder or equivalently, $n|(a-b)$ or $a = kn + b$, for some $k \in \mathbb{Z}$ we define,

$$a \equiv b \pmod{n},$$

This is read as a is congruent to b modulo n.

For example, $16 \equiv 1 \bmod 3$ (as $16 = 5 \times 3 + 1$)

Also we can see $16 \equiv 1 \equiv 4 \equiv -2 \equiv -5 \pmod{3}$

We are just adding or subtracting multiples of '3'

Johann Carl Friedrich Gauss

30 Apr 1777–23 Feb 1855
Nationality: German

6.7.1 Properties of Congruence

In what follows n, a, b, c, d, x, y are integers.

1. $a \equiv a \pmod{n}$ (Reflexive relation for all $a \in \mathbb{Z}$)
2. $a \equiv b \pmod{n} \Leftrightarrow b \equiv a \pmod{n}$ (Symmetric relation for all $a, b \in \mathbb{Z}$)
3. $a \equiv b \pmod{n}$, $b \equiv c \pmod{n} \Rightarrow a \equiv c \pmod{n}$ (Transitive relation for all Integers a, b, c)
4. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then
   (i) $a + c \equiv b + d \pmod{n}$
   (ii) $a - c \equiv b - d \pmod{n}$
   (iii) $ac \equiv bd \pmod{n}$
   (iv) $ax + cy \equiv bx + dy \pmod{n}$
5. If $a \equiv b \pmod{n}$, then
   (i) $a + c \equiv b + c \pmod{n}$
   (ii) $a - c \equiv b - c \pmod{n}$
   (iii) $ac \equiv bc \pmod{n}$
   (iv) $a + k_1 n \equiv b + k_2 n \pmod{n}$; $k_1, k_2 \in \mathbb{Z}$
   (v) $a^m \equiv b^m \pmod{n}$, $m \in \mathbb{N}$
6. $a \equiv b \pmod{c} \Rightarrow \quad f(a) \equiv f(b) \pmod{c}$
   Where f is a polynomial over $\mathbb{Z}$. *i.e.*, $f(x)$ is a polynomial with integer coefficients
7. $a\lambda \equiv b\lambda \pmod{n} \quad \Rightarrow \quad a \equiv b\left(\bmod \dfrac{n}{(\lambda, n)}\right)$
   In particular, if $\gcd(\lambda, n) = 1$, then $a\lambda \equiv b\lambda \pmod{n} \Rightarrow a \equiv b \pmod{n}$
8. If $n \neq 0$ and $(a, n) = 1$, then there exists an integer a' such that $aa' \equiv 1 \pmod{n}$ which is called the inverse of 'a' modulo n.

Example 31 *Find the largest positive integer n such that $n^3 + 100$ is divisible by $(n + 10)$.*

Solution: Using modulo $(n + 10)$ numbers, we see that

$$n + 10 \equiv 0 \; [\bmod\, (n + 10)]$$

$$i.e., \quad n \equiv -10[\bmod (n+10)]$$

$$n^3 \equiv (-10)^3 [\bmod (n+10)]$$

$$\equiv -1000[\bmod (n+10)]$$

$$\therefore n^3 + 100 \equiv (-1000+100)[\bmod (n+10)]$$

$$\equiv -900[\bmod (n+10)].$$

Now, we want $(n + 10)$ to divide $n^3 + 100$, implying that $(n + 10)$ should divide -900.

The largest such n is $900 - 10 = 890$, as $(n + 10)$ cannot be greater than $|-900| = 900$ and the greatest divisor of $|-900|$ is 900.

So the largest positive integer n, such that $n^3 + 100$ is divisible by $(n + 10)$ is $n = 890$.

Note: $900 = 3^2 \times 2^2 \times 5^2$ has 27 divisors and each divisor greater than 10, gives a corresponding value for n they are 2, 5, 8, 10, 15, 20, 26, 35, 40, 50, 65, 80, 90, 140, 170, 215, 290, 440, and 890.

Example 32 *Determine all positive integers n for which* $2^n + 1$ is *divisible by* 3.

Solution: $2^n + 1 = 2^n + 1^n$.

If n is odd, then $(2 + 1)$ is a factor. Thus for all odd values of n, $2^n + 1$ is divisible by 3.

For n even $= 2\,m$ say,

$2^n + 1 = 2^{2m} + 1 = 4^m - 1 + 2$

Now $3 = 4 - 1$ divides $4^{\mathrm{m}} - 1^n$ but $3 \nmid 2 \Rightarrow 3 \nmid (2^m + 1)$ for n even.

Aliter: $2 \equiv -1 \pmod 3$

$\Rightarrow 2^{2m+1} \equiv -1 \pmod 3$ and $2^{2m} \equiv 1 \pmod 3$

So, $2^n + 1 \equiv 0 \pmod 3$, if n is odd.

and $2^n + 1 \equiv 1 + 1 = 2 \pmod 3$, if n is even.

Therefore, $2^n + 1$ is divisible by 3, if and only if, n is an odd number.

Example 33 *What is the remainder when* 2016^{2016} *is divided by* 2017?

Solution: As $2016 \equiv -1 \pmod{2017}$

$\Rightarrow \quad 2016^{2016} \equiv 1 \pmod{2017}$

Example 34 *Find the remainder when* $452^{72^{452}}$ *is divided by* 3.

Solution: This problem doesn't require much work, just one insight leads to immediate solution, we note that $452 \equiv -1 \pmod 3$, thus

$$452^{72^{452}} \equiv (-1)^{72^{452}} \equiv 1 (\bmod 3).$$

The last congruence holds because 72^{452} is surely even.

Example 35 *Suppose* 5^{5555} *is divided by* 24, *find the remainder.*

Solution: It is not hard to find, by inspection, that $5^2 = 25 \equiv 1 \pmod{24}$. Now we can write

$$5^{5555} = 5^{5554} \cdot 5 = (5^2)^{2777} \cdot 5 \equiv 1^{2777} \cdot 5 \equiv 5$$

Finding number 'a' such that $5^a \equiv 1 \pmod{n}$ basically allows us to reduce the exponent in a problem, if the a is small such as 2, then the reduction is very drastic as seen in example above.

Example 36 *Show that* $2^{55}+1$ *is divisible by* 11.

Solution: $2^5 = 32 \equiv (-1) \pmod{11}$

$$2^{55} = (2^5)^{11} \equiv (-1)^{11} \equiv -1 \pmod{11}$$

So, $2^{55} + 1 \equiv 0 \pmod{11}$

$\therefore$ It is a multiple of 11.

Example 37 *Find the sum of all integers n, such that* $1 \le n \le 1998$ *and that* 60 *divides* $n^3 + 30n^2 + 100n$.

Solution:

(i) If $60 = 3 \times 4 \times 5$ and $4 | 100n$, then 4 should divide $n^3 + 30n^2$, *i.e.,* 4 should divide $n^2(n + 30)$. This implies that n is even. *i.e.*, 2|n

(ii) As $5 | (30n^2 + 100n)$, 5 should divide n^3. Hence, 5 should divide n.

(iii) As $3 | 30n^2$, then 3 should divide $n^3 + 100n$, *i.e.*, 3 should divide $n(n^2 + 100n) = n(n^2 + 1 + 99)$

If $n \equiv \pm 1 \pmod 3$, $n^2 \equiv 1 \pmod 3$, and $n^2 + 1 \equiv 2 \pmod 3$, so neither of $(n^2 + 1 + 99)$ and n are divisible by 3.

However, if $n \equiv 0 \pmod 3$, then $n(n^2 + 1 + 99)$ is divided by 3, *i.e., n*$(n^2 + 100)$ is divisible by 3 only if n is a multiple of 3.

From (i), (ii), and (iii), we find that n must be a multiple of $2 \times 3 \times 5 = 30$. So, we should find the sum of all multiples of 30 less than 1998

$$S_n = 30 + 60 + \cdots + 1980$$
$$= 30(1 + 2 + \cdots + 66) = 66330.$$

Example 38 *Find the last two digits of* $(56789)^{41}$.

Solution: $56789 \equiv 89 \pmod{100}$

$$\equiv -11 \pmod{100}$$

$$\therefore (56789)^{41} \equiv (-11)^{41} \pmod{100}$$
$$\equiv (-11)^{40} \times (-11) \pmod{100}$$
$$\equiv (11)^{40} \times (-11) \pmod{100}$$
$$11^2 \equiv 21 \pmod{100}$$
$$11^4 \equiv 41 \pmod{100}$$
$$11^6 \equiv 21 \times 41 \pmod{100}$$
$$\equiv 61 \pmod{100}$$
$$11^{10} \equiv 41 \times 61 \pmod{100}$$
$$\equiv 01 \pmod{100}$$
$$11^{40} \equiv (01)^{40} \pmod{100}$$
$$\equiv 1 \pmod{100}$$

$$(-11)^{41} \equiv 11^{40} \times (-11) \pmod{100}$$
$$\equiv 1 \times (-11) \pmod{100}$$
$$\equiv -11 \pmod{100}$$
$$(56789)^{41} \equiv 89 \pmod{100}$$

i.e., the last two digits of $(56789)^{41}$ are 8 and 9 in that order.

Example 39 *Prove that* $2222^{5555} + 5555^{2222}$ *is divisible by* 7.

Solution: Since $2222^{5555} + 5555^{2222}$

$= 2222^{5555} + 4^{5555} + 5555^{2222} - 4^{2222} - 4^{5555} + 4^{2222}$

$= (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - 4^{2222}\,(4^{3333} - 1)$

Now, $2222^{5555} + 4^{5555}$ is divisible by $2222 + 4 = 2226 = 7 \times 318$, $5555^{2222} - 4^{2222}$ is divisible by $5555 - 4 = 5551 = 7 \times 793$ and $4^{3333} - 1 = (4^3)^{1111} - 1 = 64^{1111} - 1$ is divisible by $64 - 1 = 63 = 7 \times 9$.
Thus $2222^{5555} + 5555^{2222}$ can be split up into three terms each of which is divisible by 7 and hence, the result.

Aliter:

$2222 \equiv 3 \pmod 7$

$\Rightarrow 2222^2 \equiv 9 \equiv 2 \pmod 7$ (1)

$\Rightarrow 2222^4 \equiv 4 \pmod 7$ (2)

$\Rightarrow 2222^6 \equiv 8 \equiv 1 \pmod 7$ (From (1) and (2))

$\Rightarrow 2222^{5555} = [(2222)^6]^{925} \times 2222^5 = [(2222)^6]^{925} \times 2222^4 \times 2222^1$
$\equiv 1 \times 4 \times 3 \pmod 7 \equiv 12 \equiv 5 \pmod 7$

Also $5555 \equiv 4 \pmod 7$

$\Rightarrow 5555^3 \equiv 4^3 \pmod 7 \equiv 1 \pmod 7$

$\Rightarrow (5555)^{2222} = (5555^3)^{740} \times 5555^2 \equiv 1 \times 4 \times 4 \pmod 7 \equiv 2 \pmod 7$

and hence, $2222^{5555} + 5555^{2222} \equiv 5 + 2 = 0 \pmod 7$ and hence, the result.

Example 40 *If a, b, c are any three integers, then show that* $abc(a^3 - b^3)(b^3 - c^3)(c^3 - a^3)$ *is divisible by* 7.

Solution: Let us find the value of $a^3 \pmod 7$ for any $a \in \mathbb{Z}$.
As, $a \pmod 7$ is $0, \pm1, \pm2, \pm3$, $a^3 \pmod 7$ will be only among $0, \pm1$.
Now, if 7 divides one of a, b, c, the given expression is divisible by 7. If not, then $a^3, b^3, c^3 \pmod 7$ will be only among 1 and -1. Hence, two of them must be the same, say a^3 and $b^3 \pmod 7$.
$\therefore\ (a^3 - b^3) \equiv 0 \pmod 7$. The given expression is divisible by 7.

Example 41 *Let* $f(x)$ *be a polynomial with integral coefficients. Suppose that both* $f(1)$ *and* $f(2)$ *are odd. Then, prove that, for any integer* n, $f(n) \neq 0$.

Solution: Let $f(n) = 0$ for some integer n

If $n \equiv 1 \pmod 2$

Then $f(n) \equiv f(1) \pmod 2 \Rightarrow 0 \equiv \text{odd} \pmod 2$ which is a contradiction

If $n \equiv 2 \pmod 2$

Then $f(n) \equiv f(2) \pmod 2 \Rightarrow 0 \equiv \text{odd} \pmod 2$ which is a contradiction

So, there exists no integer n, for which $f(n) = 0$.

Example 42 *If a, b, c are positive integers less than 13 such that*

$$2ab + bc + ca \equiv 0 \pmod{13}$$
$$ab + 2bc + ca \equiv 6abc \pmod{13}$$
$$ab + bc + 2ca \equiv 8abc \pmod{13}$$

Then determine the remainder when $a + b + c$ is divided by 13.

Solution: As 13 is prime, we may multiply each equation by $(abc)^{-1}$:

$$2c^{-1} + a^{-1} + b^{-1} \equiv 0 \pmod{13} \quad (1)$$

$$c^{-1} + 2a^{-1} + b^{-1} \equiv 6 \pmod{13} \quad (2)$$

$$c^{-1} + a^{-1} + 2b^{-1} \equiv 8 \pmod{13} \quad (3)$$

Adding (1), (2) and (3) we get

$4\,(a^{-1} + b^{-1} + c^{-1}) \equiv 14 \equiv 1 \pmod{13} \equiv 1 + 3 \times 13 \pmod{13}$

$$\Rightarrow \quad a^{-1} + b^{-1} + c^{-1} \equiv 10 \pmod{13} \quad (4)$$

From (1) and (4) we get $c^{-1} \equiv -10 \equiv 3 \pmod{13}$.

$\Rightarrow \quad c \equiv 3^{-1} \bmod 13$

$\Rightarrow \quad 3c \equiv 1 \bmod 13$

$\Rightarrow \quad 3c \equiv 1 + 2 \times 13 \pmod{13} \equiv 27 \pmod{13}$

$\Rightarrow \quad c \equiv 9 \pmod{13}$

Similarly, $a \equiv 3 \pmod{13}$ and $b \equiv 6 \pmod{13}$ and therefore our answer is $a + b + c \equiv 3 + 6 + 9 \equiv 5 \pmod{13}$.

Example 43 *Find the last three digits of* $2005^{11} + 2005^{12} + \cdots + 2005^{2006}$.

Solution: Finding last n digits of a number is done by finding the remainder when said number is divided by 10^n.

We note that $2005 \equiv 5 \pmod{1000}$, so the sum is congruent to

$$5^{11} + 5^{12} + \cdots + 5^{2006} \pmod{1000},$$

We have $5^4 = 625$ and $5 \cdot 625 \equiv 125 \pmod{1000}$, but $5 \cdot 125 = 625$, so powers of 5 modulo 1000 repeat periodically 625, 125, 625, 125, … that is to say $5^n \equiv 625 \pmod{1000}$ for even $n \geq 4$ and $5^m \equiv 125 \pmod{1000}$ for odd $m \geq 5$. So we can write the sum as

$$5^{11} + 5^{12} + \cdots + 5^{2005}$$
$$\equiv \underbrace{125+625+125+\cdots+125+625}_{1996 \text{ terms}} \pmod{1000},$$

Now

$$\underbrace{125+625+125+\cdots+125+625}_{1996 \text{ terms}} = 998 \cdot 625 + 998 \cdot 125 = 998 \cdot 750$$

Thus the sum is congruent to

$$998 \cdot 750 \equiv (-2)(-250) \equiv 500 \pmod{1000}.$$

Example 44 *Let n be a number that is made from a string of* 5*s and is divisible by* 2003*. What is the last* 6 *digits of quotient when n is divided by* 2003?

Solution: Let $2003x = 55\ldots55555555$

$\Rightarrow \quad 3x \equiv 555 \pmod{1000}$

$\Rightarrow \quad x \equiv 185 \pmod{1000}$

$\Rightarrow \quad x = 10^3 y + 185$ Say

$\Rightarrow \quad 2003(10^3 y + 185) = 55\ldots55555555$

$\Rightarrow \quad (2003000)y + 370555 = 55\ldots55555555$

$\Rightarrow \quad (2003000)y = 55\ldots55185000$

$\Rightarrow \quad 3y \equiv 185 \pmod{1000}$

$\Rightarrow \quad 3y \equiv 1185 \pmod{1000}$

$\Rightarrow \quad y \equiv 395 \pmod{1000}$

Hence, $x \equiv 395185 \pmod{1000000}$

Example 45 *If a and b are two integers such that* 11 *divides* $a^2 + b^2$*, show that* 121 *divides* $a^2 + b^2$.

Solution: Suppose 11 divides $a^2 + b^2$.

If 11 divides a^2, then 11 should also divide b^2, which implies that 11 divides a and b both, and in turn 121 divides a^2 and also b^2 and hence, 121 divides $a^2 + b^2$.

Assume 11 divides neither a^2 nor b^2.

Let $a \equiv k \pmod{11}$, where $k = 1, 2, \ldots, 10$.

Therefore, $a^2 \equiv k^2\ (mod\ 11) = l \pmod{11}$, where $l = 1, 4, 9, 5, 3$.

Similarly, $b^2 \equiv m \pmod{11}$, where $m = 1, 4, 9, 5, 3$

$\therefore \quad a^2 + b^2 \equiv (l + m) \pmod{11}$. But $l + m \not\equiv 0 \pmod{11}$

$\therefore \quad 11 \mid (a^2 + b^2)$ iff $11 \mid a^2$ and $11 \mid b^2$ and hence, $121 \mid (a^2 + b^2)$.

Example 46 *Show that if the sum of the square of two whole numbers is divisible by* 3*, then each of them is divisible by* 3.

Solution: Let x and y be any two integers

Then $\quad x \equiv 0, 1, 2 \pmod 3$

and $\quad x^2 \equiv 0, 1 \pmod 3$

Similarly, $\quad y^2 \equiv 0, 1 \pmod 3$

So $\quad x^2 + y^2 \equiv 0, 1, 2 \pmod 3 \qquad (1)$

In Eq. (1), $x^2 + y^2$ is a multiplying of 3. Iff Eq. (1) is the result of adding $x^2 \equiv 0 \pmod 3$ and $y^2 \equiv 0 \pmod 3$ implying both x^2 and y^2 are divisible by '3' and hence, both x and y are divisible by 3.

Note: In general, if $p \equiv 3 \pmod 4$ and $p \mid (a^2 + b^2)$, then $p \mid a$ and $p \mid b$.

Build-up Your Understanding 5

1. Solve the following:
   (*a*) $5x \equiv 7 \pmod{21}$
   (*b*) $19x \equiv 3 \pmod{8}$
   (c) $12x \equiv 9 \pmod{24}$
   (d) $17x \equiv 3 \pmod{210}$.
2. Find the last two digits of 3^{1234}.
3. Find the last two digits of $7^{100} - 3^{100}$.
4. Find the remainder, when $1998^{1999} + 1999^{1998}$ is divided by 7.
5. Prove that a number is divisible by 11 if and only if the difference of the sum of the odd ranked digits and the sum of the even ranked digits is divisible by 11. [*i.e.,* $11 \mid (d_1 d_2 \ldots d_k)_{10}$ if and only if $11 \mid ((d_1 + d_3 + d_5 + \ldots) - (d_2 + d_4 + d_6 + \ldots))$ where $d_1, d_2, \ldots, d_k$ are the digits of the number $(dd_2 \ldots d_k)_{10}$ written in decimal form.]
6. A number is said to be palindromic if it reads the same backwards as forward (in decimal notation). For example, 181; 5005; 1234321. Prove that any palindromic number with an even number of digits is divisible by 11.
7. Derive a divisibility test by 7.
8. Derive a divisibility test by 13.
9. Prove that $(4^{1999} + 7^{1999} - 2)$ is divisible by 9.
10. Show that $(30^{99} + 61^{100})$ is divisible by 31.
11. Prove that the number $(107^{90} - 76^{90})$ is divisible by 1891.
12. Prove that $(11^{n+2} + 12^{2n+1})$ is divisible by 133.
13. Find all sets of positive integers a, b, c satisfying the three congruences
    $$a \equiv b \pmod{c}, \quad b \equiv c \pmod{a}, \quad c \equiv a \pmod{b}.$$
14. If $\gcd(a, b) = 1$ and p is an odd prime, show that $\gcd\left(a+b, \dfrac{a^p + b^p}{a+b}\right) = 1$ or p.
15. If $a > b > 1$ and $n \in \mathbb{N}$, show that $\gcd\left(a-b, \dfrac{a^n - b^n}{a-b}\right) = \gcd(a-b, n)$.
16. Prove that if a, m, n are positive integers with $m \neq n$, then
    $$\gcd\left(a^{2^m} + 1, a^{2^n} + 1\right) = \begin{cases} 1 & \text{if a is even} \\ 2 & \text{if a is odd} \end{cases}$$
    Use this to show that there are infinitely many primes.

6.8 Complete Residue System (Modulo *n*)

Given any number n; the number of all possible remainders that can be obtained by dividing any integer by n is n.

If $\{x_1, x_2, \ldots, x_n\}$ is a set of n integers such that $x_i \not\equiv x_j \pmod{n}\ \forall i, j = 1, 2, 3, \ldots, n;$ $i \neq j$ then $\{x_i / i = 1, 2, \ldots, n\}$ is called a complete residue system modulo 'n'. There can be an infinite number of complete residue systems for a given number n.

If $n = 5$ (say) then $\{0, 1, 2, 3, 4\}$ is a complete residue system. Also known as, least non-negetive system of residues (modulo 5) and also $\{5, 6, 7, 8, 9\}$ or even $\{5, 11, 17, 23, 29\}$ are complete residue systems for modulo 5.

6.8.1 Reduced Residue System (Modulo *n*)

A related concept is reduced residue system. It is a collection of all elements of a complete residue system modulo 'n' which are co-prime with 'n'. For example, $n = 12$, one complete residue system is $\{0, 1, 2, \ldots, 11\}$. If a is an element of this system and $(a, 12) = 1$, then the corresponding member in the complete residue system given above is one of 1, 5, 7, 11. Now we define the set $\{1, 5, 7, 11\}$ to be a reduced residue system (mod 12).

Formally, a reduced residue system modulo n is a set of integers $\{r_1, r_2, \ldots, r_k\}$ satisfying the following conditions.

(i) $(r_j, n) = 1$, $1 \le j \le k$.
(ii) $r_i \not\equiv r_j \pmod{n}$, where $i \ne j$, $1 \le i, j \le k$.
(iii) For every integer x relatively prime to n, there is a 'r_j' such that $x \equiv r_j \pmod{n}$ where $1 \le j \le k$.

6.8.2 Properties

1. If $r_1, r_2, \ldots, r_n$ is a complete residue system modulo n and $(a, n) = 1$, then $ar_1, ar_2, \ldots, ar_n$ is also a complete residue system. This property also holds for reduced residue system.
2. A reduced residue system modulo n can be formed from a complete residue system modulo n by removing all integers not relatively prime to n.
3. If p is a prime number then a reduced residue system modulo p is $\{1, 2, \ldots, p - 1\}$.
4. $\phi(n)$ is the number of elements in any reduced residue system (mod n), the function ϕ is called Euler's totient function.

6.9 Some Important Function/Theorem

6.9.1 Euler's Totient Function

Leonhard Euler

15 Apr 1707–18 Sep 1783
Nationality: Swiss

The number of positive integers less than or equal to n that are coprime to n is denoted by $\phi(n)$ and is called Euler's totient function.

Euler's totient function is multiplicative, *i.e.*, if $\gcd(m, n) = 1$,

then $\phi(mn) = \phi(m) \cdot \phi(n)$.

It is also obvious that for prime p, $\phi(p) = p - 1$, $\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p - 1)$.

Also $\phi(1)=1$.

and it can be shown that

$$\phi(n) = n \cdot \prod_{p|n}\left(1 - \frac{1}{p}\right)$$

i.e., If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$$

6.9.2 Carmichael Function

Carmichael function is denoted $\lambda(n)$ and returns smallest natural number k such that

$$a^k \equiv 1 \pmod{n},$$

for every integer a, where $\gcd(n, a) = 1$.

If we know the prime factorization of n, then we can compute Carmichael function:

$$\lambda(n) = \begin{cases} \dfrac{\phi(n)}{2} & \text{for } n = 2^\alpha \text{ with } \alpha \geq 3 \\ \phi(n) & \text{for } n = p^\alpha \text{ with } p \geq 3 \\ \phi(n) & \text{for } n = 2^\alpha \text{ with } \alpha < 3 \\ \operatorname{lcm}\left(\lambda\left(p_1^{\alpha_1}\right), \ldots, \lambda\left(p_m^{\alpha_m}\right)\right) & \text{for } n = \Pi_{i=1}^m p_i^{\alpha_i}, \text{where } \alpha_i \geq 0 \end{cases}$$

6.9.3 Fermat's Little Theorem (FLT)

For any prime number p and any integer a the following congruence holds

$$a^p \equiv a \pmod{p},$$

additionally from the modular cancellation law it follows that if $\gcd(a, p) = 1$, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

6.9.4 Euler's Theorem

If $\gcd(n, a) = 1$, then

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

If $n = p$, then we obtain Fermat's little theorem.

6.9.5 Carmichael's Theorem

If $\gcd(n, a) = 1$, then

$$a^{\lambda(n)} \equiv 1 \pmod{n},$$

this theorem is stronger than Euler's theorem because $\lambda(n) \leq \phi(n)$.

Robert Daniel Carmichael

1 Mar 1879–2 May 1967
Nationality: American

Example 47 *Find the remainder when* $3^{5^{11^{7!}}}$ *is divided by* 19.

Solution: By Fermat's little theorem we have $3^{18} \equiv 1 \pmod{19}$. This hints that we will want to find $5^{11^{7!}}$ in modulo 18. Now note that $\phi(18) = 6$. So by Euler's theorem we have, $5^6 \equiv 1 \pmod{18}$.

This hints that we will want to evaluate $11^{7!}$ In modulo 6, this is easy because

$$11^{7!} \equiv (-1)^{7!} \equiv 1 \pmod{6}.$$

So there exists a such that $11^{7!} = 6a + 1$, hence

$$5^{11^{7!}} \equiv 5^{6a} \cdot 5 \equiv 5 \pmod{18},$$

This means that there exists b such that $5^{11^{7!}} = 18b + 5$, so,

$$3^{5^{11^{7!}}} \equiv 31^{8b} \cdot 3^5 \equiv 3^5 \equiv 15 \pmod{19}.$$

Example 48 *Show that* $11^{10^{1967}} \equiv 1$ *(mod* 10^{1968}*).*

Solution: Solution of this problem really shows how powerful Carmichael function is. First we can compute

$$\lambda(10^{1968}) = \text{lcm}(\lambda(2^{1968}), \lambda(5^{1968}))$$

$$\lambda(2^{1968}) = \frac{\phi(2^{1968})}{2} = 2^{1966}.$$

$$\lambda(5^{1968}) = \phi(5^{1968}) = 5^{1968} \cdot \left(1 - \frac{1}{5}\right) = 4 \cdot 5^{1967}.$$

Returning to beginning of the problem we find that

$$\lambda(10^{1968}) = \text{lcm}(2^{1966}, 4 \cdot 5^{1967}) = 2^{1966} \cdot 5^{1967}.$$

Now

$$2^{1966} \cdot 5^{1967} \mid 10^{1967}$$

So there exists a such that $10^{1967} = a \cdot \lambda(10^{1968})$ and we are done by Carmichael theorem because

$$11^{10^{1967}} = 11^{10^{1967}} = \left(11^{\lambda(10^{1968})}\right)^{a} \equiv 1 \ (\text{mod } 10^{1968}).$$

John Wilson

6 Aug 1741–18 Oct 1793
Nationality: British

6.9.6 Wilson's Theorem

Natural number $n \geq 2$ is prime number if and only if $(n - 1)! \equiv -1 \ (\text{mod } n)$.

Example 49 *Find the remainder when* 33! *is divided by* 37.

Solution: Notice that 37 is prime, Wilson's theorem states that $36! \equiv -1 \ (\text{mod } 37)$, now for simplicity let $x = 33!$, then $34 \cdot 35 \cdot 36 \cdot x \equiv -1 \ (\text{mod } 37)$.

We have $34 \cdot 35 \cdot 36 \equiv (-3)(-2)(-1) \equiv -6 \ (\text{mod } 37)$,

So $-6x \equiv -1 \ (\text{mod } 37) \Leftrightarrow 6x \equiv 1 \ (\text{mod } 37)$,

i.e., there exists a such that $6x = 37a + 1$, looking at this equation modulo 6 we find $37a \equiv a \equiv -1 \equiv 5 \ (\text{mod } 6)$, which is to say that there exists b such that $a = 6b + 5$, thus

$$6x = 37(6b + 5) + 1 \Leftrightarrow x = 37b + 31. \Rightarrow x \equiv 31 (\text{mod } 37)$$

Example 50 *What is the remainder when* 10! *is divided by* 13?

Solution:

By Wilson's Theorem

$12! \equiv -1 \ (\text{mod } 13)$

$12! \equiv -1 + 13 \ (\text{mod } 13)$

$11! \equiv 1 \ (\text{mod } 13)$ by 'dividing' by 12

$11! \equiv 1 + 5 \times 13 \ (\text{mod } 13)$

$10! = 6 \ (\text{mod } 13)$ by 'dividing' by 11

6.9.7 Chinese Remainder Theorem (CRT)

Let $n_1, \ldots, n_r$ be natural numbers such that $(n_i, n_j) = 1$ for $i \neq j$. The system of congruence

$$\begin{aligned} x &\equiv a_1 \ (\text{mod } n_1) \\ x &\equiv a_2 \ (\text{mod } n_2) \\ &\vdots \\ x &\equiv a_r \ (\text{mod } n_r) \end{aligned}$$

has a unique solution in modulo $n_1 n_2 n_3 \ldots n_r$.

Proof: $n_1 . n_2 \dots n_r = N$ (say). Writing $\frac{N}{n_j} = N_j$, simultaneous solution x_0 is given by

$$x_0 \equiv a_1 N_1 x_1 + a_2 N_2 x_2 + \cdots + a_r N_r x_r \pmod N$$

where x_i is the individual solution $N_i x_i \equiv 1 \pmod{n_i}$. For $i = 1, 2, 3, \dots, r$.

Example 51 *Find the last three digits of* 124^{1000}.

Solution: We want to evaluate the number modulo 1000. Here we find ourselves in trouble, we have gcd (124, 1000) ≠ 1, so we cannot use Euler's theorem or Carmichael's theorem. But luckily Chinese remainder theorem can help us. First we write 1000 as product of coprime numbers $1000 = 8 \cdot 125$, now we have

$$124^{1000} \equiv 0 \pmod 8,$$

and

$$124^{1000} \equiv (-1)^{1000} \equiv 1 \pmod{125}.$$

Well, what was that for? Now we have system of congruences, namely

$$\begin{cases} 124^{1000} \equiv 0 & \pmod 8 \\ 124^{1000} \equiv 1 & \pmod{125} \end{cases}.$$

And remember that according to Chinese remainder theorem this system of congruences has unique solution in modulo $8 \cdot 125 = 1000$, which is exactly what we want! Notice that Chinese remainder theorem does not tell us how to find the solution, fortunately it is nothing hard. From first congruence there exists 'a' such that $124^{1000} = 8a$, so in second congruence we have

$$8a \equiv 1 \pmod{125},$$

i.e., there exists b such that $8a = 125b + 1$, looking at this modulo 8 we find that $3b \equiv 1 \pmod 8 \Leftrightarrow 3b \equiv 1 + 8 \pmod 8 \Leftrightarrow b \equiv 3 \pmod 8$. This means that there exists c such that $b = 8c + 3$, thus

$$124^{1000} = 125(8c + 3) + 1 = 1000c + 376.$$

So the last three digits are 376.

6.9.8 Binomial Coefficient

Number $\binom{n}{k}$, where $0 \le k \le n$, $n \in \mathbb{N}$ is called binomial coefficient and we have

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}.$$

6.9.9 Binomial Theorem

The following expansion holds for any real numbers x, y:

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i,\ n \in \mathbb{N}$$

Example 52 *Find last three digits of* 9^{99}.

Solution: We wish to find remainder when 9^{99} is divided by 1000. Now we will write $9 = 10 - 1$ and use binomial theorem

$$9^{99} \equiv (10-1)^{99} \equiv -1 + \binom{99}{1}.10 - \binom{99}{2}.100 \pmod{1000}.$$

Other terms in the expansion vanish because they are divisible by 1000.

Now $\binom{99}{1} = 99$ and $\binom{99}{2} = \dfrac{99!}{97!2!} = \dfrac{98 \cdot 99}{2} = 4851$. Thus

$$9^{99} \equiv -1 + 990 - 485100 \equiv -111 \equiv 889 \pmod{1000}.$$

Aliter: We may compute $\lambda(1000) = 100$, which is very useful to even remember. We must remember to check that indeed gcd(9, 1000) = 1 and by applying Carmichael's Theorem we get $9^{100} \equiv 1 \pmod{1000}$, *i.e.*,

$$9^{99} \equiv 9^{-1} \pmod{1000},$$

Where 9^{-1} is so called modular multiplicative inverse of 9 modulo 1000, *i.e.*, we have $9 \cdot 9^{-1} \equiv 1 \pmod{100}$. For simplicity denote $9^{-1} = x$, we wish to find this number. The inverse can be generally found by noting that the congruence means that there exists a such that $9x - 1 = 1000a$.

Look at this equation modulo 9 to get $1000a \equiv a \equiv -1 \equiv 8 \pmod 9$, which is to say that there exists b such that $a = 9b + 8$, thus

$$9x = 1000(9b + 8) + 1 \Leftrightarrow x = 1000b + 889,$$

Which means

$$9^{99} \equiv x \equiv 889 \pmod{1000}.$$

6.9.10 Digit Sum Characteristic Theorem

Sum of digits of a number is congruent to the number modulo 9. The same holds for modulo 3.

Proof: Since $10^n \equiv 1 \pmod 9$ for all $n \in N$, any number written in decimal representation such as $(a_n a_{n-1} a_{n-2} \ldots a_1 a_0)_{10} \equiv a_n + a_{n-1} + \cdots + a_1 + a_0 \pmod 9$.

Example 53 *All two-digit numbers from* 10 *to* 99 *are written consecutively, i.e., N* =101112…99. *Show that* $3^2 | N$. *From which other two-digit number you should start so that N is divisible by* (a) 3 (b) 3^2.

Solution: N is divisible by 9, if the digit sum is divisible by 9.

The digital sum of *N*:
The number of 1's occurring in the digits
from 10 to 19 = 11
and from 20 to 99 = 8.
So, total of 1's is 11 + 8 = 19.
Similarly, No. of 2's, 3's,…, 9 are all equal to 19.
So, sum of all the digits = 19 (1 + 2 + 3 + ⋯ + 9)

$$= \frac{19 \times 9 \times 10}{2} = 19 \times 5 \times 9 = 855$$

and as 9|855, 1011…99 is divisible by 9.

When the numbers start from 12, the sum of the digits become 855 – 3 = 852 (Since 10, 11 account for the digital sum 3) and hence, is divisible by 3.

(a) For divisibility by 3, it could start from 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, …
(b) For divisibility by $3^2 = 9$ the numbers may start from any of 18, 19, 27, 28, 36, 37, …

Example 54 *Find the remainder when* 4333^{3333} *is divided by* 9.

Solution: $4333 \equiv 4 + 3 + 3 + 3 = 13 \pmod 9 \equiv 4 \pmod 9$

$$\therefore \quad 4333^3 \equiv 4^3 \pmod 9$$
$$\equiv 64 \pmod 9$$
$$\equiv 1 \pmod 9$$
$$\Rightarrow \quad 4333^{3333} \equiv 1 \pmod 9$$

i.e., when 4333^{3333} is divided by 9, the remainder is 1.

Example 55 *Prove that among 18 consecutive three-digit numbers there is at least one number which is divisible by the sum of its digits.*

Solution: Among 18 consecutive integers there are two numbers which are divisible by 9.

The sum of the digits of these two numbers must be 9, 18 or 27.

If the sum of the digits is 9, then the number is divisible by the sum of the digits, so there is nothing to prove.

If the sum of the digits is 27, then the three-digit numbers should be $999 = 9 \times 111 = 9 \times 3 \times 37$ and hence, the result. Let both the numbers have 18 as the sum of their digits. Let those numbers be a and b with $a < b$.

If a is odd and sum of its digits is 18, it is divisible by 9 but not by 18. However, the other number b is also divisible by 9 and b should be $a + 9 \Rightarrow b$ is even and sum of its digits is 18, and hence, b is an even number as well as divisible by $9 \Rightarrow b$ is divisible by 18.

Example 56 Suppose $\delta(n)$ denotes digit sum of n. Find $\delta(\delta(\delta(5^{2013})))$.

Solution: First repeatedly using modulo 9 yields

$$\delta(\delta(\delta(5^{2013}))) \equiv \delta(\delta(5^{2013})) \equiv \delta(5^{2013}) \equiv 5^{2013} \pmod 9$$

Thus finding 5^{2013} in modulo 9 will help us. This can be done by finding $\phi(9) = 6$. So by Euler's theorem we get

$$5^{2013} \equiv (5^6)^{335} \cdot 5^3 \equiv 5^3 \equiv 8 \pmod 9.$$

Now it suffices to realize that the sought number will probably be very small because digit sum of a big number is much smaller than the number. So it suffices to establish some upper bounds on the number sought. We can of course establish sharp bounds, but it is not needed for this problem, we have $5^{2013} < 10^{2013}$, so

$$\delta(5^{2013}) < 9 \cdot 2014 = 18126$$

Number less than 18126 with greatest digit sum is 9999, so

$$\delta(\delta(5^{2013})) \le 9 + 9 + 9 + 9 = 36,$$

Again number less than or equal to 36 that has greatest digit sum is 29, thus

$$\delta(\delta(\delta(5^{2013}))) \le 11.$$

But only positive number less than or equal to 11 and congruent to 8 modulo 9 is 8. Thus the number sought was 8.

Build-up Your Understanding 6

1. (a) Prove that $1001 \mid (300^{3000} - 1)$

(b) Prove that $13 \mid (2^{70} + 3^{70})$.

(c) Prove that $11 \cdot 31 \cdot 61 \mid (20^{15} - 1)$.

(d) Prove that $169 \mid (3^{3n+3} - 26n - 27)$ for all $n \in \mathbb{N}$.

(e) Prove that $19 \mid (2^{2^{6n+2}} + 3)$ for all $n \in \mathbb{N}_0$.

2. Prove that the square of any prime number larger than 3 leaves a remainder 1 when divided by 12.

3. Show that the eighth power of any number N is written in one of the forms $17m$ or $17m \pm 1$.

4. Find the remainder when 2^{1990} is divided by 1990. **[RMO, 1990]**

5. Find the remainder when 19^{92} is divided by 92. **[INMO, 1992]**

6. Find the three last digits of 7^{9999}.

7. What is the fifth digit from the end (*i.e.*, the ten thousand's digit) of the number $5^{5^{5^{5^{5}}}}$.

8. Show that $(19^{93} - 13^{99})$ is positive and divisible by 162. **[RMO, 1993]**

9. Find all positive integers n for which $120|(n^5 - n)$.

10. Prove that for all natural number n, $\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is a natural number.

11. If $p > 5$ is prime, prove that $30|(p^2 - 1)$ or $30|(p^2 - 19)$.

12. Prove that for every prime $p > 7$, $p^6 - 1$ is divisible by 504.

13. If p is a *prime* and $a^p \equiv b^p \pmod{p}$, prove that $a^p \equiv b^p \pmod{p^2}$.

14. Let k be a positive integer. Find the largest power of 3 which divides $10^k - 1$.

15. Find the smallest four consecutive positive integers such that the least is divisible by 4, the next by 9, the next by 25 and the greatest by 49.

16. Solve the system of congruences simultaneously

$2x \equiv 1 \pmod{5}$

$3x \equiv 9 \pmod{6}$

$4x \equiv 1 \pmod{7}$

$5x \equiv 9 \pmod{11}$

17. A photographer comes to take a group photograph of the students of the final year class in a school. He tries to arrange then in equal rows. But with *2,* 3 or 4 rows, he finds that there is one person left over each time. However, when he puts them into 5 equal rows, there is no such problem. What is the smallest number of students in the class consistent with this situation?

18. Here is an ancient Chinese problem. A gang of 17 pirates steal a sack of gold coins. When they try to divide the loot equally, there are three coins left over. They fight over these extra coins and one pirate is killed. They try to divide the coins equally a second time, but now there are 10 left over. Again they fight and another of the gang meets an untimely end. Fortunately for the remainder of the gang, when they try to divide the loot, a third time an equal distribution results. What is the smallest number of coins they can have stolen?

19. Let $Q(n)$ be the sum of digits of n. Prove that $Q(n) = Q(2n)$ implies $9|n$.

20. (a) Take any 2222 digit number that is divisible by 9. Let the sum of its digits equals to a. The sum of the digits in a equals to b and the sum of the digits in b equals c. What does c equal to?

(b) When 4444^{4444} is written in decimal notation the sum of its digits is A. Let B be the sum of digits of A. Find sum of digits of B.

21. For a given positive integer k, denote the square of the sum of its digit by $f_1(k)$ and $f_{n+1}(k) = f_1[f_n(k)]$ Find the value of $f_{1995}(2^{1995})$.

22. Prove the existence of a positive integer divisible by 1998, the sum of whose decimal digits is 1998.

23. A composite number m that satisfies $a^{m-1} \equiv 1 \pmod{m}$ is called a pseudo prime to the base a, and if m is pseudo prime to every base a whenever gcd$(a, m) = 1$, the m is called a Carmichael number. Show that 341 is a pseudo prime to the base 2 and 561 is a Carmichael number. In fact, they are the smallest numbers of their respective kind.

24. (a) Prove that if p|($(p - 1)! + 1$) and $p > 1$, then p is prime.

(b) Prove that $(p-1)! \equiv p-1 \pmod{1+2+\cdots+(p-1)}$ if p is a prime.

(c) Show that $(p - 2)! - 1 = p^n$ has no solution if p is a prime > 5 and $n \in \mathbb{N}$.

(d) Show that $(n - 1)! + 1$ is a power of n if and only if $n = 2, 3$ or 5.

(e) If p is a prime and $0 \le k \le p-1$, prove that $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$.

25. (a) If n is an even perfect number, then prove that $n - \phi(n)$ is a square. Where even perfect numbers are given, by $2^{p-1}(2^p - 1)$ where p, $2^p - 1$ being prime numbers.

(b) Prove that the sum of all positive integers less than or equal to n and co-prime to n equals $n\phi(n)/2$.

(c) Find all positive integers n such that $\phi(n) \mid n$.

(d) If $\phi(m) = \phi(mn)$ and $n > 1$, prove that $n = 2$ and m is *odd*.

(e) For any integers a, m, prove that $a^m \equiv a^{m-\phi(m)} \pmod{m}$.

6.10 Scales of Notation

Every natural number that we use is expressed in expanded notation in the form of $a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10^1 + a_0$, where $0 \le a_i \le 9$ and $a_n \ne 0$ and we can write it as $(a_n a_{n-1} \ldots a_1 a_0)_{10}$ and call $a_n, a_{n-1}, \ldots, a_1, a_0$ as the digits of the number.

Here a_n means there are 'a_n' 10^n's in the number and so on.

Thus, we have a place value for every digit. The numbers, that we use, are also called number in base 10 or number in decimal system.

Bases other than 10 can also be used to represent numbers. Supposing $b > 1$ is the base, for the different place values we have different non-negative integral powers of b.

Thus, every natural number m can be represented in base 'b', $b > 1$, $b \in \mathbb{N}$ as given below:

$m = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b^1 + a_0$ where $0 \le a_i \le b - 1$ for each $i = 0, 1, 2, \cdots, n - 1$ and $1 \le a_n \le b - 1$.

1. Here 'b' is called the base for the representation.
2. Usually, we write the above as

$$(a_n a_{n-1} \ldots a_1 a_0)_b \tag{1}$$

3. In base b system we use 'b' different numerals $(0, 1, 2, 3, \ldots, b-1)$.
4. Given any numbers n, (say in base 10) and 'b' the base in which the number n is to be represented, we can find the number in the form given in Eq. (1) by the repeated application of the rule

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}.$$

5. Base 2, base 8 and base 16 are very often used in computers and they are called binary, octal and hexadecimal systems, respectively.

Example 57 *Express* 29_{10} *in base* 2, *base* 3 *and base* 5 *systems.*

Solution: We can write 29_{10} as

$$\begin{aligned}(29)_{10} &= 2^4 + 2^3 + 2^2 + 1\\ &= 1.2^4 + 1.2^3 + 1.2^2 + 0.2 + 1\\ &= (11101)_2\end{aligned}$$

or

$$\begin{array}{r|l} 2 & 29 \\ \hline 2 & 14, 1 = a_0 \\ \hline 2 & 7, 0 = a_1 \\ \hline 2 & 3, 1 = a_2 \\ \hline 2 & 1, 1 = a_3 \\ \hline & 0, 1 = a_4 \end{array}$$

$$(29)_{10} = (11101)_2$$

$$\begin{aligned}(29)_{10} &= 3^3 + 2\\ &= 1.3^3 + 0.3^2 + 0.3^2 + 2\\ &= (1002)_3\end{aligned}$$

or

$$\begin{array}{r|l} 3 & 29 \\ \hline 3 & 9, 2 = a_0 \\ \hline 3 & 3, 0 = a_1 \\ \hline 3 & 1, 0 = a_2 \\ \hline & 0, 1 = a_3 \end{array}$$

$\therefore$ $$(29)_{10} = (1002)_3$$

$$(29)_{10} = 5^2 + 4 = 1.5^2 + 0.5 + 4 = (104)_5$$

or

$$\begin{array}{r|l} 5 & 29 \\ \hline 5 & 5, 4 = a_0 \\ \hline 5 & 1, 0 = a_1 \\ \hline & 0, 1 = a_2 \end{array}$$

$\therefore$ $$(29)_{10} = (104)_5.$$

Note that $\begin{array}{r|l} \text{divisor} & \text{dividend} \\ \hline & \text{quotient, remainder} \end{array}$

Example 58 *Express* $(1042)_{10}$ *in base* 12 *system.*

Solution: In base 12 we have 12 numerals. We take them as 0, 1, *2,* 3, 4, 5, 6, 7, 8, 9, A, B, where

$$A = (10)_{10}$$
$$B = (11)_{10}$$
$$\begin{array}{r|l} 12 & 1042 \\ \hline 12 & 86, 10 = a_0 \\ \hline 12 & 7, 2 \quad = a_1 \\ \hline & 0, 7 = a_2 \end{array}$$

$\therefore \qquad (1042)_{10} = (72\text{A})_{12}$

Example 59 *A three-digit number in base* 11*, when expressed in base* 9*, has its digits reversed. Find the number.*

Solution:
$$(xyz)_{11} = (zyx)_9$$
$$11^2x + 11y + z = 9^2z + 9y + x$$
$$\Rightarrow \quad 120x + 2y - 80z = 0$$
$$\Rightarrow \quad 60x + y - 40z = 0$$
$$\Rightarrow \quad 40z - 60x = y$$
$$\Rightarrow \quad 20(2z - 3x) = y,$$

So $20|y$, but as $0 \le y < 9$, $y = 0$

Therefore, $2z = 3x$. As $0 \le x, z < 9$, the solutions are $x = 2, z = 3$ and $x = 4, z = 6$. Thus the two possible solutions are $(203)_{11}$ and $(406)_{11}$.

Exercise *Verify that these numbers when converted to base* 9 *get reversed.*

Example 60 *Show that* $N = (1\ 2\ 3\ 4\ 3\ 2\ 1)_b$ *written in base b,* $b > 4$ *is a square number for all b.*

Solution: $(1\ 2\ 3\ 4\ 3\ 2\ 1)_b$ in the expanded notation is $b^6 + 2b^5 + 3b^4 + 4b^3 + 3b^2 + 2b + 1$, $b > 4$.

Now, $N = (b^3 + b^2 + b + 1)^2$.

This is true for all real number b and hence, is true for all $b > 4$, $b \in N$ also.

Example 61 *If* $100^{25} - 25$ *is written in decimal notation, find the sum of its digits.*

Solution: Since $100^{25} = (10^2)^{25} = 10^{50} = 10000\ldots0$ (50 zeroes)

$$100^{25} - 25 = 10^{50} - 25$$
$$= \underbrace{10000...00 - 25}_{(50\text{ zeroes})}$$
$$= \underset{(48\text{ nines})}{999...9975}$$

So the sum of its digits $= 48 \times 9 + 12 = 432 + 12 = 444$.

Example 62 *When the numbers from 1 to n are written in decimal notation, it is found that the total number of digits in writing all these is 1998. Find n.*

Solution: To write the first nine single-digit number from 1 to 9, both inclusive the number of digits used = 9.

To write the two-digit numbers from 10 to 99, number of digits used $= (99 - 9) \times 2 = 180$.

So, the number of digits used to write numbers from 1 to 99 is 189.

Total number of digits used in writing up to n is 1998.

The total number of digits used in writing all the three-digit numbers $= (999 - 99) \times 3 = 2700 > 1998$.

So, n should be less than 999.

Number of digits used to write the three-digit numbers up to n is

$$1998 - 189 = 1809.$$

In each three-digit number, we use three digits.

So, the number of three-digit numbers in $n = \dfrac{1809}{3} = 603$.

Therefore, $n = 100 + (603 - 1) \cdot 1 = 702$.

Example 63 *Find the smallest natural number n, which has the following properties:*

(a) *Its decimal representation has 6 as the last digit.*

(b) *When its last digit is removed and placed in front of the remaining digits, the resulting number is four times the original number.*

Solution: If $a, b, c, d, \ldots, k$ are the digits of a number written in decimal system, $abcd\ldots lk$, then

$10(abcd\ldots l) + k = abcd\ldots lk = 100(abcd\ \ldots) + (lk)$, etc.

Now, let the unit digit of the number be 6 and all the other digits on the left of 6 is taken as x, then the number is $x6$.

When 6 is written in front, the number becomes $6x$ and it is equal to $4 \times x6$.

Note: $6x$ is not $6 \times x$, here 6 is the extreme left digit of the number.

If $6x$ is a two-digit number, then $6x = 60 + x$, if it is a three-digit number, then x is the last two digits of $6x$ and $6x = 600 + x$, because 6 is in the hundreds place.

Similarly, if it is a four-digit number, $6x$ is $6000 + x$ and thus, $6x = 60 + x$ or $600 + x$, or $6000 + x$ and so on, according to the number of digits in x (*i.e.,* the place value of 6 may be $10, 10^2, 10^3$ or $10^4 \ldots$ according to the number of digits of the given number.)

However, $x6 = 10x + 6$, whatever be the number of digits x has.

Thus, $4 \times x6 = 4(10x + 6) = 6x$

$$= 6 \times 10^k + x$$

where k is the number of digits in x

$$39x = 6 \times 10k - 24$$
$$13x = 2 \times 10^k - 8.$$

To find the smallest value for x, we need to find the smallest power k for which $13|(2 \times 10^k - 8)$,

i.e., $2 \times 10^k \equiv 8 \pmod{13}$ or $10^k \equiv 4 \pmod{13}$

As $\quad 10 \equiv -3 \pmod{13}$

$\Rightarrow \quad 10^2 \equiv 9 \pmod{13} \equiv -4 \pmod{13}$

$\Rightarrow \quad 10^4 \equiv 16 \pmod{13} \equiv 3 \pmod{13}$

$\Rightarrow \quad 10^5 \equiv -9 \pmod{13} \equiv 4 \pmod{13}$

$\therefore \quad 13|2 \times 10^5 - 8$

So, x has 5 digits and is given by $\dfrac{200000-8}{13} = \dfrac{199992}{13} = 15384$.

$\therefore$ The given number is 153846.

Clearly, $615384 = 153846 \times 4$.

Build-up Your Understanding 7

1. Find all perfect squares whose base 9 representation consists only of 1's.
2. (a) In base 9, find the greatest perfect square of 4 digits.
   (b) In base 16, find the greatest perfect square of 4 digits.
3. If the different letters used in the following expressions, denote uniquely a different digit in base 10, and if V × VEXATION = EEEEEEEEE. Find the value of V + E + X + A + T + I + O + N.
4. Find the numerical value of each of the letters in the following expression TWO + TWO = FOUR in (a) base 10 and (b) base 7.
5. Let a be the integer $a = \underbrace{111\ldots1}_{m\text{ times}}$ and $b = \underbrace{1000\ldots05}_{m\text{-1 zeroes}}$

   Prove that $ab + 1$ is a square integer. Express the square root of $ab + 1$ in the same form as a and b are expressed.
6. Let n be a five digit number (whose first digit is non-zero) and let m be the four digit number formed from n by deleting its middle digit. Determine all n such that $\frac{m}{n}$ is an integer.
7. For which positive integral bases b is 1367631, will be a perfect cube?
8. (a) Find all positive integers with initial digit 6 such that the integer formed by deleting this '6' is $\frac{1}{25}$ of the original integer.

   (b) Show that there is no integer such that the deletion of the first digit produces a result which is $\frac{1}{35}$ of the original digit.
9. $\frac{(a\,b)_{10}}{(c\,a)_{10}} = \left(\frac{b}{c}\right)_{10}$ where $(a\ b)_{10}$ and $(c\ a)_{10}$ are two digit numbers in base ten

   [*i.e.*, a, b are the digits of the number $(a\ b)_{10}$ and c, a are the digits of the number $(c\ a)_{10}$. We get $\frac{b}{c}$, by cancelling those digit 'a' of the numerator with the unit digit 'a' of the denominator]. Find all such two digit numbers.

   For example, $\frac{64}{16} = \frac{4}{1}$ is the correct answer so here $a = 6$, $b = 4$, and $c = 1$.

   In the above problem, having found a, b, c, verify if $\frac{aab}{caa}, \frac{aaab}{caaa}, \ldots$ can also give the answer $\frac{a}{c}$ by cancelling the common digits or not.
10. If $a_1\ a_2, \ldots, a_k$ are the digits of the number $(a_1\ a_2 \ldots a_k)_d$ in base $d > 2$, show that $(d-1)\,|(a_1\ a_2 \ldots a_k)_d$ if and only if $(d-1)\,|\,(a_1 + a_2 + \cdots + a_k)$.
11. If $a_1\ a_2, \ldots, a_k$ are the digits of the number $(a_1\ a_2 \ldots a_k)_d$ in base $d > 2$, show that $(d + 1)\,|\,(a_1\ a_2 \ldots a_k)_d$ if and only if the difference between the sum of the odd ranked digits and the sum of the even ranked digits is divisible by $(d + 1)$.

6.11 Greatest Integer Function

For a given x, an integer k such that $k \le x < k + 1$, $k \in \mathbb{Z}$ is called Greatest integer of x. $\lfloor x \rfloor$ represents the greatest integer less than or equal to x. $f(x) = \lfloor x \rfloor$ is called the greatest integer function or floor function.

A related concept $\{x\}$, the fractional part of x, is defined as $\{x\} = x - \lfloor x \rfloor$.

For example, $\lfloor \sqrt{3} \rfloor = 1, \lfloor 10 \rfloor = 10, \lfloor -\pi \rfloor = -4$ and $\lfloor -10 \rfloor = -10$.

$$\{4.7\} = 0.7,\ \{3.1\} = 0.1,\ \{-7.9\} = 0.1,\ \{-6.3\} = 0.7.$$

6.11.1 Properties of Greatest Integer Function

(i) $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$ and $x - 1 < \lfloor x \rfloor \le x,\ 0 \le x - \lfloor x \rfloor < 1$.

(ii) If $x \ge 0, \lfloor x \rfloor = \sum_{1 \le i \le x} 1$

(iii) $\lfloor x + m \rfloor = \lfloor x \rfloor + m$, if m is an integer.

(iv) $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$

(v) $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0, \text{if } x \text{ is an integer} \\ -1, \text{otherwise} \end{cases}$

(vi) $\left\lfloor \dfrac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \dfrac{x}{m} \right\rfloor$, if m is a positive integer.

(vii) $-\lfloor -x \rfloor$ is the least integer greater than or equal to x. This is denoted as $\lceil x \rceil$ (read as 'ceiling x'). For example, $\lceil 2.5 \rceil = 3, \lceil -2.5 \rceil = -2$.

(viii) $\lfloor x + 0.5 \rfloor$ is the nearest integer to x. If x is midway between two integers, $\lfloor x + 0.5 \rfloor$ represents the even number of the two integers.

(ix) The number of positive integers less than or equal to n and divisible by m is given by $\left\lfloor \dfrac{n}{m} \right\rfloor$.

(x) The number of perfect k^{th} powers form 1 to n is $\left\lfloor n^{\frac{1}{k}} \right\rfloor$.

(xi) If p is a prime number and e is the largest exponent of p such that $p^e \| n!$, (Read it 'p^e completely divides $n!$') then $e = \sum_{i=1}^{\infty} \left\lfloor \dfrac{n}{p^i} \right\rfloor$. This formula known as **Legendre formula**.

Note: $p^e \| n! \Rightarrow p^e | n!$ and $p^{e+1} \nmid n!$

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18 Sep 1752–10 Jan 1833
Nationality: French

Example 64 *If n and k are positive integers and $k > 1$, prove that*

$$\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n+1}{k} \right\rfloor \le \left\lfloor \frac{2n}{k} \right\rfloor.$$

Solution: Let $n = qk + r,\ 0 \le r < k$.

$$\text{Now, } \frac{n}{k} = \frac{qk + r}{k} = q + \frac{r}{k};\ \frac{n+1}{k} = \frac{qk + r + 1}{k} = q + \frac{r+1}{k};$$

$$\frac{2n}{k} = \frac{2qk + 2r}{k} = 2q + \frac{2r}{k};\ 0 \le r < k.$$

Thus,

(i) r may be equal to $k-1$, or (ii) r may be $< k-1$.

If $r = k-1$, we have

$$\left\lfloor \frac{n}{k} \right\rfloor = q, \quad \left\lfloor \frac{n+1}{k} \right\rfloor = \left\lfloor q + \frac{k}{k} \right\rfloor = q+1$$

$$\left\lfloor \frac{2n}{k} \right\rfloor = \left\lfloor 2q + \frac{2k-2}{k} \right\rfloor = 2q+1 \quad \left[\text{since } k > 1, \frac{2}{k} \le 1\right].$$

So, by adding and equating, we get

$$\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n+1}{k} \right\rfloor = 2q+1 = \left\lfloor \frac{2n}{k} \right\rfloor$$

(ii) If, $r < k-1$ we have

$$\left\lfloor \frac{n}{k} \right\rfloor = q, \left\lfloor \frac{n+1}{k} \right\rfloor = q$$

$$\left\lfloor \frac{2n}{k} \right\rfloor = \left\lfloor 2q + \frac{2r}{k} \right\rfloor \ge 2q.$$

So, by adding, we get

$$\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n+1}{k} \right\rfloor \le \left\lfloor \frac{2n}{k} \right\rfloor$$

Combining (i) and (ii), we get

$$\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n+1}{k} \right\rfloor \le \left\lfloor \frac{2n}{k} \right\rfloor.$$

Note: When $k = 2$, the above inequality holds as an equality. (*verify*).

Example 65 *Prove that* $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x+y \rfloor$

Solution:

$$x + y = \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\}$$

$$\Rightarrow \quad \lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor$$

$$\Rightarrow \quad \lfloor x+y \rfloor \ge \lfloor x \rfloor + \lfloor y \rfloor$$

This can be generalized for n numbers:

$$\lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \cdots + \lfloor x_n \rfloor \le \lfloor x_1 + x_2 + \cdots + x_n \rfloor$$

Example 66 *Prove that* $\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 8x \rfloor + \lfloor 16x \rfloor + \lfloor 32x \rfloor = 12345$ *has no solution.*

Solution: $12345 \le x + 2x + 4x + 8x + 6x + 32x = 63x$

$$\therefore \quad x \ge \frac{12345}{63} = 195\frac{20}{21}.$$

When $x = 196$, the L.H.S of the given equation becomes $12348 \Rightarrow x < 196$

$$\Rightarrow \quad 195\frac{20}{21} \le x < 196.$$

Consider x in the interval $\left(195\frac{31}{32}, 196\right)$. The LHS expression of the given equation

$$= 195 + 0 + 390 + 1 + 780 + 3 + 1560 + 7 + 3120 + 15 + 6240 + 31$$
$$= 12342 < 12345$$

When $x < 195\frac{31}{32}$, the LHS is less than 12342.

$\therefore$ For no value of x, the given equality will be satisfied.

Example 67 *How many zeroes are there at the end of* 2000!?

Solution: If k be the highest power of 5 and l be the highest power of 2 contained in 2000!, then the highest power of 10 contained in 2000! is the minimum of k and l, as the highest power of 2 contained in any factorial is greater than the highest power of 5 contained in it.

For example, consider 10!

$$10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10.$$

So, the highest power of 2 contained in 10! is $2 \times 2^2 \times 2 \times 2^3 \times 2$ of 2, 4, 6, 8, and 10 of the factors, *i.e.*, $2^8 = 256$ and the highest power of 5 in 10! is $5^1 \times 5^1$ of 5 and 10 $= 5^2 = 25$.

If $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, then the highest power of 5 contained in 2000! is

$$\left\lfloor \frac{2000}{5} \right\rfloor + \left\lfloor \frac{2000}{5^2} \right\rfloor + \left\lfloor \frac{2000}{5^3} \right\rfloor + \cdots + \left\lfloor \frac{2000}{5^n} \right\rfloor \tag{1}$$

where $5^n \le 2000$, for otherwise, $\left\lfloor \frac{2000}{5^n} \right\rfloor = 0$ and hence, the sum in (1) is not an infinite sum.

Therefore, $k = 400 + 80 + 16 + 3 + 0 + 0 \cdots = 499$.

So, the number of zeroes at the end of 2000! is 499.

Example 68 *How many zeroes does* 6250! *end with?*

Solution: We need to find the largest e such that $10^e | 6250!$. But as $10 = 2 \times 5$, this implies that we need to find the largest e such that $5^e | 6250!$ (clearly a larger power of $2 | 6250!$).

$$\text{But} \quad e = \sum_{i=1}^{\infty} \left\lfloor \frac{6250}{5^i} \right\rfloor = 1250 + 250 + 50 + 10 + 2 = 1562.$$

Hence, 6250! ends with 1562 zeroes.

Example 69 *If* n! *has exactly* 20 *zeroes at the end, find* n*. How many such* n *are there?*

Solution: If e is the maximum power of 5 in n!, then

$$e = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{5^i} \right\rfloor < \sum_{i=1}^{\infty} \left(\frac{n}{5^i} \right) = \frac{n}{5} + \frac{n}{5^2} + \frac{n}{5^3} + \cdots$$

$$\Rightarrow e < \frac{\frac{n}{5}}{1 - \frac{1}{5}} = \frac{n}{4}$$

$$\therefore \qquad n > 4e.$$

Here e is given to be 20.

$\therefore \quad n \geq 80$. For 80, $e = 19$.

Therefore, 85 is the required answer. 86, 87, 88, 89 are also valid values of n. If solution exists for this type of problem, there will be five solutions.

Example 70 *Find all n such that n! ends with exactly* 497 *zeroes.*

Solution: If $e = 497$, then $n \geq 1988$. (As e $< \dfrac{n}{4}$ from previous example.)

Consider 1990.

For $n = 1990$, $e = 495$.

For $n = 1995$, $e = 496$. But when $n = 2000$, e jumps to 499 as 2000 is a multiple of 125.

$\therefore \quad$ For no $n \in \mathbb{N}$, n! ends with exactly 497 zeroes.

Example 71 *Find all n such that n! has* 1998 *zeroes at the end of n!*

Solution: You know that the greatest power of $a > 1$, $a \in$ prime, dividing n is given by

$$\sum_{i=1}^{\infty}\left\lfloor \frac{n}{a^i} \right\rfloor. \tag{1}$$

But $$\sum_{i=1}^{\infty}\left\lfloor \frac{n}{a^i} \right\rfloor < \sum_{i=1}^{\infty}\frac{n}{a^i} = n\left(\frac{1}{a-1}\right) \tag{2}$$

We want to find n, such that

$$\sum_{i=1}^{\infty}\left\lfloor \frac{n}{5^i} \right\rfloor = 1998$$

By Eq. (2) $\displaystyle\sum_{i=1}^{\infty}\left\lfloor \frac{n}{5^i} \right\rfloor < n\left(\frac{1}{5-1}\right) = \frac{n}{4}.$

So $\dfrac{n}{4} > 1998 \quad \Rightarrow \quad n > 7992.$

By trial and error, we take $n = 7995$ and then search for the correct value. If $n =$ 7995, then the number of zeroes at the end of 7995 is by Eq. (1)

$$\frac{7995}{5} + \frac{7995}{5^2} + \cdots$$
$$= 1599 + 319 + 63 + 12 + 2 = 1995.$$

So true for $n = 8000$, we get the number of zeroes at the end of 8000! = 1600 + 320 + 64 + 12 + 2 = 1998.

All such $n = 8000, 8001, 8002, 8003, 8004$

Note: Corresponding to 1997 zeroes at the end, there exist no n, as 7995! has 1995 zeroes and the next multiple of 5, *i.e.,* 8000 is a multiple of 125, it adds 3 more zeroes to 1995 given 1998 zeroes at the 9 end.

Build-up Your Understanding 8

1. Prove that $\lfloor 2x\rfloor - 2\lfloor x\rfloor = 0$ or 1.
2. Prove that $\lfloor 2x\rfloor + \lfloor 2y\rfloor \geq \lfloor x\rfloor + \lfloor y\rfloor + \lfloor x+y\rfloor$.
3. Prove that for any positive integer n and any real x, $\left\lfloor \frac{\lfloor nx\rfloor}{n}\right\rfloor = \lfloor x\rfloor$.
4. For $\alpha \in (0,1)$, prove that $\lfloor x\rfloor - \lfloor x-\alpha\rfloor = 1$ or 0 according as $\{x\} < \alpha$ or $\{x\} > \alpha$, where $\{x\}$ is the fractional part of x.
5. Prove that for any positive integer n, $\left\lfloor \frac{n}{3}\right\rfloor + \left\lfloor \frac{n+2}{6}\right\rfloor + \left\lfloor \frac{n+4}{6}\right\rfloor = \left\lfloor \frac{n}{2}\right\rfloor + \left\lfloor \frac{n+3}{6}\right\rfloor$.
6. Find all the triples (x, y, z) of real numbers, satisfying the three equations
$x + \lfloor y\rfloor + \{z\} = 200.2$, $\{x\} + y + [z] = 200.1$, $\lfloor x\rfloor + \{y\} + z = 200$
7. Find the number of positive integers x which satisfy: $\left\lfloor \frac{x}{99}\right\rfloor = \left\lfloor \frac{x}{101}\right\rfloor$.
[RMO, 2001]
8. Find all real 'x' satisfying, $\frac{1}{\lfloor x\rfloor} + \frac{1}{\lfloor 2x\rfloor} = \{x\} + \frac{1}{3}$. **[RMO, 1997]**
9. For all $n \in \mathbb{N}$, prove that $\lfloor \sqrt{n} + \sqrt{n+1}\rfloor = \lfloor \sqrt{4n+1}\rfloor$
10. If $n \in \mathbb{N}$ and $x \in \mathbb{R}$, prove that $\lfloor x\rfloor + \left\lfloor x + \frac{1}{n}\right\rfloor + \left\lfloor x + \frac{2}{n}\right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n}\right\rfloor = \lfloor nx\rfloor$.
(This is known as **Hermits Identity**)
11. Prove that for $n = 1, 2, 3, \ldots$, $\left\lfloor \frac{n+1}{2}\right\rfloor + \left\lfloor \frac{n+2}{4}\right\rfloor + \left\lfloor \frac{n+4}{8}\right\rfloor + \left\lfloor \frac{n+8}{16}\right\rfloor + \cdots = n$
12. Determine the number of distinct integers in the following sequence:
$\left\lfloor \frac{1^2}{1999}\right\rfloor, \left\lfloor \frac{2^2}{1999}\right\rfloor, \left\lfloor \frac{3^2}{1999}\right\rfloor, \ldots, \left\lfloor \frac{1999^2}{1999}\right\rfloor$.
13. Find the highest power of 7 dividing 1998!.
14. How many zeroes are at the end of 1005!?
15. Find n such that there are 300 zeroes at the end of n!?
16. How many zeros are at the end of $(5^5)!$?
17. Prove that $n!$ for $n > 1$ cannot be a square or cube or any power of an integer.
18. Show that the number $4! + 5! + 6! + \cdots + 1998!$ is divisible by 24 but not by 25.
19. Show that $\left\lfloor (1+\sqrt{3})^{2n} + 1\right\rfloor$ and $\left\lfloor (1+\sqrt{3})^{2n+1}\right\rfloor$ and are both divisible by 2^{n+1}. Is this the highest power of 2 dividing either of the numbers?
20. Prove that the two numbers $\lfloor an\rfloor, \lfloor bn\rfloor$ for $n = 1, 2, 3, \ldots$ comprise of all integers $1, 2, 3, \ldots$, without repetition if a and b are positive irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$.
21. For positive integers n, define $A(n)$ to be $\frac{(2n)!}{(n!)^2}$. Determine the sets of positive integers n for which, (i) $A(n)$ is an even number, (ii) $A(n)$ is a multiple of 4.

Charles Hermite

24 Dec 1887–14 Jan 1901
Nationality: French

22. For such integer $n \geq 1$, define $a_n = \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor$. Find the number of all n in the set $\{1, 2, 3, \ldots, 2010\}$ for which $a_n > a_{n+1}$. **[RMO, 2010]**

23. Let n be an integer greater than prime p. Show that p divides $\binom{n}{p} - \left\lfloor \frac{n}{p} \right\rfloor$ **[RMO, 2003]**

24. Let $m, n \in \mathbb{N}$. Prove that $\frac{(2m)!(2n)!}{m!n!(m+n)!}$ is an integer. **[IMO, 1972]**

6.12 Diophantine Equations

An equation of the form $f(x_1, x_2, x_3, \ldots, x_n) = 0$ where f is an n-variable function with $n \geq 2$ is called diophantine equation. If f is polynominal with integral coefficients, then it is called algebraic diophantine equation.

An n-tuple $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ satisfying the equation called a solution to the equation.

In diophantine equation we basically concern with solvability of the equation, number of solution finite or infinite and determining all solutions.

Please observe following examples:

Example 72 *Determine the integer n for which* $n^2 + 19n + 92$ *is a square.* **[RMO, 1992]**

Solution: Let $n^2 + 19n + 92 = x^2$, where x is a positive integer.

Now, $4(n^2 + 19n + 92) = 4x^2$

$\Rightarrow$ $(2n + 19)^2 + 7 = 4x^2$

$\Rightarrow$ $(2x)^2 - (2n + 19)^2 = 7$

$\Rightarrow$ $(2x + 2n + 19)(2x - 2n - 19) = 7$

As x is positive both cannot be negetive.
Hence, both must be positive. There are two possibilities.

$$2x + 2n + 19 = 1$$
$$\text{and } 2x - 2n - 19 = 7$$
$$\Rightarrow n = -11$$
$$\text{or } 2x + 2n + 19 = 7$$
$$\text{and } 2x - 2n - 19 = 1$$
$$\Rightarrow \quad n = -8$$

Hence, $n = -8, -11$.

Example 73 *Find all unordered pairs of natural numbers, the difference of whose square is* 45.

Solution: Let x and y be the natural numbers such that $x^2 - y^2 = 45$, where $x > y$.

$\Rightarrow$ $(x - y)(x + y) = 45$

So, both $(x - y)$ and $(x + y)$ are the divisors of 45, and $x + y > x - y$, where x and y are positive integers.

So, $x - y = 1$, and $x + y = 45$ (1)

or $x - y = 3$, and $x + y = 15$ (2)

or $x - y = 5$, and $x + y = 9$ (3)

Diophantus of Alexandria

AD 201–215 to AD 285–299
Nationality: Greek

His epitaph: This tomb hold Diophantus, Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifty year after his marriage He granted him a son. Alas! Late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.

Solving (1), (2) and (3), we get

$$x = 23, y = 22 \text{ and}$$
$$x = 9, y = 6 \text{ and}$$
$$x = 7, y = 2$$

So, the pairs of numbers satisfying the condition are (23, 22), (9, 6), (7, 2).

Example 74 Find all *positive integers n for which* $n^2 + 96$ *is a perfect square.*

Solution: Let $n^2 + 96 = k^2$, where $k \in \mathbb{N}$.

Then $k^2 - n^2 = 96$

$$(k - n)(k + n) = 96 = 3^1 \times 2^5.$$

Clearly $k > n$ and hence, $k + n > k - n > 0$.

Since 3 is the only odd factor, both k and n are integers. We must have $k + n$ and $k - n$ both to be either even or odd. (If one is odd and the other even, then k and n do not have integer solutions). Also both $k + n$ and $k - n$ cannot be odd as the product is given to be even. So the different possibilities for $k + n$, $k - n$ are as follows.

$$k - n = 2 \quad \text{and} \quad k + n = 48 \tag{1}$$
$$\text{or} \quad k - n = 4 \quad \text{and} \quad k + n = 24 \tag{2}$$
$$\text{or} \quad k - n = 6 \quad \text{and} \quad k + n = 16 \tag{3}$$
$$\text{or} \quad k - n = 8 \quad \text{and} \quad k + n = 12 \tag{4}$$

So, solving separately Eqs. (1), (2), (3) and (4), we get $n = 23, 10, 5, 2$.

So, there are exactly four values of n for which $n^2 + 96$ is a perfect square.

$$n = 23 \quad \text{gives} \quad 23^2 + 96 = 625 = 25^2$$
$$n = 10 \quad \text{gives} \quad 10^2 + 96 = 196 = 14^2$$
$$n = 5 \quad \text{gives} \quad 5^2 + 96 = 121 = 11^2$$
$$n = 2 \quad \text{gives} \quad 2^2 + 96 = 100 = 10^2$$

Example 75 *Find all the ordered pairs of integers* (x, z) *such that* $x^3 = z^3 + 721$.

Solution: Since $x^3 - z^3 = 721$

$$\Rightarrow \quad x^3 - z^3 = (x - z)(x^2 + xz + z^2) = 721$$

For integers x, z; $x^2 + xz + z^2 > 0$

$$\Rightarrow \quad x - z > 0.$$

So $(x - z)(x^2 + xz + z^2) = 721 = 1 \times 721$

$$= 7 \times 103 = 103 \times 7 = 721 \times 1.$$

Case 1: $x - z = 1 \quad \Rightarrow \quad x = 1 + z$

and $x^2 + xz + z^2 = (1 + z)^2 + (1 + z)z + z^2 = 721$

$$\Rightarrow \quad 3z^2 + 3z - 720 = 0$$
$$\Rightarrow \quad z^2 + z - 240 = 0$$
$$\Rightarrow \quad (z + 16)(z - 15) = 0$$
$$\Rightarrow \quad z = -16 \quad \text{or} \quad z = 15.$$

Solving, we get

$$x = -15 \text{ or } 16.$$

So $(-15, -16)$ and $(16, 15)$ are two of the ordered pairs.

Case 2: $x - z = 7$ or $x = 7 - z$

and $$x^2 + xz + z^2 = 103$$

$\Rightarrow$ $$(7 + z)^2 + (7 + z)z + z^2 = 103$$

$\Rightarrow$ $$3z^2 + 21z - 54 = 0$$

$\Rightarrow$ $$z^2 + 7z - 18 = 0$$

$\Rightarrow$ $$(z + 9)(z - 2) = 0$$

$\Rightarrow$ $$z = -9 \quad \text{or} \quad z = 2.$$

So, the corresponding values of x are -2 and 9.

So, the other ordered pairs are $(-2, -9)$ and $(9, 2)$.

Corresponding to $x - z = 103$ and $x - z = 721$, the values are imaginary and hence, there are exactly four ordered pairs of integers $(-15, -16)$, $(16, 15)$, $(-2, -9)$ and $(9, 2)$, satisfying the equation $x^3 = z^3 + 721$.

Example 76 *Let d be any positive integer not equal to 2, 5 or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a square.* **[IMO, 1986]**

Solution: Here we should show that there does not exist any positive integer d, which makes $(2d - 1)$, $(5d - 1)$, $(13d - 1)$ to be a square number simultaneously.

Assuming the contrary,

$$2d - 1 = x^2$$
$$5d - 1 = y^2$$
$$13d - 1 = z^2,$$

where x, y and z are positive integers, $x^2 = 2d - 1$ is an odd number, $\Rightarrow x$ is odd $\Rightarrow x^2 \equiv 1 \pmod 8$.

$\Rightarrow 2d - 1 \equiv 1 \pmod 8$

$\Rightarrow 2d \equiv 2 \pmod 8 \Rightarrow d \equiv 1 \pmod 4$

hence, d must be odd. Hence, y and z are even.

Now, $$z^2 - y^2 = 8d$$

$\Rightarrow$ $$(z - y)(z + y) = 8d.$$

Therefore, either $(z - y)$ or $(z + y)$ is divisible by 4.

If $z - y$ is divisible by 4, then $z + y = (z - y) + 2y$ is also divisible by 4 because $(z - y)$ and $2y$ are divisible by 4.

Similarly, if $z + y$ is divisible by 4, then $z - y = (z + y) - 2y$ is also divisible by 4.

Thus, $(z - y)(z + y)$ is divisible by $4 \times 4 = 16$.

Thus, $16 \mid 8d$, where d is an odd number.

This is a contradiction and hence, $(2d - 1)$, $(5d - 1)$ and $(13d - 1)$ cannot simultaneously be square integers.

Example 77 *Find all the positive integers x, y, and z satisfying*

$$x^{y^z} \cdot y^{z^x} \cdot z^{x^y} = 5xyz.$$

Solution: x, y, and z are integers and 5 is a prime number and given equation is $x^{y^z} \cdot y^{z^x} \cdot z^{x^y} = 5xyz$.

Dividing both sides of the equation by xyz

$$x^{y^z - 1} \cdot y^{z^x - 1} \cdot z^{x^y - 1} = 5$$

So, the different possibilities are

$$\left.\begin{aligned} x^{y^z-1} &= 5\\ y^{z^x-1} &= 1\\ z^{x^y-1} &= 1 \end{aligned}\right| \quad \text{or} \quad \left.\begin{aligned} x^{y^z-1} &= 1\\ y^{z^x-1} &= 5\\ z^{x^y-1} &= 1 \end{aligned}\right| \quad \text{or} \quad \left.\begin{aligned} x^{y^z-1} &= 1\\ y^{z^x-1} &= 1\\ z^{x^y-1} &= 5 \end{aligned}\right|$$

Taking the first column

$$x = 5,\ y^z - 1 = 1;\ y^z = 2,\ y = 2, \text{ and } z = 1$$

and these values are satisfying the other expressions in the first column.

Similarly, from the second column, we get $y = 5$, $z = 2$, and $x = 1$ and from the third column, we get $z = 5$, $x = 2$, and $y = 1$.

$$\Rightarrow \quad (x, y, z) \equiv (5, 2, 1), (2, 1, 5), (1, 5, 2)$$

Example 78 *Find all pairs of integers x, y, such that* $(xy - 1)^2 = (x + 1)^2 + (y + 1)^2$.

Solution: We have, $(xy - 1)^2 = (x + 1)^2 + (y + 1)^2$

$\Rightarrow \quad (xy - 1)^2 - (x + 1)^2 = (y + 1)^2$

$\Rightarrow \quad (xy - x - 2)(xy + x) = (y + 1)^2$

$\Rightarrow \quad x(xy - x - 2)(y + 1) = (y + 1)^2 \qquad (1)$

$\Rightarrow \quad (y + 1)[x(xy - x - 2) - (y + 1)] = 0 \qquad (2)$

If $y = -1$, then x takes all the values from the set of integers.

Similarly, we also get

$$(x + 1)\,[y(xy - y - 2) - (x + 1)] = 0 \qquad (3)$$

If $x = -1$, then y takes all the values from the set of integers.

If $x \neq -1$, $y \neq -1$, then from Eq. (1)

$$x(xy - x - 2)(y + 1) = (y + 1)^2$$

$\Rightarrow \quad x(xy - x - 2) = (y + 1) \quad (\because\ y \neq -1)$

$\Rightarrow \quad x^2y - x^2 - 2x - y - 1 = 0$

$\Rightarrow \quad y(x - 1)(x + 1) = (x + 1)^2$

Since $x \neq -1$, we have $y(x - 1) = (x + 1)$

$\Rightarrow \quad y = \dfrac{x+1}{x-1} = 1 + \dfrac{2}{x-1}$

$\Rightarrow \quad (x-1)|2 \ \Rightarrow\ x - 1 = \pm 1, \pm 2$

$$\Rightarrow \quad x = 0, 2, -1, 3$$

$$\begin{aligned} \text{Now, } x = 0 &\ \Rightarrow\ y = -1\\ x = 2 &\ \Rightarrow\ y = 3\\ x = 3 &\ \Rightarrow\ y = 2 \end{aligned}$$

Hence, the solution set is (3, 2), (2, 3), $(x, -1)$, $(-1, y)$.

Example 79 *Find all integral solutions of* $x^2 - 3y^2 = -1$.

Solution: We have, $x^2 - 3y^2 = -1$

$\Rightarrow \quad x^2 = 3y^2 - 1 \equiv -1 \pmod 3$

$\qquad\qquad \equiv 2 \pmod 3$

But, for any $x \in Z$, $x^2 \equiv 0 \pmod 3$, or $x^2 \equiv 1 \pmod 3$

And hence, there is no solution for the given equation.

Example 80 *Show that* $15x^2 - 7y^2 = 9$ *has no integral solutions.*

Solution: Since the RHS is odd, x and y must be opposite parity (*i.e.*, one even and the other odd). As $3|15$ and $3|9$, 3 must divide $7y^2 \quad \Rightarrow \quad 3|y$.

$$\therefore \qquad y = 3y_1$$

Substituting and simplifying, we get $5x^2 - 21y_1^2 = 3$.

Again, $3|5x^2$, therefore, $x = 3x_1$ leading to the new equation

$$15x_1^2 - 7y_1^2 = 1$$

Take mod 3 of the equation, we get

$$0 - y_1^2 \equiv 1 \bmod 3$$

$$\text{or } y_1^2 \equiv -1 \pmod 3$$

But for any number n, $n^2 \equiv 0, 1 \bmod 3$ which is a contradiction.

Therefore, $15x_1^2 = 7y_1^2 + 1$ has no solution in integers.

Hence, the given equation has no integral solution.

Example 81 *Show that the quadratic equation* $x^2 + 7x - 14(q^2 + 1) = 0$, *where* q *is an integer, has no integral root.*

Solution: Assume its contrary that n be an integer root of $x^2 + 7x - 14(q^2 + 1) = 0$.

Then, $$n^2 + 7n - 14(q^2 + 1) = 0 \qquad (1)$$

$\Rightarrow \quad n^2 = -7(n + 2q^2 + 2)$

$\Rightarrow \quad 7|n^2$ and hence, $7|n$ as 7 is a prime number.

Let, $$n = 7n_1.$$

Then, Eq. (1) can be written as

$$49n_1^2 + 49n_1 = 14(q^2 + 1)$$

$$\Rightarrow \qquad 7n_1^2 + 7n_1 = 2(q^2 + 1)$$

So, $7|2(q^2 + 1)$ and hence, $7|(q^2 + 1)$

$\Rightarrow \quad q^2 + 1 \equiv 0 \pmod 7$

$q^2 \equiv 6 \pmod 7$

As $q = 0, \pm 1, \pm 2, \pm 3 \pmod 7$

$q^2 = 0, 1, 4, 2 \pmod 7$, respectively.

Hence, $q^2 \not\equiv 6 \pmod 7$ for any integer.

Therefore, there exists no integral root for the given quadratic equation.

Example 82 *Find all the integral solutions of* $x^3 + 5y^3 + 25z^3 - 15xyz = 0$.

Solution: We shall use the identity

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

Writing $a = x$, $b = 5^{1/3}y$, $c = 5^{2/3}z$ in the given equation, it can be written as

$$x^3 + (5^{1/3}y)^3 + (5^{2/3}z)^3 - 3 \times x \times 5^{1/3}y \times 5^{2/3}z = 0$$

$$\therefore\quad \frac{1}{2}(x+5^{1/3}y+5^{2/3}z)\times[(x-5^{1/3}y)^2+(5^{1/3}y-5^{2/3}z)^2+(5^{2/3}z-x)^2]=0$$

$$\Rightarrow\qquad (x+5^{1/3}y+5^{2/3}z)=0$$

$$\text{or}\qquad [(x-5^{1/3}y)^2+(5^{1/3}y-5^{2/3}z)^2+(5^{2/3}z-x)^2]=0.$$

If $x+5^{1/3}y+5^{2/3}z=0$, then $5^{1/3}y+5^{2/3}z=-x$.

Clearly, the left-hand side is irrational, when y and z are integers other than zero, and the right-hand side is always an integer.

So, $x=y=z=0$ is a solution.

If $(x-5^{1/3}y)^2+(5^{1/3}y-5^{2/3}z)^2+(5^{2/3}z-x)^2=0$, then $x=5^{1/3}y$, $y=5^{1/3}z$ and $x=5^{2/3}z$.

Again, this is possible only when $x=y=z=0$ as we need integer values for x, y, and z.

Aliter: Number theoretic solution

$$x^3+5y^3+25z^3-15xyz=0$$

$$\Rightarrow\qquad x^3=5(3xyz-y^3-5z^3)\qquad (1)$$

$$\Rightarrow\qquad 5\,|\,x^3 \text{ and hence, } 5\,|\,x$$

Let, $x=5x_1$, then $x^3=125x_1^3$

so that the equation becomes

$$y^3=5x_1yz-25x_1^{\,3}-5z^3$$

$$\Rightarrow\qquad 5\,|\,y \text{ and let } y=5y_1$$

Again, the equation becomes $z^3=15zx_1y_1-5x_1^{\,3}-25y_1^{\,3}$

$\Rightarrow 5\,|\,z$ and taking $z=5z_1$.

We get,

$$x_1^3+5y_1^3+25z_1^3-15x_1y_1z_1=0\qquad (2)$$

This implies that if (x, y, and z) is an integral solution, then $\left(\frac{x}{5},\frac{y}{5},\text{and }\frac{z}{5}\right)$ is also an integral solution to Eq. (1).

Arguing in the same way, we find

$$x_2=\frac{x_1}{5},\ y_2=\frac{y_1}{5},\ z_2=\frac{z_1}{5}$$

$$\text{or}\qquad x_2=\frac{x}{5^2},\ y_2=\frac{y}{5^2},\ z_2=\frac{z}{5^2}$$

is also an integral solution and thus, by induction method, we get

$$x_n=\frac{x}{5^n},\ y_n=\frac{y}{5^n},\ z_n=\frac{z}{5^n}$$

is an integral solution for all $n\geq 0$.

This means that x, y, and z are multiples of 5^n, for all $n\in N$.

This is possible only when x, y, and z are all zero.

Example 83 *Find all integers values of 'a' such that the quadratic expressions* $(x+a)(x+1991)+1$ *can be factored as* $(x+b)(x+c)$*, where b and c are integers.*

[RMO, 1991]

Solution: $(x + a)(x + 1991) + 1 = (x + b)(x + c)$

$\Rightarrow \quad 1991 + a = b + c$

and $\quad 1991a + 1 = bc$

$$\therefore \quad (b - c)^2 = (b + c)^2 - 4bc$$
$$= (1991 + a)^2 - 4(1991a + 1)$$
$$= (1991 + a)^2 - 4 \times 1991a - 4$$
$$= (1991 - a)^2 - 4$$

or $(1991 - a)^2 - (b - c)^2 = 4$.

If the difference between two perfect squares is 4, then one of them is 4 and the other is zero. (*Prove this*)

Therefore, $1991 - a = \pm 2$, $(b - c)^2 = 0$

$\Rightarrow \quad a = 1991 + 2 = 1993 \quad$ and $\quad b = c$

or $\quad a = 1991 - 2 = 1989 \quad$ and $\quad b = c$.

So, the only two values of a are 1993 and 1989.

Example 84 *Find all the integral solutions of* $y^2 = 1 + x + x^2$.

Solution: If $x > 0$, then $x^2 < x^2 + 1 + x < x^2 + 2x + 1 = (x + 1)^2$

So $x^2 + x + 1$ lies between the two consecutive square integers and hence, cannot be a square.

If $x = 0$, $y^2 = 1 + 0 + 0 = 1$ is a square number, the solutions in this case are $(0, 1)$, $(0, -1)$.

Again if $x < -1$, then $x^2 > x^2 + x + 1 > x^2 + 2x + 1$, and hence, there exist no solution.

For $x = -1$, we have

$$y^2 = 1 - 1 + (-1)^2 = 1$$

$$\therefore \quad y = \pm 1.$$

for $x \in (-1, 0)$, $x^2 + x + 1 \in \left(\frac{3}{4}, 1\right)$, hence no such y.

Thus, the only integral solutions are $(0, 1)$, $(0, -1)$, $(-1, 1)$, $(-1, -1)$.

Example 85 *Find all integers x for which $x^4 + x^3 + x^2 + x + 1$ is a perfect square.*

Solution: If $x^4 + x^3 + x^2 + x + 1$ is a perfect square, then let

$$y^2 = x^4 + x^3 + x^2 + x + 1.$$

$$\text{consider} \left(x^2 + \frac{x}{2}\right)^2 = x^4 + x^3 + \frac{x^2}{4}$$
$$= x^4 + x^3 + x^2 + x + 1 - \left(\frac{3}{4}x^2 + x + 1\right)$$
$$= y^2 - \frac{1}{4}(3x^2 + 4x + 4)$$

As the discriminant of $3x^2 + 4x + 4$ is less than 0, so $3x^2 + 4x + 4$ is always greater than zero.

Thus,

$$\left(x^2 + \frac{x}{2}\right)^2 < y^2 \quad \text{or} \quad \left|x^2 + \frac{x}{2}\right| < |y|$$

But, $x^2+\dfrac{x}{2}=x\left(x+\dfrac{1}{2}\right)$ is non-negative for all $x \in \mathbb{Z}$

$\therefore$ $$\left|x^2+\frac{x}{2}\right|=x^2+\frac{x}{2}<|y|$$

If x is even, then

$$|y| \geq x^2+\frac{x}{2}+1$$

$$\Rightarrow\ y^2 \geq x^4+x^3+x^2+x+1+\frac{5}{4}x^2=y^2+\frac{5}{4}x^2$$

Not possible, if $x \neq 0$. $x=0$ is the only solution when x is even.

If x is odd, then $x^2+\dfrac{x}{2}+\dfrac{1}{2}$ is an integer.

So, $$|y| \geq \left(x^2+\frac{x}{2}\right)+\frac{1}{2}$$

In this case, $y^2 \geq x^4+x^3+x^2+x+1+\left(\dfrac{x^2}{4}-\dfrac{x}{2}-\dfrac{3}{2}\right)$

that is, $y^2 \geq y^2+\left(\dfrac{x^2}{4}-\dfrac{x}{2}-\dfrac{3}{4}\right)=y^2+\dfrac{1}{4}(x^2-2x-3)$

hence, $$\frac{1}{4}(x^2-2x-3) \leq 0$$

$$\Rightarrow \quad x^2-2x-3 \leq 0$$

$$\Rightarrow \quad (x-3)(x+1) \leq 0$$

$$\therefore \quad -1 \leq x \leq 3$$

The odd integral values of x are -1, 1 and 3 of which 1 does not give a perfect square.

Hence, there are exactly 3 integral values of x, namely, 0, -1 and 3, for which the expression is a perfect square.

Aliter: $y^2=x^4+x^3+x^2+x+1$, (1)

obviously $x=0 \Rightarrow y=\pm 1$

Let $x \neq 0$

Now, $4y^2=4x^4+4x^3+4x^2+4x+4=(2x^2+x)^2+3x^2+4x+4$

As $3x^2+4x+4>0 \quad \forall x \in \mathbb{R} \Rightarrow 4y^2>(2x^2+x)^2$

Also $(2x^2+x)^2+3x^2+4x+4<(2x^2+x)^2+8x^2+4x+4=(2x^2+x+2)^2$

$\Rightarrow\ 4y^2<(2x^2+x+2)^2$

As $(2x^2+x)^2<4y^2<(2x^2+x+2)^2 \Rightarrow\ 4y^2=(2x^2+x+1)^2$.

Now solving it with Eq. (1) we get $x=1, 3$.

Example 86 *Find all solutions in positive integers of the equation*

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{15}$$

Solution: Without loss of generality, let us assume that $x \le y \le z$.

Then $$\frac{1}{x} \ge \frac{1}{y} \ge \frac{1}{z}$$

$$\therefore \quad \frac{1}{x} < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \le \frac{3}{x}$$

$$\therefore \quad \frac{1}{x} < \frac{7}{15} \le \frac{3}{x} \Rightarrow \frac{15}{7} < x \le \frac{45}{7}$$

$$\therefore \quad 3 \le x \le 6.$$

Hence, x can take values 3, 4, 5 or 6 only.

Case 1: $x = 3$, $\frac{1}{y} + \frac{1}{z} = \frac{7}{15} - \frac{1}{3} = \frac{2}{15}$

$$\frac{1}{y} < \frac{1}{y} + \frac{1}{z} \le \frac{2}{y} \quad \text{and} \quad \frac{1}{y} < \frac{2}{15} \le \frac{2}{y} \Rightarrow \frac{15}{2} < y \le 15$$

$$\therefore \quad 8 \le y \le 15$$

Also $$\frac{1}{z} = \frac{2}{15} - \frac{1}{y}$$

$$\Rightarrow \quad z = \frac{15y}{2y - 15}.$$

For y = 8, 9, 10, 12 and 15 we get z = 120, 45, 30, 20 and 15 respectively. For other values of y, z is not integer. Thus, the solutions when $x = 3$ are (3, 8, 120), (3, 9, 45), (3, 10, 30), (3, 12, 20) and (3, 15, 15). Similarly for $x = 4$, we have (4, 5, 60), (4, 6, 20) and $x = 5$ we have (5, 5, 15), (5, 6, 10). For $x = 6$ no solution.

Example 87 *For any positive integer n, let $s(n)$ denote the number of ordered pairs (x, y) of positive integers for which* $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$.

For instance if $n = 2$, we have $s(n) = 3$.

For $$\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \quad \frac{1}{3} + \frac{1}{6} = \frac{1}{2}, \quad \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

thus the three ordered pairs are (4, 4), (3, 6), (6, 3) *and hence,* $s(2) = 3$. *Determine the set of positive integers n for which $s(n) = 5$.*

Solution: Let us consider the general case

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}.$$

Here both x and y are greater than n and let

$$x = n + p \quad \text{and} \quad y = n + q.$$

Therefore, $$\frac{1}{(n+p)} + \frac{1}{(n+q)} = \frac{1}{n}$$

$$\Rightarrow \quad n(n + p) + n(n + q) = (n + p)(n + q)$$

$$\Rightarrow \qquad n(2n + p + q) = n^2 + n(p + q) + pq$$

$$\Rightarrow \qquad n^2 = pq.$$

Here, $s(n)$ = the number of $p's$ (or $q's$) such that $n^2 = pq$ and it is easily seen that p ranges over the divisors of n^2.

Thus $s(n)$ = the number of divisors of $n^2 = \tau(n^2)$.

To find n such that $s(n) = 5$, we note that $\tau(n^2) = 5$ and 5 is a prime number.

$\therefore$ n can have only one prime factor (say) p and $n = p^l$ (say).

$$\therefore \qquad \tau(p^{2l}) = 2l + 1 = 5$$

or, $$l = 2.$$

The possible values for n such that $s(n) = 5$ is all squares of primes.

Example 88 *The sum of several consecutive positive integers is equal to* 1000. *Find the numbers.*

Solution: Let $x + (x + 1) + \cdots + (x + (n - 1)) = 1000$ (1)

We have to find n and x, hence, the numbers that add up to 1000.

From Eq (1), we have

$$nx + 1 + 2 + \cdots + (n - 1) = 1000, x, n \in \mathbb{N}, n > 1$$

$$\Rightarrow \qquad nx + \frac{n(n-1)}{2} = \frac{n}{2}(2x + n - 1) = 1000$$

$$\Rightarrow \qquad n(2x + n - 1) = 2000 = 2^4 \times 5^3 \quad (1)$$

$$\Rightarrow \qquad n|(2^4 \times 5^3). \quad (2)$$

Also $\quad n^2 < n(2x + n - 1) = 2000$

$$\Rightarrow \qquad n < 45 \quad (3)$$

Case 1: If n is odd, then from (2) and (3)

$$n = 5, \quad 5^2$$

If $n = 5$, then $2x + 4 = 2^4 \times 5^2 = 400$ (From (1))

$$\Rightarrow x = 198.$$

So, in this case the numbers are 198, 199, 200, 201 and 202.

If $n = 5^2 = 25$, then $2x + 24 = 2^4 \times 5 = 80$ (From (1))

$$\Rightarrow x = 28.$$

So the numbers are 28, 29, …, 52.

Case 2: If n is even, then from (1) n must be divisible by 16 as $2x + n - 1$ will be odd.

So, $n = 16$ (as any other multiple of 16 which satisfies Eq (2) exceeds 45)

$$\therefore \qquad 16(2x + 15) = 2^4 \times 5^3$$

$$\Rightarrow \qquad 2x + 15 = 5^3 = 125$$

$$\Rightarrow \qquad 2x = 110 \quad \Rightarrow \quad x = 55.$$

So, the consecutive numbers in this case are 55, 56, …, 70.

Example 89 *Determine all non-negative integral pairs* (x, y) *for which* $(xy - 7)^2 = x^2 + y^2$.

Solution: $(xy - 7)^2 = x^2 + y^2$ is a symmetric equation in x, y. So, whenever (a, b) is a solution, (b, a) is also a solution.

Again, if (a, a) is a solution, then

$$(a^2 - 7)^2 = 2a^2$$

$$\Rightarrow \quad a^4 - 14a^2 + 49 = 0$$

$$\Rightarrow \quad \Delta = 256 - 196 = 60$$

is not a perfect square and hence, a is irrational.

So, we will find all solutions (x, y) for which $0 \le x < y$.

$$(xy - 7)^2 = x^2 + y^2$$

$$\Rightarrow \quad x^2y^2 - 14xy + 49 = x^2 + y^2 \qquad (1)$$

Dividing Eq. (1) by y^2, both the sides we get

$$x^2 = 14\frac{x}{y} - \frac{49}{y^2} + \frac{x^2}{y^2} + 1$$

or $\quad x^2 < 14\frac{x}{y} + \left(\frac{x}{y}\right)^2 + 1 < 14 \cdot 1 + 1^2 + 1$

as we have assumed $x < y$.

$\Rightarrow x^2 < 16,$

$$\therefore \quad x < 4.$$

It means x can take the values 0, 1, 2 and 3.

$x = 0 \quad \Rightarrow \quad y = 7$

$x = 1 \quad \Rightarrow \quad (y - 7)^2 = 1 + y^2$

$\Rightarrow \quad -14y = -48$

$\Rightarrow y = \dfrac{48}{14}$, not an integer.

$x = 2 \quad \Rightarrow \quad (2y - 7)^2 - y^2 + 4$

$\Rightarrow \quad 3y^2 - 28y + 45 = 0$

$\Rightarrow \quad y$ is irrational because $28^2 - 4 \times 45$ is not a perfect square.

$x = 3 \quad \Rightarrow \quad (3y - 7)^2 = y^2 + 9$

$\Rightarrow \quad 8y^2 - 42y + 40 = 0$

$\Rightarrow \quad 4y^2 - 21y + 20 = 0$

$\Rightarrow \quad (y - 4)(4y - 5) = 0$

$\Rightarrow \quad y = 4 \quad$ or $\quad y = \dfrac{5}{4}.$

Neglecting $y = \dfrac{5}{4}$, we get the following pairs (0, 7), (7, 0), (3, 4) and (4, 3) to be the only solutions.

Example 90 *Find all integers x, y satisfying* $(x - y)^2 + 2y^2 = 27$.

Solution: $(x - y)^2, 2y^2 > 0$ and since, $2y^2$ is even, $(x - y)^2$ is odd and hence, $(x - y)$ should be odd.

So, the different possibilities for $(x - y)^2$ and y^2 are (1, 13), (9, 9), (25, 1) corresponding to $y^2 = 13$. There is no solution as y is an integer. So, taking the other two-ordered pairs, we have

$$x - y = \pm 3, y = \pm 3 \qquad (1)$$

$$x - y = \pm 5, y = \pm 1 \qquad (2)$$

Solving the systems given in (1), we get: (0, 3), (6, 3), (0, –3), (–6, –3).

Solving the systems given in (2), we get: (6, 1), (– 4, 1), (–6, –1), (4, –1).

Example 91 *Solve the following systems of equations in natural numbers:* $a^3 - b^3 - c^3 = 3abc;\ a^2 = 2(b + c)$.

Solution: Since, a, b and c are positive integers, $a^3 - b^3 - c^3 = 3abc$ gives $a^3 > (b^3 + c^3)$ and hence, $a^3 > b^3$ also $a^3 > c^3$

or $\quad a > b$ and $a > c$

$\Rightarrow \quad 2a > (b + c)$

$\Rightarrow \quad 4a > 2(b + c) = a^2$

$\Rightarrow \quad 4 > a$

or $a < 4$. But, from second equation, a^2 is even and hence, a is even numbers.

So, $a = 2$. But, $b < a$ and $c < a$ gives $b = 1$ and $c = 1$.

The only solution is $a = 2$, $b = c = 1$, which satisfied the given system.

Aliter: $a^3 - b^3 - c^3 - 3abc = 0$

$\Rightarrow (a - b - c)(a^2 + b^2 + c^2 + ab - bc + ac) = 0$

Now, $a^2 + b^2 + c^2 + ab - bc + ac = \dfrac{1}{2}[(a+b)^2 + (b-c)^2(c+a)^2] \neq 0$

$\Rightarrow b + c = a$

$\Rightarrow a^2 = 2a$

$\Rightarrow a = 2$

$\Rightarrow b = c = 1$.

Example 92 *A leaf is torn from a paperback novel. The sum of the remaining pages is* 15,000. *What are the page numbers on the torn leaf?* **[RMO, 1994]**

Solution: Let the number of pages in the novel be n. Since, the number of pages after a leaf is torn is 15,000, the sum of the numbers on all the pages must exceed 15,000.

i.e., $\quad \dfrac{n(n+1)}{2} > 15{,}000$

$\Rightarrow \quad n(n + 1) > 30{,}000$

$\therefore \quad (n + 1)^2 > n(n + 1) > 30{,}000 > 29929 = 173^2$

$\Rightarrow \quad (n + 1) > 173$

$\Rightarrow \quad n > 172 \qquad (1)$

The sum of the numbers on the page torn should be less than or equal to $(n - 1) + n = 2n - 1$.

Hence, $(1 + 2 + \cdots + n) - (2n - 1) \leq 15{,}000$.

$\Rightarrow \quad n(n + 1) - 2(2n - 1) \leq 30{,}000$

$\Rightarrow \quad n^2 - 3n + 2 \leq 30{,}000$

$\Rightarrow \quad (n - 2)(n - 1) \leq 30{,}000$

$\Rightarrow \quad (n - 2)^2 < (n - 2)(n - 1) \leq 30{,}000 < 30276 = 174^2$

$\Rightarrow \quad (n - 2) < 174$

$\Rightarrow \quad n < 176.$

By Eq. (1) and (2), we get

$$172 < n < 176.$$

So, n could be one of 173, 174 or 175.

If $n = 173$, then

$$\frac{n(n+1)}{2} = \frac{173 \times 174}{2} = 15{,}051$$

Thus, the sum of the numbers on the torn pages = 15,015 – 15,000 = 51, and this should be $x + (x + 1) = 2x + 1 = 51$.

So, the page numbers on the torn pages $= \frac{51-1}{2} = 25$ and $\frac{51+1}{2} = 26$.

If $n = 174$, then

$$\frac{n(n+1)}{2} = \frac{174 \times 175}{2} = 15{,}225.$$

So, the sum of the numbers on the torn pages = 15,225 – 15,000 = 225, and in this case,

the numbers on torn pages $= \frac{225-1}{2} = 112$ and $\frac{225+1}{2} = 113$.

But, actually the smaller number on the torn page should be odd and hence, though it is theoretically correct, but not acceptable in reality.

If $n = 175$, then $\frac{n(n+1)}{2} = \frac{175 \times 176}{2} = 15{,}400$

and the sum of the numbers on the torn page is 400 = (15,400 – 15,000) which is impossible, because the sum should be an odd number. Hence, this value of n also should be rejected.

So, the numbers on the torn page should be 25 and 26 and the number of pages is 173.

Example 93 *Find all primes p for which the quotient* $\frac{2^{p-1}-1}{p}$ *is a square.*

[INMO, 1995]

Solution: If $p = 2$, $\frac{2^{p-1}-1}{p} = \frac{1}{2}$ is not even an integer.

Let p be a prime of the form $4k + 1$.

Then, if $\frac{2^{p-1}-1}{p} = \frac{2^{4k}-1}{4k+1} = m^2$ for some odd integer m then $2^{4k} - 1 = (4k + 1)m^2$.

Since m^2 is an odd number, $m^2 \equiv 1 \pmod 4$ as all odd squares leave a remainder 1 when divided by 4 and hence, of the form $4l + 1$ (say) then

$$2^{4k} - 1 = (4k + 1)(4l + 1) = 1 \pmod 4$$

But the left hand side

$$\begin{aligned} 2^{4k} - 1 = (16^k - 1) &\equiv -1 \pmod 4 \\ &\equiv 3 \pmod 4 \end{aligned}$$

and it is a contradiction and hence, p cannot be of the form $4k + 1$.

So, let p be of the form $4k + 3$.

Firstly, let us take $k = 0$, then $p = 3$

So, $\frac{2^{p-1}-1}{3} = \frac{2^2-1}{3} = 1$ is a square.

Therefore, $p = 3$ is one of the solutions.

Let p be $4k + 3$ with $k > 0$.

$2^{p-1} - 1 = 2^{4k+2} - 1 = (2^{2k+1} - 1)(2^{2k+1} + 1)$ and $2^{2k+1} - 1$ and $2^{2k+1} + 1$ being consecutive odd numbers are relatively prime.

So, $2^{p-1} - 1 = pm^2$

$$\Rightarrow \qquad (2^{2k+1} - 1)(2^{2k+1} + 1) = (4k + 3)m^2 = pm^2$$

So, pm^2 could be written as $pu^2 \times v^2$ where pu^2 and v^2 are relatively prime.

Case 1: $2^{2k+1} - 1 = pu^2$ and $2^{2k+1} + 1 = v^2$

$$\Rightarrow \qquad 2^{2k+1} = v^2 - 1 = (v + 1)(v - 1).$$

So, $(v + 1)$ and $(v - 1)$ are both powers of 2.

Two powers of 2 differ by 2 only if they are 2 and 2^2. In all other cases, the difference will be greater than 2.

So, $v - 1 = 2^1 = 2$

$$v + 1 = 2^2 = 4 \quad \Rightarrow \quad v = 3$$

i.e., $2^{2k+1} = 2^3 = 8.$

Hence, $k = 1$ and $p = 4k + 3 = 7$.

Therefore, the only other possibility is $p = 7$.

Thus for $p = 7, \dfrac{2^{p-1} - 1}{p} = \dfrac{2^{7-1} - 1}{7} = \dfrac{63}{7} = 9$ which is a perfect square.

Case 2: $2^{2k+1} - 1 = v^2$ and $2^{2k+1} + 1 = pu^2$

As $\quad v =$ odd and $k > 1$

$2^{2k+1} - 1 = v^2 \quad \Rightarrow \quad -1 \equiv 1 \pmod 8$ Contradiction, not possible.

Thus the only primes satisfying the given conditions are 3 and 7.

Build-up Your Understanding 9

1. Show that there is no integral solution for the equation $19x^3 - 84y^2 = 1984$.
2. Prove that the equation $4x^3 - 7y^3 = 2010$ has no solution in integers.
3. Show that there is no integral solution for the equation $x^4 - 3y^4 = 1994$.
4. Show that $x^2 + 3xy - 2y^2 = 1992$ has no solutions in integers.
5. Show that $x^2 + 9xy + 4y^2 = 1995$ has no solutions in integers.
6. Show that $x^4 + y^4 - z^4 = 1993$ has no solutions in integers.
7. Show that, there are no integers (m, n) such that, $m^2 + (m + 1)^2 = n^4 + (n + 1)^4$.
8. Determine all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ if any apart from permutations of the Diophantine equation $n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599$.
9. Prove that the equation $x^3 - y^3 = xy + 1995$ has no solution in integers.
10. Determine all integral solutions of $a^2 + b^2 + c^2 = a^2b^2$.

[USA MO, 1976]

11. Discover all integers, which can be represented in the form $\dfrac{(x + y + z)^2}{xyz}$.
12. Find all positive integers x such that $x(x + 180)$ is a square.
13. Find all positive integers $n < 200$, such that $n^2 + (n + 1)^2$ is a perfect square.
14. Find all positive integer 'n' such that, $(n + 9)$, $(16n + 9)$, $(27n + 9)$ are all perfect squares.
15. a, b, c are distinct digits. Find all (a, b, c) such that, the 3 digit numbers abc and cba are both divisible by 7.

16. Prove that the equation $x^2 + y^2 + 2xy - mx - my - m - 1 = 0$, m is a positive integer, has exactly m solutions (x, y) for which x and y are both positive integers.

17. The equation $a^2 + b^2 + c^2 + d^2 = abcd$ has the solution $(a, b, c, d) = (2, 2, 2, 2)$. Find infinitely many other solutions in positive integers.

18. In a book with page numbers 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages were torn off?

[RMO, 2009]

19. Find all triplets (x, y, z) of positive integers such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{4}{5}$.

20. Find all solutions of $x^3 + 2y^3 = 4z^3$ where x, y, z are integers.

21. Find all integer solution to $x^2 + 615 = 2^n$.

22. Find all integers x, y, z such that $2^x + 3^y = z^2$. **[INMO, 1992]**

23. Find integers x, y, z such that, $x^2z + y^2z + 4xy = 40$ and $x^2 + y^2 + xyz = 20$.

24. Find all positive integer solution of the equation $(2x - 1)^3 + 16 = y^4$.

25. Find all the triples of positive integers (x, y, z) satisfying $2^x + 2^y + 2^z = 2336$.

26. Find all pairs(x, y), where (x, y) are integers, such that $x^3 + 11^3 = y^3$.

27. Find all integers (x, y, z) such that $x^2 + y^2 = z^2$, and that $(x, y) = (y, z) = (x, z) = 1$.

28. Find the primitive solutions of the equation $x^2 + 2y^2 = z^2$ in integers.

29. Find the primitive solution of the equation $x^2 + 3y^2 = z^2$ in integers.

Solved Problems

Problem 1 *What is the three digit number that is equal to 4 times the product of its digits?*

Solution:

$$100a + 10b + c = 4abc \quad \Rightarrow \quad c = 2k,\ 1 \le k \le 4$$

Then $5(10a + b) = k(4ab - 1)$

$\Rightarrow \quad 5|4ab - 1 \quad \Rightarrow \quad 4ab - 1 \equiv 0 \bmod 5 \quad \Rightarrow \quad 4ab \equiv 1 \bmod 5 \Rightarrow -ab = 1 \bmod 5$

$\Rightarrow \quad ab \equiv 4 \text{ (mod)}$

$\Rightarrow$ There are 16 possible values of $(a, b) = (1, 4), (1, 9), (2, 2), (2, 7), (3, 3), (3, 8), (4, 1), (4, 6), (6, 4), (6, 9), (7, 2), (7, 7), (8, 3), (8, 8), (9, 1), (9, 6)$.

Out of which only $a = 3$, $b = 8$ gives $c = 4$ which satisfies the given constrains and sought number is 384.

Problem 2 *On New Year's day, few kids get together and decide to play a simple math game. They write the year 2016 on the blackboard. Every minute they decide to do the following: the written number is erased and the product of its digits plus 12 is written on its place. What number will be written on the blackboard after 24 hours?*

Solution: The pattern just cycles 12, 14, 16, 18, 20, 12, 14, 16, … with a period of 5 and 12 being written on the first minute.

Since $24(60) \equiv 5 \pmod 5$, the number written on the 24th hour will be the 5th number in the sequence, which is 20.

Problem 3 *Find the product of*

$$101 \times 10001 \times 100000001 \times \cdots \times (1000\ldots01)$$

where the last factor has $2^7 - 1$ zeroes between the ones. Find the number of ones in the product.

Solution: Since $101 \times 10001 \times \cdots \times 1000\ldots01$

$$= (10^{2^1}+1)(10^{2^2}+1)\cdots(10^{2^7}+1)$$
$$= (10^2+1)(10^4+1)\cdots(10^{128}+1).$$

Multiply and divide by $10^2 - 1$

$$\frac{(10^2-1)(10^2+1)}{(10^2-1)}(10^4+1)(10^8+1)\cdots(10^{2^7}+1)$$
$$= \frac{1}{(10^2-1)}(10^4-1)(10^4+1)(10^8+1)\cdots(10^{2^7}+1)$$
$$= \frac{1}{(10^2-1)}(10^8-1)(10^8+1)\cdots(10^{2^7}+1)$$
$$= \frac{1}{(10^2-1)}(10^{2^8}-1) = \frac{[(10^2)^{128}-1]}{10^2-1}$$
$$= \frac{(10^2-1)[(10^2)^{127}+(10^2)^{126}+\cdots+10^2+1]}{99}$$
$$= (10^2)^{127}+(10^2)^{126}+\cdots+1.$$
$$= \underbrace{10^{254}+10^{252}+\cdots+10^2+1}_{128\,\text{terms}}$$
$$= 101010...101.$$

(There are 128, 1's alternating zeroes and there are 127 zeroes in between.)

Problem 4 *Show that there exist no rational numbers a, b, c, d such that*

$$(a+b\sqrt{2})^{100}+(c+d\sqrt{2})^{100} = 7+5\sqrt{2}.$$

Solution: Any number in the form $(a+b\sqrt{p})^n$, where p is prime and a and b are rational will again be in the form $\alpha+\beta\sqrt{p}$ where α and β are rational. and $(a+b\sqrt{p})^n = \alpha+\beta\sqrt{p} \Leftrightarrow (a-b\sqrt{p})^n = \alpha-\beta\sqrt{p}$ This can be proved by induction on natural number.

So $(a+b\sqrt{2})^{100} = a_1 + b_1\sqrt{2}$ (say)

then $(a-b\sqrt{2})^{100} = a_1 - b_1\sqrt{2}$,

where both $a_1+b_1\sqrt{2}$ and $a_1-b_1\sqrt{2}$ are both greater than zero, as on LHS the power is an even number 100.

Similarly, $(c+d\sqrt{2})^{100} = c_1+d_1\sqrt{2} > 0$

and $(c-d\sqrt{2})^{100} = c_1-d_1\sqrt{2} > 0.$

Now,
$$(a+b\sqrt{2})^{100}+(c+d\sqrt{2})^{100} \tag{1}$$
$$= (a_1+b_1\sqrt{2})+(c_1+d_1\sqrt{2})$$
$$= (a_1+c_1)+(b_1+d_1)\sqrt{2}$$
$$= 7+5\sqrt{2}.$$

Also $(a-b\sqrt{2})^{100}+(c-d\sqrt{2})^{100}$

$$=(a_1+c_1)-(b_1+d_1)\sqrt{2}>0 \tag{2}$$

and taking conjugate of Eq. (1), we get

$$(a_1+c_1)-(b_1+d_1)\sqrt{2}=7-5\sqrt{2}<0. \tag{3}$$

But this is a contradiction to Eq. (2) and hence, there do not exist rational numbers a, b, c, d to satisfy the given equation.

Problem 5 *Prove that $\log_3 2$ is irrational.*

Solution: If possible, let $\log_3 2$ be a rational number $\frac{p}{q}$, where p, q are integers, $q \neq 0$.

$$\log_3 2 = \frac{p}{q}$$

$$\Rightarrow \quad 3^{p/q} = 2$$

$$\Rightarrow \quad 3^p = 2^q$$

$3 \mid 3^p$ but $3 \nmid 2^q$ and also $2 \mid 2^q$ and $2 \nmid 3^p$ and hence, it is a contradiction.

[or 3^p is an odd number and 2^q is an even number but an odd number equals to an even number is a contradiction.]

Problem 6 *Show that any circle with centre $(\sqrt{2},\sqrt{3})$ cannot pass through more than one lattice point.*

[Lattice points are points in Cartesian plane, whose abscissa and ordinate both are integers.]

Solution: If possible, let (a, b), (c, d) be two lattice points on the circle with $(\sqrt{2},\sqrt{3})$ as centre and radius 'R'.

$$(a-\sqrt{2})^2+(b-\sqrt{3})^2 = R^2 = (c-\sqrt{2})^2+(d-\sqrt{3})^2$$

$$\Rightarrow a^2+b^2-c^2-d^2 = 2(\sqrt{2}a+\sqrt{3}b)-2(\sqrt{2}c+\sqrt{3}d)$$

$$= 2\sqrt{2}(a-c)+2\sqrt{3}(b-d). \ (1)$$

Let $a^2+b^2-c^2-d^2 = r$, $2(a-c) = p$

and $2(b-d) = q$; $p, q, r \in \mathbb{Z}$

From Eq (1) we get $p\sqrt{2}+q\sqrt{3} = r$ (2)

$$\Rightarrow \quad 2pq\sqrt{6} = r^2-2p^2-3q^2$$

$$\Rightarrow \quad \text{for } pq \neq 0,\ \sqrt{6} = \frac{r^2-2p^2-3q^2}{2pq} \quad \text{contradiction}$$

$$\Rightarrow \quad pq = 0 \quad \Rightarrow \quad p = 0 \text{ or } q = 0$$

using Eq. (2) we get $p = q = r = 0$

Hence, $a = c$, $b = d$ $\Rightarrow$ Circle cannot pass through more than one lattice point.

Problem 7 *Let $m_1\, m_2, m_3, \ldots, m_n$ be a rearrangement of numbers $1, 2, 3, \ldots, n$, suppose that n is odd. Prove that $(m_1-1)\times(m_2-2)\times\cdots\times(m_n-n)$ is an even integer.*

Solution: Since n is odd, there are $\frac{n-1}{2}$ even integers and $\frac{n+1}{2}$ odd integers, *i.e.*, there is one more odd integer than even integers. Thus, even after pairing of each even integer m, with an odd integer i there exists an m_k and k, both of which are odd integer, so, $(m_k - k)$ is even and hence, the product is even.

Problem 8 *There are n necklaces such that the first necklace contains 5 beads, the second contains 7 beads and, in general, the ith necklace contains i beads more than the number of beads in (i − 1)th necklace. Find the total number of beads in all the n necklaces.*

Solution: Let us write the sequence of the number of beads in the 1st, 2nd, 3rd, …, nth necklaces

$$= 5, 7, 10, 14, 19, \ldots$$

$$= (4 + 1), (4 + 3), (4 + 6), (4 + 10), (4 + 15), \ldots, \left[4+\frac{n(n+1)}{2}\right]$$

S_n = Total number of beads in the n necklaces

$$S_n = \left(\underbrace{4+4+\cdots+4}_{n \text{ times}}\right)+1+3+6+\cdots+\frac{n(n+1)}{2}$$

$$= 4n + \text{Sum of the first } n \text{ triangular numbers}$$

$$= 4n+\frac{1}{2}\sum(n^2+n)$$

$$= 4n+\frac{1}{2}\left(\sum n^2+\sum n\right)$$

$$= 4n+\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}\right]+\frac{1}{2}\frac{n(n+1)}{2}$$

$$= 4n+\frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4}$$

$$= \frac{1}{12}[48n+2n(n+1)(n+2)]$$

$$= \frac{n}{6}[n^2+3n+26].$$

Problem 9 *Show that for* $f(m)=\frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{2m+1}+\left(3-2\sqrt{2}\right)^{2m+1}-6\right]$ *both* $f(m) + 1$ *and* $2f(m) + 1$ *are perfect squares for all* $m \in N$ *by showing that* $f(m)$ *is an integer.*

Solution: First let us show that the expression is an integer.

$$f(m)=\frac{1}{8}\left((3+2\sqrt{2})^{2m+1}+(3-2\sqrt{2})^{2m+1}-6\right)$$

$$=\frac{1}{8}\times 2\left[\binom{2m+1}{0}3^{2m+1}+\binom{2m+1}{2}3^{2m-1}\cdot(2\sqrt{2})^2+\binom{2m+1}{4}3^{2m-3}\cdot(2\sqrt{2})^4\right.$$
$$\left.+\cdots+\binom{2m+1}{2m}3\cdot(2\sqrt{2})^{2m}-3\right]$$

All terms in the above expression except $3^{2m+1} - 3$ are multiples of 4, as the even powers of $2\sqrt{2}$ is a multiple of 4. Now $3^{2m+1} - 3 = 3(9^m - 1)$ and $9 \equiv 1 \pmod 4 \Rightarrow 9^m \equiv 1 \pmod 4 \Rightarrow 4 \mid (9^m - 1) \Rightarrow f(m)$ is an integer.

Now,

$$f(m)+1 = \frac{1}{8}\times\left[(3+2\sqrt{2})^{2m+1} + (3-2\sqrt{2})^{2m+1} - 6\right]+1$$

$$= \frac{1}{8}\times[\{(1+\sqrt{2})^2\}^{2m+1} + \{(1-\sqrt{2})^{2m+1}\}^2 - 6 + 8]$$

$$(\text{as } 3 \pm 2\sqrt{2} = (1\pm\sqrt{2})^2)$$

$$= \frac{1}{8}\times[\{(1+\sqrt{2})^{2m+1}\}^2 + \{(1-\sqrt{2})^{2m+1}\}^2 + 2]$$

$$= \frac{1}{8}\times[\{(1+\sqrt{2})^{2m+1}\}^2 + \{(1-\sqrt{2})^{2m+1}\}^2 - 2(-1)]$$

$$= \frac{1}{8}\left[\left\{\left(1+\sqrt{2}\right)^{2m+1}\right\}^2 + \left\{\left(1-\sqrt{2}\right)^{2m+1}\right\}^2 - 2\times(1+\sqrt{2})^{2m+1}(1-\sqrt{2})^{2m+1}\right]$$

Since,

$$(1+\sqrt{2})^{2m+1}\,(1-\sqrt{2})^{2m+1} = [(1+\sqrt{2})(1-\sqrt{2})]^{2m+1} = (-1)^{2m+1} = -1.$$

So, the given expression is equal to

$$\left\{\frac{(1+\sqrt{2})^{2m+1} - (1-\sqrt{2})^{2m+1}}{2\sqrt{2}}\right\}^2$$ which is a perfect square of an integer.

Note that $\dfrac{(1+\sqrt{2})^{2m+1} - (1-\sqrt{2})^{2m+1}}{2\sqrt{2}}$ is an integer, as all the left over terms contain $2\sqrt{2}$ as a factor in the numerator.

Now,

$$2f(m)+1 = \frac{1}{4}\times[(3+2\sqrt{2})^{2m+1} + (3-2\sqrt{2})^{2m+1} - 6]+1$$

$$= \frac{1}{4}\times[(3+2\sqrt{2})^{2m+1} + (3-2\sqrt{2})^{2m+1} - 2]$$

Since $f(m)$ is shown to be an integer, so $2f(m) + 1$ is also an integer. Now, $2f(m) + 1$ can be written as

$$\frac{1}{4}\times[\{(1+\sqrt{2})^{2m+1}\}^2 + \{(1-\sqrt{2})^{2m+1}\}^2 - 2]$$

$$= \frac{1}{4}\times\left[\{(1+\sqrt{2})^{2m+1}\}^2 + \{(1-\sqrt{2})^{2m+1}\}^2 + 2\times\{(1+\sqrt{2})^{2m+1}(1-\sqrt{2})\}^{2m+1}\right]$$

$$= \left\{\frac{(1+\sqrt{2})^{2m+1} + (1-\sqrt{2})^{2m+1}}{2}\right\}^2$$

which is a perfect square of an integer.

By a similar reasoning, the expression

$$\frac{(1+\sqrt{2})^{2m+1} + (1-\sqrt{2})^{2m+1}}{2}$$

is an integer. Hence, the result.

Problem 10 *Show that* $\frac{1}{32}[(17+12\sqrt{2})^n+(17-12\sqrt{2})^n-2]$ *a perfect square of the form* $\frac{m(m+1)}{2}$, *where* $m \in N$, *(i.e., the expression is a triangular integer which also a square integer.)*

Solution:

As $17+12\sqrt{2}=(3+2\sqrt{2})^2, 17-12\sqrt{2}=(3-2\sqrt{2})^2$

and $\quad (3+2\sqrt{2})(3-2\sqrt{2})=1$

So, given expression becomes

$$\frac{1}{32}\times[(3+2\sqrt{2})^{2n}+(3-2\sqrt{2})^{2n}-2\times(3+2\sqrt{2})\ (3-2\sqrt{2}]$$

$$=\left[\frac{(3+2\sqrt{2})^n-(3-2\sqrt{2})^n}{4\sqrt{2}}\right]^2$$

$$=\left[\frac{\{(1+\sqrt{2})^n\}^2-\{(1-\sqrt{2})^n\}^2}{4\sqrt{2}}\right] \quad [\because 3\pm2\sqrt{2}=(1\pm\sqrt{2})^2]$$

$$=\left[\left\{\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}\right\}\left\{\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}\right\}\right]^2 \quad (1)$$

Which is clearly a square number.

In the expansion of

$$\frac{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n}{2}=\frac{2}{2}\left[\binom{n}{0}+\binom{n}{2}\left(\sqrt{2}\right)^2+\binom{n}{4}(\sqrt{2})^4+\cdots\right]$$ is clearly an integer.

Similarly $\dfrac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}=\dfrac{2}{2\sqrt{2}}\left[\binom{n}{1}\sqrt{2}+\binom{n}{3}(\sqrt{2})^3+\binom{n}{5}(\sqrt{2})^5+\cdots\right]$ is an integer also as $\sqrt{2}$ will get cancelled.

Now we will show the Eq. (1) can be written as $\frac{1}{2}m(m+1)$. Consider

$$\frac{1}{32}\times[(17+12\sqrt{2})^n+(17-12\sqrt{2})^n-2]$$

$$=\left\{\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}\right\}^2\times\left\{\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}\right\}^2$$

$$=\frac{1}{2}\left[\frac{\left\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\right\}^2}{4}\times\frac{\left\{(1+\sqrt{2})^n+(1-\sqrt{2})^n\right\}^2}{4}\right]$$

For all n, we shall show that

$$\frac{\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\}^2}{4}, \frac{\{(1+\sqrt{2})^n+(1-\sqrt{2})^n\}^2}{4}$$

are consecutive integers.

Now,

$$\frac{((1+\sqrt{2})^n+(1-\sqrt{2})^n)^2}{4}=\frac{(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}+2(-1)^n}{4} \quad (1)$$

$$=\frac{(3+2\sqrt{2})^n+(3-2\sqrt{2})^n+2(-1)^n}{4} \quad (2)$$

and similarly,

$$\frac{((1+\sqrt{2})^n-(1-\sqrt{2})^n)^2}{4}=\frac{(3+2\sqrt{2})^n+(3-2\sqrt{2})^n-2(-1)^n}{4} \quad (3)$$

$$(3+2\sqrt{2})^n+(3-2\sqrt{2})^n=2\left[\tbinom{n}{0}3^n+\tbinom{n}{2}3^{n-2}(2\sqrt{2})^2+\cdots\right]$$

$=$ even integer $= 2k$ (say)

From Eqs. (2) and (3), we find that one of them $\frac{2k-2}{4}$ and other $\frac{2k+2}{4}$ or $\frac{1}{2}(k-1)$ and $\frac{1}{2}(k+1)$ and both are integers also, they differ by $\frac{1}{2}(k+1)-\frac{1}{2}(k-1)=1$.

Note that $\frac{1}{32}\times[(17+12\sqrt{2})^n+(17-12\sqrt{2})^n-2]$ gives you an infinite family of square and triangular numbers.

Problem 11 *Show that* $n=\frac{1}{8}\times[(17+12\sqrt{2})^m+(17-12\sqrt{2})^m+6]$ *is an integer for all* $m\in N$ *and hence, show that both* $(n-1)$ *and* $(2n-1)$ *are perfect squares for all* $m\in N$.

Solution: The terms containing $\sqrt{2}$ vanishes in the expansion of $(17+12\sqrt{2})^m+(17-12\sqrt{2})^m$ and integral terms are all multiples of 8 and hence, n is an integer.

$$n-1=\frac{1}{8}\times[(17+12\sqrt{2})^m+(17-12\sqrt{2})^m+6-8]$$

$$=\frac{1}{8}\times[(17+12\sqrt{2})^m+(17-12\sqrt{2})^m-2]$$

As $\quad 17\pm12\sqrt{2}=(3\pm2\sqrt{2})^2,$

again both $(17+12\sqrt{2})(17-12\sqrt{2})$ and $(3+2\sqrt{2})\times(3-2\sqrt{2})$ are equal to 1.

So,

$$\frac{1}{8}\times[(17+12\sqrt{2})^m+(17-12\sqrt{2})^m-2]$$

$$= \frac{1}{8} \times \left[\{(3+2\sqrt{2})^m\}^2 + \{(3-2\sqrt{2})^m\}^2 - 2 \times \{(3+2\sqrt{2})^m (3-2\sqrt{2})^m\} \right]$$

$$= \left[\frac{(3+2\sqrt{2})^m - (3-2\sqrt{2})^m}{2\sqrt{2}} \right]^2$$

and $$2n - 1 = \frac{1}{4} \times [(17+12\sqrt{2})^m + (17-12\sqrt{2})^m + 6 - 4]$$

$$= \frac{1}{4} \times [(17+12\sqrt{2})^m + (17-12\sqrt{2})^m + 2]$$

$$= \left[\frac{(3+2\sqrt{2})^m + (3-2\sqrt{2})^m}{2} \right]^2$$

and hence, the result.

Problem 12 $S = 1! + 2! + 3! + 4! + \cdots + 1997!$. *Find the unit digit and tens digit of S.*

Solution: From 5!, all the numbers will have the unit digit zero and from 10!, all the unit and tens digit will be zero.

So, the unit digit of the number S is the unit digit of

$$1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33.$$

That is unit digit of S is 3.

The tens digit of S, is the tens digit of

$$\begin{aligned} &1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! \\ &= 33 + 120 + 720 + 5040 + 40320 + 362880. \end{aligned}$$

So to get the tens digit of S, add only the tens digit of $33 + 120 + \cdots + 362880$ which is $3 + 2 + 2 + 4 + 2 + 8 = 21$

So, the tens digit of S is 1.

Problem 13 *Show that the square of an integer cannot be in the form* $4n + 3$ *or* $4n + 2$ where $n \in \mathbb{N}$.

Solution: Let us take the square of an even integer, say, $2a$.

$$m = 2a$$

$$\Rightarrow \quad m^2 = 2a \times 2a = 4a^2$$

and $4a^2$ is not in the form of $4n + 3$ or $4n + 2$.

If m is an odd number, then $m = 2a + 1$

and $$m^2 = (2a + 1)^2 = 4a^2 + 4a + 1$$

$$= 4a(a + 1) + 1 = 4n + 1.$$

Here again the square is not in the form of $4n + 3$ or $4n + 2$.

In other words, any number in the form of $4n + 3$ or $4n + 2$ cannot be a square number.

Note: When m is odd, $m^2 = 4a(a + 1) + 1$.

As either a or $a + 1$ is even, $m^2 = 8k + 1$ for some $k \in \mathbb{N}$.

$\therefore$ The square of an odd number is in the form $8k + 1$.

Problem 14 *Show that no square number can end with 4 ones or 4 nines.*

Solution: Let n ends with 4 ones

i.e., $$n = 10000k + 1111$$
$$= 8l + 7 \quad (\text{as } 1111 = 8 \times 138 + 7)$$

Similarly, let n ends with 4 nines,

i.e., $$n = 10000k + 9999 = 8m + 7$$

In both the cases, n cannot be a square number, because the square of an odd number is in the form of $8k + 1$.

Note: A perfect square number can have only 0, 1, 4, 5, 6, 9 in its units place.

Similarly, the last two digits of a perfect square number are 00, 01, 21, 41, 61, 81, 04, 24, 44, 64, 84, 25, 16, 36, 56, 76, 96, 09, 29, 49, 69, 89.

Observe that if last digit is '6' then second last digit can be any odd digit out of 1, 3, 5, 7, 9, if last digit is a perfect square, *i.e.*, 1, 4, or 9 then second last digit can be any even digit out of 0, 2, 4, 6, 8, if last digit 0 then second last digit will be '0' if last digit 5 then second last digit will be '2'.

Problem 15 *A four-digit number has the following properties:*

(a) *It is a perfect square*
(b) *The first two digits are equal*
(c) *The last two digits are equal*

Find the number. **[RMO, 1991]**

Solution: Let $N = aabb$ be the representation of such a number.

$$1 \le a \le 9, \quad 0 \le b \le 9.$$

Then $N = 1000a + 100a + 10b + b = 1100a + 11b$
$= 11\,(100a + b)$

Since N is a perfect square and 11 is a factor of N,

$11^2|N \Rightarrow 11|(100a + b) \Rightarrow 11|(a + b) \Rightarrow a + b = 11k$

But $1 \le a + b \le 18 \Rightarrow a + b = 11 \Rightarrow b = 11 - a \Rightarrow b > 0$

The last two non-zero digits of a perfect square where both the digits are equal is only 44. So, $b = 4$

$\therefore$ $$a = 7$$

$\therefore$ $N = 7744$ is the only possibility.

$$N = 11 \times 704 = 11 \times 11 \times 64 = 88^2.$$

$\therefore$ This is the only solution.

Problem 16 *Prove that the product of four consecutive positive integers increased by 1 is a perfect square.*

Solution: Let the consecutive positive integers be n, $n + 1$, $n + 2$ and $n + 3$.

Consider the expression

$$N = n\,(n + 1)(n + 2)(n + 3) + 1$$
$$= (n^2 + 3n)(n^2 + 3n + 2) + 1$$

$$= (n^2 + 3n)^2 + 2(n^2 + 3n) + 1$$
$$= [(n^2 + 3n) + 1]^2 = (n^2 + 3n + 1)^2$$

and hence, the result.

Problem 17 *Three consecutive positive integers raised to the first, second and third powers, respectively, when added, make a perfect square, the square root of which is equal to the sum of the three consecutive integers. Find these integers.*

Solution: Let $(n - 1)$, n, $(n + 1)$ be the three positive consecutive integers $(n > 1)$.

Then $(n-1)^1 + n^2 + (n+1)^3 = (n - 1 + n + n + 1)^2 = (3n)^2 = 9n^2$

$\Rightarrow$ $n - 1 + n^2 + n^3 + 3n^2 + 3n + 1 = 9n^2$

$\Rightarrow$ $n^3 - 5n^2 + 4n = 0$

$\Rightarrow$ $n(n - 1)(n - 4) = 0$

$\Rightarrow$ $n = 0$ or $n = 1$ or $n = 4$,

As $n > 1$, $n = 4$, corresponding to which the consecutive integers are 3, 4 and 5.

Problem 18 *Prove that the product of* 8 *consecutive natural numbers is never a perfect* 4*th power of an integer.*

Solution: Let, x be the least of the 8 consecutive natural numbers. Let, their product be P.

Then, $P = x(x + 1)(x + 2)(x + 3)(x + 4)(x + 5)(x + 6)(x + 7)$

$$= x(x + 7)(x + 1)(x + 6)(x + 2)(x + 5)(x + 3)(x + 4)$$
$$= (x^2 + 7x)(x^2 + 7x + 6)(x^2 + 7x + 10)(x^2 + 7x + 12)$$

Let, $x^2 + 7x + 6$ be a.

Then, $P = (a - 6)\, a(a + 4)(a + 6) = (a^2 - 36)(a^2 + 4a)$

$$= a^4 + 4a^3 - 36a^2 - 144a$$
$$= a^4 + 4a(a^2 - 9a - 36) = a^4 + 4a\,(a + 3)(a - 12)$$

Now, $a = x^2 + 7x + 6$ and $x \geq 1 \Rightarrow a \geq 14. \Rightarrow a - 12 > 0$

and hence, $P = a^4 + 4a(a + 3)(a - 12) > a^4$.

Again,

$(a + 1)^4 = a^4 + 4a^3 + 6a^2 + 4a + 1 > a^4 + 4a^3 - 36a^2 - 144a$.

Thus, $a^4 < P < (a + 1)^4$ and so, P lies between 4th power of consecutive integers and hence, cannot be a perfect 4th power.

Problem 19 *Show that a positive integer n good if there are n integers, positive or negative and not necessarily distinct, such that their sum and product both equal to n.*

Example 8 *is good as*

$$8 = 4 \times 2 \times 1 \cdot 1 \cdot 1 \cdot 1(-1) \cdot (-1)$$
$$= 4 + 2 + 1 + 1 + 1 + 1 + (-1) + (-1) = 8.$$

Show that the integers of the form $(4k + 1)$ *where* $k \geq 0$ *and* $4l (l \geq 2)$ *are good.*

Solution:

Case 1: $n = 4k + 1$

$$n = 4k + 1 = (4k + 1) \times (1)^{2k} \times (-1)^{2k}$$
$$= (4k + 1) + \underbrace{(1 + 1 + \cdots + 1)}_{2k \text{ times}} + \underbrace{[(-1) + (-1) + \cdots + (-1)]}_{2k \text{ times}}$$

Case 2: $n = 4l$, in this place there are two cases where (a) l is even with $l \geq 2$ and (b) l is odd with $l \geq 3$

(a) $n = 4l$, l is even.

Consider, integers w and v, such that

$$n = 4l = 2l \times 2 \times (1)^w \times (-1)^v$$
$$= 2l + 2 + \underbrace{(1+1+\cdots+1)}_{w\text{ times}} + \underbrace{[(-1)+(-1)+\cdots+(-1)]}_{v\text{ times}}$$

Now, by the definition of good integer, we have $2 + w + v = 4l$(there are $2 + w + v$ factors).

$\Rightarrow$ $$w + v = 4l - 2 \qquad (1)$$

Again, since $4l = 2l + 2 + w - v$, we get

$$w - v = 2l - 2 \qquad (2)$$

Solving Eqs. (1) and (2), we get $w = 3l - 2$ and $v = l$.

(b) l is odd. With $l \geq 3$.

Choose w and v, such that

$$n = 4l = (2l) \times (-2) \times (1)^w (-1)^v$$
$$= 2l + (-2) + \underbrace{(1+1+\cdots+1)}_{w\text{ times}} + \underbrace{[(-1)+(-1)+\cdots+(-1)]}_{v\text{ times}}$$

Again, since there are $w + v + 2$ factors, we have

$$w + v + 2 = 4l \quad \text{or} \quad w + v = 4l - 2$$

and $4l = 2l - 2 + w - v$ (by definition of good integer)

$\Rightarrow$ $$w - v = 2l + 2$$

Solving $w = 3l$ and $v = l - 2$

Since, l is odd and $l \geq 3$

$$l - 2 \geq 1$$

Now, $n = 4l = 2l \times (-2) \times (1)^{3l} \times (-1)^{l-2}$

$$= 2l + (-2) + \underbrace{(1+1+\cdots+1)}_{3l\text{ times}} + \underbrace{[(-1)+(-1)+\cdots+(-1)]}_{(l-2)\text{ times}}$$
$$= 2l - 2 + 3l - (1 - 2) = 4l.$$

Check Your Understanding

1. Show that the number of divisors of an integer is odd if and only if this integer is a square.
2. Represent in all possible ways (a) 1547 and (b) 1768 as difference of two squares.
3. Prove if a three digit integer n is relatively prime to 10 then 101th power of n ends with the same three digits of n.
4. Find natural numbers x, y such that $\sqrt{x} + y = 7$ and $x + \sqrt{y} = 11$.
5. Prove that $a^3 - b^3 = 2011$ has no integer solutions.
6. Prove that if integer a is not divisible by 2 or 3 then $a^2 - 1$ is divisible by 24.
7. Show that for any natural number n, $n^2 + 2n + 12$ and $n^2 + 3n + 5$, both are not divisible by 121.
8. Show that for any natural number n, $n^2 - 3n - 19$ is not divisible by 289. **[RMO, 2009]**

9. Prove that there is one and only one natural number n such that $2^8 + 2^{11} + 2^n$ is a perfect square.

10. What is the largest n for which $4^{27} + 4^{1000} + 4^n$ is a perfect square?

11. Prove that the equation $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=1$ has no solution in positive integers x, y, z.

12. Find all pairs (x, y) of integers such that $x^3 = y^3 + 2y^2 + 1$. **[Bulgarian MO, 1999]**

13. Let m be a 2002 digit number each digit of which is 6. What is the remainder obtained when m is divided by 2002?

14. Show that $\left\lfloor\left(2+\sqrt{3}\right)^n\right\rfloor$ is odd for every positive integer n.

Note: For any real number x, $\lfloor x\rfloor$ denotes the largest integer less than or equal to x.

15. Show that there exists no integer n, such that the sum of the digits of n^2 is 2000.

16. Find the number of perfect square divisors of the number 12!

17. Show that every integer in the infinite sequence 49, 4489, 444889, 44448889, … is a square.

18. Find the number of 2 digit natural numbers, which, when increased by 11, has the order of digits reversed.

19. Prove that $\sqrt{3}\sqrt[3]{3}$ is irrational. (Do not assume $\sqrt{3}$ as irrational to prove this.)

20. Show that, there is no three digit number abc $(a \neq c)$ such that, abc-cba is a perfect square.

21. N is a natural number, such that it is the product of three distinct prime numbers. Find all such prime numbers, so that, the sum of all its composite divisors is equal to $2N + 1$.

22. Prove that there exist arbitrarily long sequence of consecutive positive integers, none of which is a power of an integer with an integer exponent greater than 1.

23. Given m and n as relatively prime positive integers greater than one, show that $\frac{\log_{10} m}{\log_{10} n}$ is not a rational number.

24. The nonzero real numbers (a, b) satisfy the equation $a^2b^2(a^2b^2 + 4) = 2(a^6 + b^6)$; Prove a, b cannot both be rational under this condition.

25. Show that, in the year 1996, no one could claim on his birthday, that his age was the sum of the digits of the year, in which, he was born. Find also the last year, prior to 1996, which had this property.

26. If $a^2 + b^2 + c^2 = D$ where a, b are consecutive positive integers and $c = ab$, show that $\sqrt{D}$ is always an odd integer.

27. Sequences A and B, both contain the same number 95. Find the next number in the sequence A which is also in B.

A: 19, 95, 171, 247, …

B: 20, 45, 70, 95, …

28. A sequence is generated, starting with the first term t_1, as a 4 digit natural number. The second, third and the fourth terms are obtained by squaring the sum of the digits of the previous terms; for example, if $t_1 = 9999$, $t_2 = 36^2 = 1296$, $t_3 = 18^2 = 324$, $t_4 = 9^2 = 81$, and so on. Start with 2012, *i.e.*, let $t_1 = 2012$. Form the sequence and find the sum of the first 2013 terms.

29. A sports meet was organized for 4 days. If on each day, half of the existing medals and one more medal was awarded, find the number of medals awarded for each day.

30. There are two natural numbers, whose product is 192. It is given that the quotient of the AM to the HM of their greatest common measure and least common multiple is $\frac{169}{48}$; Find these numbers.

31. Find all integers a, b, c, d satisfying the condition
(i) $1 \le a \le b \le c \le d$
(ii) $ab + cd = a + b + c + d + 3$ **[RMO, 2002]**

32. Does there exist a positive integer whose prime factors include at most the primes 2, 3, 5 and 7 and which ends in the digits 11? If so, find the smallest such positive integer; if not, show why none exists.

33. Show that if n is a positive integer such that $2n + 1$ and $3n + 1$ are both squares then n is a multiple of 40.

34. The digital sum $D(n)$ of a positive integer n, expressed in base ten, is defined recursively as follows:
$D(n) = n$ if $1 < n < 9$
$D(n) = D(a_0 + a_1 + a_2 + \cdots + a_m)$ if $n > 9$ (where $a_0, a_1, a_2, \ldots, a_m$ are all digits of n in the scale of 10, *i.e.*, $n = a_m 10^m + a_{m-1} 10^{m-1} + \cdots + a_1 10 + a_0$.
For example, $D(989) = D(26) = D(8) = 8$
(i) Check whether $D((1234)_5) = D(5) = 5$.
(iii) Hence prove the result: $D((123)_5 \times (34)_5) = D(D((123)_5 \times D((34)_5))$.

35. Show that the cube roots of three distinct prime numbers cannot be the three terms (not necessarily consecutive) of an arithmetic progression. **[USA MO, 1973]**

36. Find the number of 4-digit numbers in base 10, having non-zero digits and which are divisible by 4 but not by 8. **[RMO, 2010]**

37. Let $E(m)$ denote the number of even digits in m. For example, $E(2) = 1$; $E(19) = 0$; $E(5672) = 2$, etc. Prove the following result:
$E(E(101) \times E(201) \times E\,(301) \times \cdots \times E(2001)) = 1$.

38. In 1930, a correspondent proposed the following question: 'A man's age at death, was (1/29) of the year of his birth'. How old was he in 1900?

39. Find the number of triples (x, y, z) such that, when any of these numbers is added to the product of the other two, then, the result is 2.

40. Find all pairs of positive integers (a, b) with $a > b$, such that, the sum of their sum, difference, product and quotient is 36.

41. Let a, b, c, d, e be consecutive positive integers, such that, $(b + c + d)$ is a perfect square and $(a + b + c + d + e)$ is a perfect cube. Find the smallest value of c.

42. Determine whether integers x, y exist such that, $(x + y)$ and $(x^2 + y^2)$ are consecutive integers.

43. Find the number of all integer-sided isosceles obtuse-angled triangles with perimeter 2008. **[RMO, 2008]**

44. If $n_1, n_2, n_3, \ldots, n_p$ are 'p' positive integers, whose sum is an even number, prove that the number of odd integers, among them, cannot be odd.

45. Show that there do not exist any distinct natural numbers a, b, c, d such that $a^3 + b^3 = c^3 + d^3$ and $a + b = c + d$.

46. Prove that if the coefficients of the quadratic equation $ax^2 + bx + c = 0$ are odd integers, then the roots of the equation cannot be rational numbers.

47. Prove that $x^2y^2 = x^2 + y^2$ has no integral solution except $x = y = 0$.

48. Prove that the sequence $\sqrt{24n+1}$ with $n \in \mathbb{N}$ contains all prime except 2 and 3.

49. Let A denote a subset of the set $\{1, 11, 21, 31, \ldots, 541, 551\}$, having the property that no two elements of A add up to 552. Prove that A cannot have more than 28 elements.

50. Prove that the ten's digit of any power of 3 is even. **[RMO, 1993]**

51. Consider the equation in positive integers $x^2 + y^2 = 2000$ with $x < y$.

(i) Prove that $31 < y < 45$

(ii) Rule out the possibility that, one of x, y even and the other is odd.

(iii) Rule out the possibility that, both x, y are odd.

(iv) Prove that, y is a multiple of 4.

(v) Obtain all solutions to this problem.

52. N is a 50-digit number (in the decimal notation). All the digits except the 26th digit (from the left) are 1. If N is divisible by 13, find the 26th digit. **[RMO, 1990]**

53. Show that the equation $x^2 + 3 = 4y(y + 1)$ has no integral solution.

54. Show that there exists no positive integers m and n such that both $m^2 + n^2$ and $m^2 - n^2$ are perfect squares.

55. Find three consecutive integers each divisible by a perfect square greater than 1.

56. Find three consecutive numbers, the first of which is divisible by a square, the second by a cube and the third by a fourth power.

57. Solve the equation $y^3 = x^3 + 8x^2 - 6x + 8$ for positive integers x and y. **[RMO, 2000]**

58. Suppose N is an n-digit positive integer such that (a) all the n-digits are distinct and (b) the sum of any three consecutive digits is divisible by 5. Prove that n is at most 6. Further, show that starting with any digit one can find a six-digit number with these properties. **[RMO, 1996]**

59. (i) Consider two positive integers a and b which are such that $a^a b^b$ is divisible by 2000. What is the least possible value of the product ab.

(ii) Consider two positive integers a and b which are such that $a^b b^a$ is divisible by 2000. What is the least possible value of the product ab. **[RMO, 2000]**

60. Prove that if $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$ is expressed as a fraction, where $p \geq 5$ is a prime, then p^2 divides the numerator.

Challenge Your Understanding

1. Let a and b be two positive rational numbers, such that, $\sqrt[3]{a} + \sqrt[3]{b}$ is rational. Prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ themselves are rational. **[INMO, 1998]**

2. We call an integer 'FORTUNATE' if it can be expressed in the form $n = 54x^2 + 37y^2$ for some integers x and y. Prove that, if 'n' is 'fortunate', then, $1999n$ is also 'fortunate'.

3. We define 'Funny Numbers' as follows

(i) Every single digit prime is 'Funny'.

(ii) A prime number with two or more digits is 'Funny' if the numbers obtained by deleting either its leading digit or its unit digit are both 'Funny'. Discover all 'Funny Numbers' in the set $\mathbb{N}$.

4. A natural number n is said to be a 'superstar' if the number is less than 10 times the product of its digits.
(i) Examine if 10 and 200 are 'superstar' numbers.
(ii) Find the number of 'superstar' numbers between 10 and 200.

5. Show that $m^5 + 3m^4n - 5m^3n^2 - 15m^2n^3 + 4mn^4 + 12n^5 - 33 = 0$ has no solution in integers m, n.

6. Find the least natural number whose last digit is 7 such that it becomes 5 times larger when this last digit is carried to the beginning of the number.

7. Consider the set A of numbers $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2013}\right\}$ we delete two of them, say 'a' and 'b' and in their place, we put only one number $(a + b + ab)$. After performing the operation 2012 times, what is the number that is left?

8. Prove that for every natural number $m \geq 2$ there exists m distinct natural numbers $n_1, n_2, \ldots, n_m$ such that $\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_m} = \frac{1008}{2012}$.

9. An integer n will be called 'good' if we can write $n = a_1 + a_2 + \ldots + a_k$ where $a_1\, a_2, \ldots,$ a_k are positive integers (not necessarily distinct) satisfying $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1$.

Given the information that the integers 33 through 73 are good, prove that every integer greater than or equal to 33 is good.

[USA MO, 1978]

10. Three nonzero real numbers a, b, c are said to be in harmonic progression if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$. Find all three-term harmonic progression a, b, c of strictly increasing positive integers in which $a = 20$ and b divides c. **[RMO, 2008]**

11. Prove that for every positive integer n there exists a positive integer x such that each of the terms of the infinite sequence $x+1, x^x+1, x^{x^x}+1, \ldots$ is divisible by n.

12. Determine the 3-digit numbers, which are equal to eleven times the sum of the squares of their digits. **[IMO, 1960]**

13. 7-digit numbers are formed by the digits 1, 2, 3, 4, 5, 6, 7. In each number, no digit is repeated. Prove that among all these numbers, there is no number, which is a multiple of another number.

14. Prove that among any 39 consecutive natural numbers it is always possible to find one whose sum of digits is divisible by 11.

15. Find one pair of positive integers a, b such that,
(i) $ab(a + b)$ is not divisible by 7.
(ii) $(a + b)^7 - a^7 - b^7$ is divisible by 7^7. **[IMO, 1984]**

16. Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the numbers on all the faces, find all the possible values of T.

[RMO, 2004]

17. Find all natural numbers n, such that, $n + s(n) + s(s(n)) = 2010$, where $s(n)$ = sum of the digits of n. (Example $n = 238$; $s(n) = 13$; $s(s(n) = 4$.)

18. Find the smallest n, such that, any sequence $a_1, a_2, a_3, \ldots, a_n$ whose values are relatively prime square-free integers between 2 and 1995, must contain a prime.

19. What is the smallest perfect square that ends with 9009?

20. Let $S_n = \{1, n, n^2, n^3, \dots\}$ where n is an integer greater than 1. Find the smallest number $k = k(n)$ such that there is a number which may be expressed as a sum of k (possibly repeated) elements of S_n in more than one way. Rearrangements are considered the same.

21. Find all positive integers n, such that $n^2 + 3^n$ is perfect square.

22. Prove that there is an infinite number of non-congruent triangles T such that
(i) the lengths of the sides of T are consecutive integers and
(ii) the area of T is an integer.

23. Prove that the area of a right triangle with integral sides can never be a perfect square.

24. Prove that every even integer can be written in the form $(x + y)^2 + 3x + y$ with x, y non-negative integers.

25. Find the positive integers n with exactly 12 divisors $1 = d_1 < d_2 < d_3 < \cdots < d_{12} = n$ such that the divisor with index $d_4 - 1$ (that is d_{d_4-1}) is $(d_1 + d_2 + d_4)\, d_8$.
[Russian MO, 1989]

26. The geometric mean of any set of m non-negative numbers is the mth root of their product.
(i) For which positive integers n is there a finite set S_n of n distinct positive integers such that the geometric mean of any subset of S_n is an integer?
(ii) Is there an infinite set S of distinct positive integers such that the geometric mean of any finite subset of S is an integer?
[USA MO, 1984]

27. What is the smallest integer n, greater than 1, for which the root mean square of the first n positive integers is an integer?
[USA MO, 1986]

28. Let α and β be the roots of the quadratic equation $x^2 + mx - 1 = 0$, where m is an odd integer. Let $\lambda_n = \alpha^n + \beta^n$, and $n \geq 0$. Prove that for $n \geq 0$, (a) λ_n is an integer and (b) $\gcd(\lambda_n, \lambda_{n+1}) = 1$. **[RMO, 2004]**

29. Find the least natural number n such that, if the set $A_n = (1, 2, 3, \dots, n)$ is arbitrarily divided into two non-intersecting subsets, then one of the subsets contains 3 distinct numbers such that the product of two of them equals the third.
[IMO Shortlisted Problem, 1988]

30. For the Fibonacci sequence defined by $a_{n+1} = a_n + a_{n-1}$ $(n \geq 1)$, $a_0 = 0$, $a_1 = a_2 = 1$ find the greatest common divisors of 1960th and 1988th terms of the Fibonacci sequences. **[IMO Shortlisted Problem, 1988]**

31. (i) Given any positive integer n, show that there exist distinct positive integers x and y such that $x + i$ divides $y + j$ for $j = 1,2,3, \dots, n$.
(ii) If for some positive integers x and y, $x + j$ divides $y + j$ for all positive integers j, then $x = y$. **[INMO, 1996]**

32. Determine the set of all positive integers n for which 3^{n+1} divides $2^{3^n} + 1$. Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n. **[INMO, 1991]**

33. In any set of 181 square integers, prove that one can always find a subset of 19 numbers, sum of whose elements is divisible by 19. **[INMO, 1994]**

34. Let $(a_1, a_2, \dots, a_{2011})$ be a permutation (that is a rearrangement) of the numbers 1, 2, …, 2011. Show that there exists two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$. **[RMO, 2011]**

35. Suppose the integers 1, 2, 3, …, 10 are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that $a_1 < a_2 < a_3 < a_4 < a_5$ and $b_1 > b_2 > b_3 > b_4 > b_5$.

(i) Show that the larger number in any pair $\{a_j, b_j\}$, $1 \le j \le 5$, is at least 6.

(ii) Show that $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$ for every such position. **[RMO, 2002]**

36. A natural number n is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting k from n and the larger is obtained by adding l to n. Prove that $n - kl$ is a perfect square. **[RMO, 2011]**

37. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer. **[RMO, 2010]**

38. In a group of ten persons, each persons is asked to write a sum of the ages (in integers) of all the other 9 persons. If all the ten sums form the 9 element set $\{82, 83, 84, 85, 87, 89, 90, 91, 92\}$ find the individual ages of the persons. **[RMO, 1993]**

39. Let A be a set of 16 positive integers with the property that product of any 2 distinct members of A does not exceed 1994. Show that there are numbers a and b in A such that $\gcd(a, b) > 1$. **[RMO, 1994]**

40. Prove that there exists infinite sequences $\langle a_n \rangle_{n\ge1}$ and $\langle b_n \rangle_{n\ge1}$ of positive integers such that following conditions hold simultaneously,

(i) $1 < a_1 < a_2 < a_3 < \ldots$;

(ii) $a_n < b_n < a_n^2$, for all $n \ge 1$;

(iii) $a_n - 1$ divides $b_n - 1$, for all $n \ge 1$;

(iv) $a_n^2 - 1$ divides $b_n^2 - 1$, for all $n \ge 1$. **[RMO, 2008]**

41. Let a, b, c be three natural numbers such that $a < b < c$ and $\gcd(c - a, c - b) = 1$. Suppose there exists an integer d such that $a + d$. $b + d$, $c + d$ forms the sides of a right triangle. Show that there exists integers l, m such that $c + d = l^2 + m^2$. **[RMO, 2007]**

42. Prove that there are infinitely many positive integers n such that $n(n + 1)$ can be expressed as sum of squares of two positive integers in at least two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation). **[RMO, 2006]**

43. A 6×6 square is dissected into 9 rectangles by lines parallel to its sides such that all these rectangles have integer sides. Prove that there are always two congruent rectangles. **[RMO, 2006]**

44. Determine all triples (a, b, c) of positive integers such that $a \le b \le c$ and $a + b + c + ab + bc + ca = abc + 1$. **[RMO, 2005]**

45. Find all triples (a, b, c) of positive integers such that $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) = 3$. **[RMO, 1996]**

46. Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 positive odd integers, by a multiple of 2001. **[RMO, 2001]**

47. Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint non empty subsets A and B of X such that

(i) $A \cup B = X$;

(ii) prod (A) is divisible by prod (B), where for any finite set of numbers C, prod (C) denotes the product of all numbers in C;

(iii) the quotient prod (A)/prod (B) is as small as possible. **[RMO, 2003]**

48. Prove that the only solutions in positive integers of the equation $m^n = n^m$ are $m = n$ and $\{m, n\} = \{2, 4\}$.

49. 52 is the sum of two squares and 3 less than 52 is also a square. Prove that there exist infinitely such numbers, n such that n is the sum of two squares and $(n - 3)$ is also a square.

50. Find the number of quadratic polynomials, $ax^2 + bx + c$, which satisfy the following conditions:

(i) a, b, c are distinct,

(ii) $a, b, c \in \{1, 2, 3, \ldots, 1999\}$ and

(iii) $x + 1$ divides $ax^2 + bx + c$. **[RMO, 1999]**

51. Find all solutions in integers m, n of the equation $(m-n)^2 = \dfrac{4mn}{m+n-1}$.

[RMO, 1999]

52. If A is a fifty-element subset of the set $\{1, 2, 3, \ldots, 100\}$ such that no two numbers from A add up to 100, show that A contains a square. **[RMO, 1996]**

53. Given any positive integer n show that there are two positive rational numbers a and b, $a \neq b$, which are not integers and which are such that $a - b$, $a^2 - b^2$, $a^3 - b^3$, $\ldots$, $a^n - b^n$ are all integers. **[RMO, 1996]**

54. Find all natural number n for which every natural number, whose decimal representation has $(n - 1)$ digits 1 and one digit 7, is prime.

55. If $2+2\sqrt{28n^2+1}$ is an integer, prove that it must be a square.

56. Show that the equation $a^3 + 2b^3 + 4c^3 = 9d^3$ has no non-trivial integer solutions.

57. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of integers defined as follows:

$$x_0 = 1, x_1 = 1, x_{n+1} = x_n + 2x_{n-1}, n = 1, 2, 3, \ldots$$

$$y_0 = 1, y_1 = 7, y_{n+1} = 2y_n + 3y_{n-1}, n = 1, 2, 3, \ldots$$

Thus, the first few terms of the sequence are

x : 1, 1, 3, 5, 11, 21, …

y = 1, 7, 17, 55, 161, 487, …

Prove that except for the 1 there is no term which occurs in both the sequences.

[USA MO, 1973]

58. Let gcd(a, b) = 1.

(i) Show that the equation $ax + by = n$ has no solution in non-negative integers x and y if $n = ab - a - b$, but has a solution if $n > ab - a - b$.

(ii) Show that exactly one of the equations $ax + by = m$, $ax + by = n$ has a solution in non-negative integers x and y if $m + n = ab - a - b$.

(iii) Show that there are $\dfrac{1}{2}(a-1)(b-1)$ positive integers n not, expressible in the form $ax + by$ with $x, y \in \mathbb{N}_0$.

(iv) Show that the sum of such integers as in part (iii) is $\frac{1}{12}(a-1)(b-1)(2ab - a - b - 1)$.

59. Find all $x \in \mathbb{N}$ for which the product of the digits $d(x)$ of x, when x is writen in decimal notation equals $x^2 - 10x - 22$.

60. Prove that $y^2 = x^3 + 7$ has no integral solution.

Chapter 7

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\binom{9}{0} \quad \binom{9}{1} \quad \binom{9}{2} \quad \binom{9}{3} \quad \binom{9}{4} \quad \binom{9}{5} \quad \binom{9}{6} \quad \binom{9}{7} \quad \binom{9}{8} \quad \binom{9}{9}$$
$$\binom{10}{0} \quad \binom{10}{1} \quad \binom{10}{2} \quad \binom{10}{3} \quad \binom{10}{4} \quad \binom{10}{5} \quad \binom{10}{6} \quad \binom{10}{7} \quad \binom{10}{8} \quad \binom{10}{9} \quad \binom{10}{10}$$

Combinatorics

7.1 Definition of Factorial

The falling product of first n natural numbers is called the "n factorial" and is denoted by n! or $\lfloor n$.

That is, $n! = n(n-1)(n-2) \ldots 3 \times 2 \times 1$

For example, $4! = 4 \times 3 \times 2 \times 1 = 24$; $\quad 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$;

$$\frac{(2n)!}{n!} = \frac{1 \cdot 2 \cdot 3 \cdots (2n-1)(2n)}{n!} \quad \text{(by using the definition of factorials)}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \cdots (2n-1)\}\{2 \cdot 4 \cdot 6 \cdots 2n\}}{n!} = \frac{\{1 \cdot 3 \cdot 5 \cdots (2n-1)\}\, 2^n n!}{n!}$$

(By taking 2 out from all terms of the second factor in Numerator)

$$= \{1 \cdot 3 \cdot 5 \cdots (2n-1)\}\, 2^n$$

Factorials of proper fractions and of negative integers are not defined. Factorial n is defined only for whole numbers.

7.1.1 Properties of Factorial

(a) $0! = 1$ (by definition)

(b) $n! = 1 \times 2 \times \cdots \times (n-1) \times n = [1 \times 2 \times \cdots \times (n-1)]\, n = (n-1)!\, n$

Thus, $n! = n\,((n-1)!)$

(c) If two factorials, *i.e.*, $x!$ and $y!$ are equal, then

$$(x, y) = (0, 1) \text{ or } (1, 0) \text{ or } (k, k)\ \forall\, k \in \mathbb{N}_0$$

(d) $n!$ ends in 0, for all $n > 4$. (Number of 5's in $n!$, $n > 4$, is always less than the number of 2's. Therefore for every 5, there is a 2. Hence $n!$, $n > 4$, ends in 0).

Example 1 *If* $\dfrac{n!}{2!(n-2)!}$ *and* $\dfrac{n!}{4!(n-4)!}$ *are in the ratio* 2:1, *then find the value of n.*

Solution: $\dfrac{\dfrac{n!}{2!(n-2)!}}{\dfrac{n!}{4!(n-4)!}} = \dfrac{2}{1} \Rightarrow \dfrac{n! \times 4!(n-4)!}{2!(n-2)! \times n!} = \dfrac{2}{1} \Rightarrow \dfrac{4 \times 3}{(n-2) \times (n-1)} = \dfrac{2}{1}$

$$\Rightarrow (n-2)\,(n-3) = 6 \Rightarrow n^2 - 5n = 0 \Rightarrow n = 0, 5$$

But, for $n = 0$, $(n - 2)!$ and $(n - 4)!$ are not meaningful

So, $n = 5$.

7.2 Basic Counting Principles

7.2.1 Addition Principle

Let $A = \{ a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ be sets.

Let A and B be disjoint (or mutually exclusive) set, *i.e.*, $A \cap B = \phi$ (the empty set).

Then an element of A or an element of B can be chosen in $n + m$ ways.

It can be extended as

Let a set A_i have k_i elements and any two sets A_i's be disjoint, $i = 1, 2, \ldots, n$. Then any element of A_1 or A_2 or … or A_n can be chosen in $k_1 + k_2 + \cdots + k_n$ ways.

In set theoretic notation, the extended form is stated as:

If A_i, $i = 1, 2, \ldots, n,$ are n finite pair-wise disjoint (or mutually exclusive) sets, *i.e.*, $A_i \cap A_j = \phi$ for $i \neq j$; $i, j = 1, 2, \ldots n$; then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

That is, the cardinality of the union of finite number of pair-wise disjoint finite sets is the sum of the cardinalities of the individual sets.

Here $|A_i|$ is the number of elements of the set A_i. Other notations for number of elements of the set A_i are $n(A_i)$ *or* $\#(A_i)$, etc.

In other words:

If there are

n_1 ways for the event E_1 to occur

n_2 ways for the event E_2, to occur

…

…

…

n_k ways for the event E_k, to occur

where $k \geq 1$, and if these are pair-wise disjoint (or mutually exclusive), then the number of ways for at least one of the events $E_1, E_2, \ldots, E_k$ to occur is $n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i$.

Example 2 *There are 15 gates to enter a city from north and 10 gates to enter the city from east. In how many ways a person can enter the city?*

Solution: Number of ways to enter the city from north = 15.

Number of ways to enter the city from east = 10.

A person can enter the city from north or from east.

So, number of ways to enter the city = 15 + 10 = 25.

Example 3 *There are 15 students is a class in which 10 are boys and 5 are girls. The class teacher selects either a boy or a girl for monitor of the class. In how many ways the class teacher can make this selection?*

Solution: A boy can be selected for the post of monitor in 10 ways.

A girl can be selected for the post of monitor in 5 ways.

Number of ways in which either a boy or a girl can be selected = 10 + 5 = 15.

Example 4 *Find the number of two digit numbers (having different digits) which are divisible by 5.*

Solution: Any number of required type either ends in 5 or in 0. Number of two digit numbers (with different digits) ends with 5 is 8 and that of ends with 0 is 9.
Hence, by addition principle the required number of numbers is 8 + 9 = 17.

7.2.2 Multiplication Principle

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ be sets.

An ordered pair (a, b), where $a \in A$, $b \in B$, can be formed in $n \times m$ ways.

It can further be extended as

Let a set A_i have k_i elements, $i = 1, 2, \ldots, n$.

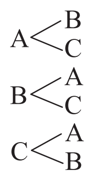
An ordered n-tuple $(a_{1,} a_2, \ldots, a_n)$ where $a_i \in A_i$ for each i, can be formed in $k_1 \times k_2 \times k_3 \times \cdots \times k_n$ ways.

In set theoretic notation, the above principle is stated as:

$\prod_{i=1}^{r} A_i = A_1 \times A_2 \times \cdots \times A_r = \{(a_1, a_2, a_3, \ldots, a_r) : a_i \in A_i, i = 1, 2, 3, \ldots, r\}$ denotes the cartesian product of the finite sets $A_1, A_2, \ldots, A_r$ then $\left|\prod_{i=1}^{r} A_i\right| = \prod_{i=1}^{r} |A_i|$.

In other words:
If an event E can be decomposed into r *ordered* sub events $E_1, E_2, \ldots, E_r$ and if there are n_1 ways (independent to other sub events) for E_1 to occur, n_2 ways (independent to other sub events) for the event E_2 to occur, $\ldots$, n_r ways (independent to other sub events) for E_r to occur, then the total number of ways for the event E to occur is given by $n_1 \times n_2 \times \cdots \times n_r$.

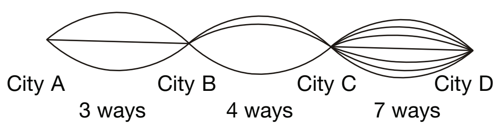
Example 5 *A Hall has 3 gates. In how many ways can a man enter the hall through one gate and come out through a different gate?*



Solution: Suppose the gates are A, B and C. Now there are 3 ways (A, B or C) of entering into the hall. After entering into the hall, the man come out through a different gate in 2 ways. Hence, by the multiplication principle, total number of ways is $3 \times 2 = 6$ ways.

Example 6 *There are 3 routes to travel from City A to City B and 4 routes to travel from City B to City C and 7 routes from C to D. In how many different ways (routes) a man can travel from City A to City D via City B and City C.*

Solution:



The man can perform the task of travelling from City A to City B in ways = 3.

The man can perform task of travelling from City B to City C in ways = 4.

Similarly from City C to City D in ways = 7.

Using fundamental principle of counting, total routes to travel from A to D via B and via C = $m \times n \times p = 3 \times 4 \times 7 = 84$ routes.

Example 7 *If $S = \{a, b, c, \ldots, x, y, z\}$, find the number of five-letter words that can be formed from the elements of the set S, such that the first and the last letters are distinct vowels and the remaining three are distinct consonants.*

Solution:

Place:	□	□	□	□	□
Number of choices:	5	21	20	19	4

As there are 5 vowels and 21 consonants, position 1 and 5 can be filled in 5 and 4 ways respectively and 2, 3, 4 can be filled in 21, 20 and 19 ways respectively. Therefore, the total number of ways

$$= 5 \times 4 \times 21 \times 20 \times 19$$
$$= 400 \times 399 = 159600.$$

Example 8 *A city has 12 gates. In how many ways can a person enter the city through one gate and come out through a different gate?*

Solution: Since, there are 12 ways to enter into the city. After entering into the city, the man can come out through a different gate in 11 ways.
Hence, by the fundamental principle of counting.
Total number of ways is $12 \times 11 = 132$ ways.

Example 9 *A basket contains 12 apples and 10 oranges. Ram takes an apple or an orange. Then Shyam takes an apple and an orange. In which case does shyam have more choice: When Ram takes an apple or when he takes an orange? (Consider apples and similarly oranges are distinguishable.) In how many ways both of them can take the fruits?*

Solution:

Case 1: Ram takes an apple
Shyam has to take one apple and one orange from 11 apples and 10 oranges.
Number of ways in which Shyam can take his fruits = $11 \times 10 = 110$.

Case 2: Ram takes an orange
Shyam has to take one apple and one orange from 12 apples and 9 oranges.
Number of ways in which Shyam can take his fruits = $12 \times 9 = 108$.
Shyam has more choice when Ram takes an apple.
Using addition principle, number of ways in which both can take a fruit
$= 12 \times 110 + 10 \times 108$
$= 1320 + 1080 = 1400$

Example 10 *A number lock has 3 concentric rings on which the digits 0, 1, 2, …, 9 are engraved. Only one particular arrangement on the rings, say ABC, against an arrow opens the lock. What is the number of unsuccessful attempts to open the lock?*

Solution: Total number of numbers formed by the digits 0, 1, 2, …, 9 on the three rings
$= 10 \times 10 \times 10$ (by multiplication principle) and number of successful attempts = 1
$\Rightarrow$ Number of unsuccessful attempts = $10^3 - 1$
$= 999$

Note: Here the method for counting used is called **i**ndirect method of counting**.**)

Example 11 *A binary sequence consists of 0's or 1's only. Find the number of binary sequences having n terms.*

Solution: Since every term of the binary sequence has two options (0 or 1), therefore the number of binary sequences of n terms = $\underbrace{2\times2\times2\times\cdots\times\ 2}_{\text{n times}} = 2^n$ (using multiplication principle).

Example 12 *How many* (i) 5 *digit* (ii) 3-*digit numbers can be formed using* 1, 2, 3, 7, 9 *without any repetition of digits.*

Solution:

(i) **5-digit numbers:**
Making a 5 digit number is equivalent to filling 5 places.
The last place (unit's place) can be filled in 5 ways using any of the five given digits.
The ten's place can be filled in four ways using any of the remaining 4 digits.
The number of choices for other places can be calculated in the same way.
Number of ways to fill all five places
$= 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$
$\Rightarrow$ 120 five-digit numbers can be formed.

Place:					
Number of choices:	1	2	3	4	5

(ii) **3-digit numbers:**
Making a three-digit number is equivalent to filling three places (unit's, ten's, hundred's).
Number of ways to fill all the three places $= 5 \times 4 \times 3 = 60$
$\Rightarrow$ 60 three-digit numbers can be formed.

Place:			
Number of choices:	1	2	3

Example 13 *How many* 3-*letter words can be formed using a, b, c, d, e if:*
(i) *Repetition is not allowed*
(ii) *Repetition is allowed?*

Solution:

(i) **Repetition is not allowed:**
The number of words that can be formed is equal to the number of ways to fill the three places.
First place can be filled in five ways using any of the five letters (a, b, c, d, e).
Similarly second and third places can be filled using 4 and 3 letters respectively.
$\Rightarrow$ Total number of ways to fill $= 5 \times 4 \times 3 = 60$.
Hence 60 words can be formed.

Place:			
Number of choices:	5	4	3

(ii) **Repetition is allowed:**
The number of words that can be formed is equal to the number of ways to fill the three places.
First place can be filled in five ways (a, b, c, d, e).
If repetition is allowed, each of the remaining places can be filled in five ways using a, b, c, d, e.
Total number of ways to fill $= 5 \times 5 \times 5 = 125$.
Hence 125 words can be formed.

Place:			
Number of choices:	5	5	5

Example 14 *How many four-digit numbers can be formed using the digits* 0, 1, 2, 3, 4, 5 *without repetition?*

Solution: For a four-digit number, we have to fill four places and 0 cannot appear in the first place (thousand's place).

Place:	☐	☐	☐	☐
Number of choices:	5	5	4	3

For the first place, there are five choices (1, 2, 3, 4, 5); Second place can then be filled in five ways (0 and remaining four-digits); Third place can be filled in four ways (remaining four-digits); Fourth place can be filled in three ways (remaining three-digits).

Total number of ways $= 5 \times 5 \times 4 \times 3 = 300$

$\Rightarrow$ 300 four-digits numbers can be formed.

Example 15: *In how many ways can six persons be arranged in a row?*

Solution: Arranging a given set of n different objects is equivalent to fill n places. So arranging six persons along a row is equivalent to fill 6 places.

Place:	☐	☐	☐	☐	☐	☐
Number of choices:	6	5	4	3	2	1

Number of ways to fill all places $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$.

Example 16 *How many 5-digit odd numbers can be formed using digits* 0, 1, 2, 3, 4, 5 *without repetition?*

Place:	☐	☐	☐	☐	☐
Number of choices:	4	4	3	2	3

Solution: Making a 5-digit number is equivalent to fill 5 places

To make odd numbers, fifth place can be filled by either of 1, 3, 5, *i.e.*, 3 ways.

Number of ways first place can be filled in = 4 (excluding 0 and the odd number used for the fifth place).

Similarly second, third and fourth places can be filled in 4, 3, 2 ways respectively.

Using fundamental principle of counting, total number of ways to fill 5 places.

= Total 5-digit odd numbers that can be formed $= 4 \times 4 \times 3 \times 2 \times 3 = 288$ ways.

Example 17 *How many 5-digit numbers divisible by* 2 *can be formed using digits* 0, 1, 2, 3, 4, 5 *without repetition.*

Solution: To find 5-digit numbers divisible by 2,

We will make 2 cases. In first case, we will find number of numbers divisible by 2 ending with either 2 or 4. In second case, we will find even numbers ending with 0.

Place:	☐	☐	☐	☐	☐
Number of choices:	4	4	3	2	2

Case 1: Even numbers ending with 2 or 4:

Making a 5 digit number is equivalent to filling 5 places

Fifth place can be filled by 2 or 4, *i.e.*, 2 ways.

First place can be filled in 4 ways (excluding 0 and the digit used to fill fifth place)

Similarly places second, third and fourth can be filled in 4, 3, 2 ways respectively.

Using fundamental principle of counting, total number of ways to fill all 5 places together $= 4 \times 4 \times 3 \times 2 \times 2 = 192$. (1)

Place:	☐	☐	☐	☐	☐
Number of choices:	5	4	3	2	1

Case 2: Even numbers ending with 0:

Making a 5-digit number is equivalent to fill 5 place.

Fifth place is filled by 0, hence can be filled in 1 way.

First place can be filled in 5 ways (Using either of 1, 2, 3, 4, 5).

Similarly places second, third and fourth can be filled in 4, 3, 2 ways respectively.

Using fundamental principle of counting, total number of ways to fill 5 places $= 5 \times 4 \times 3 \times 2 \times 1 = 120$ (2)

Combining (1) and (2),

Total number of 5 digit numbers divisible by $2 = 192 + 120 = 312$.

Example 18 *How many 5-digit numbers divisible by 4 can be formed using digits* 0, 1, 2, 3, 4, 5 *without repetition?*

Solution: Making a 5-digit number is equivalent to fill 5 places.
A number would be divisible by 4 if the last 2 places are filled by either of 04, 12, 20, 24, 32, 40, 52.

Case 1:
Last 2 places are filled by either of 04, 20, 40.

Place:	□	□	□	□□
Number of choices:	4	3	2	3

Fourth and fifth places can be filled in 3 ways. (either of 04, 20, 40).
First place can be filled in 4 ways (excluding the digits used to fill fourth and fifth place.
Similarly second and third place can be filled in 3 and 2 ways respectively.
Using fundamental principle of counting, total number of ways to fill 5 places $= 4 \times 3 \times 2 \times 3 = 72$ ways (1)

Case 2:
Last 2 places are filled by either of 12, 24, 32, 52

Place:	□	□	□	□□
Number of choices:	3	3	2	4

Fourth and fifth place can be filled in 4 ways (either 12, 24, 32, 52).
First place can be filled in 3 ways (excluding 0 and the digits used to fill fourth and fifth place)
Similarly, second and third place can be filled in 3 and 2 ways respectively.
Using fundamental principle of counting, total number of ways to fill 5 place $= 3 \times 3 \times 2 \times 4 = 72$ ways. (2)
Combining (1) and (2),
Total number of ways to fill 5 places = Total 5-digit numbers divisible by 4
$= 72 + 72 = 144$.

Example 19 *How many six-digit numbers divisible by* 25 *can be formed using digits* 0, 1, 2, 3, 4, 5 *without repetition?*

Solution: Numbers divisible by 25 must end with 25 or 50.

Case 1: Number ending with 25

Place:	□	□	□	□	□	□
Number of choices:	3	3	2	1	1	1

Using fundamental principle of counting, total 6 digit numbers divisible by 25 ending with 25
$= 3 \times 3\,! = 18$ numbers are possible.

Case 2: Number ending with 50

Place:	□	□	□	□	□	□
Number of choices:	4	3	2	1	1	1

Using fundamental principle of counting, total 6 digited numbers divisible by 25 ending with 50
$= 4! = 24$ numbers are possible.
Hence, total numbers of multiples of 25
$= 18 + 24 = 42$.

Example 20 *Find the number of* 4*-digit numbers greater than* 3400*, when digits are chosen from* 1, 2, 3, 4, 5, 6 *with repetition allowed.*

Solution: To count the number of numbers greater than 3400, we consider the following two cases:

Place:				
Number of choices:	3	6	6	6

Case 1: Thousand's place filled by 4 or 5 or 6
(That is, thousand's place can be filled in 3 ways) each digit (of last three digits) has 6 options (*i.e.*, they can be filled by any of 1, 2, 3, 4, 5, 6). Using multiplication principle, the number of such numbers $= 3 \times 6 \times 6 \times 6 = 648$

Place:	3			
Number of choices:	1	3	6	6

Case 2: Thousand's place filled by 3 and hundred's place filled by 4 or 5 or 6.
(That is, thousand's place can be filled in 1 way and hundred's place can be filled in 3 ways)

Using multiplication principle, the number of such numbers $= 1 \times 3 \times 6 \times 6 = 108$

Cases I and II are *mutually exclusive* (*i.e., cannot occur together*) *and exhaustive* (*i.e., all possibilities are covered*)

$\therefore$ Using addition principle, the number of 4-digit numbers greater than 3400, (formed by 1, 2, 3, 4, 5, 6) $= 648 + 108 = 756$.

Example 21 *Find the number of odd integers between* 30,000 *and* 80,000 *in which no digit is repeated.*

Solution:
Let *abcde* be the required odd integers.

a can be chosen from 3, 4, 5, 6 and 7 and *e* can be chosen from 1, 3, 5, 7, 9. Note that 3, 5 and 7 can occupy both the positions *a* and *e*.

So, let us consider the case where one of 3, 5, 7 occupies the position *a*.

Case 1: If *a* gets one of the values 3, 5, 7, then there are 3 choices for *a*, but then, *e* has just four choices as repetition is not allowed. Thus, *a* and *e* can be chosen in this case in $3 \times 4 = 12$ ways.

The 3 positions *b*, *c*, *d* can be filled from among the remaining 8 digits in $8 \times 7 \times 6$ ways. Total number of ways in this case $= 12 \times 8 \times 7 \times 6 = 4{,}032$.

Case 2: If *a* takes the values 4 or 6, then there are two choices for *a* and there are five choices for *e*.

There are again eight choices altogether for the digits *b*, *c*, *d* which could be done in $8 \times 7 \times 6$ ways.

Therefore in this case, the total numbers are $2 \times 5 \times 8 \times 7 \times 6 = 3{,}360$.

Hence, total number of odd numbers between 30,000 to 80,000, without repetition of digits is $4{,}032 + 3{,}360 = 7{,}392$.

Example 22 *A number of four digits is to be formed from* 1, 2, 3, 4, 5 *and* 6. *Find the number of* 4*-digit numbers*

(i) *if repetition of a digit is allowed.*
(ii) *if no repetition of a digit is allowed.*
(iii) *How many of the numbers are divisible by 4, if*
  (a) *repetition is allowed?*
  (b) *repetition is not allowed?*

Solution:

(i) Since each digit of a 4 - digit number can be one from 1, 2, 3, 4, 5, 6, therefore using multiplication principle, the number of 4 digit numbers (repetition is allowed) $= 6 \times 6 \times 6 \times 6 = 6^4$

(ii) Using multiplication principle, the number of 4-digit numbers (repetition is not allowed) $= 6 \times 5 \times 4 \times 3 = 360$

(iii) If a number is divisible by 4, then the last two digits must form one of the following numbers :

12, 16, 24, 32, 36, 44, 52, 56, 64 (9 in all)

(a) Number of numbers, divisible by 4 (repetition is allowed) $= 6 \times 6 \times 9 = 324$

(b) Number of numbers, divisible by 4 (repetition is not allowed) $= 4 \times 3 \times 8 = 96$

Note that in (a), to fill the places of last two digits (considered it as one 2-digit number) 9 options are available as stated above.

Note that in (b), since repetition is not allowed, so the number formed by the last two digits cannot be 44. So it can be one from the remaining 8 options.

Example 23 *Find the sum of all five-digit numbers that can be formed using digits* 1, 2, 3, 4, 5 *if repetition is not allowed?*

Solution: There are 5! = 120 five digit numbers and there are 5 digits. Hence by symmetry or otherwise we can see that each digit will appear in any place (unit's or ten's or …) $\frac{5!}{5}$ times.

Let X = Sum of digits in any place

$$\Rightarrow X = \frac{5!}{5}\times 5+\frac{5!}{5}\times 4+\frac{5!}{5}\times 3+\frac{5!}{5}\times 2+\frac{5!}{5}\times 1$$

$$\Rightarrow X = \frac{5!}{5}\times(5+4+3+2+1) = \frac{5!}{5}(15) = 5!\times 3$$

$\Rightarrow$ The sum of the all numbers $= X + 10X + 100X + 1000X + 10000X$

$= X(1 + 10 + 100 + 1000 + 10000)$

$= 5! \times 3\ (1 + 10 + 100 + 1000 + 10000)$

$= 120 \times 3\ (11111) = 3999960.$

Example 24 *Find the sum of the four digit numbers obtained in all possible permutations of the digits* 1, 2, 3, 4.

Solution: There are 4! (= 24) 4-digit numbers made up of 1, 2, 3, 4. In these 24 numbers, in unit place all 1, 2, 3, 4 appear 3! (=6) times. Similarly, in the ten's, hundred's, thousand's places too, they appear 6 times.

Sum $= 6(4+3+2+1) + 10 \times 6\ (4+3+2+1)+100 \times 6(4+3+2+1)+1000 \times 6(4+3+2+1)$

$= 60 + 600 + 6000 + 60000 = 66,\ 660$

Example 25 *Find the sum of 5-digit numbers obtained by permuting* 0, 1, 2, 3, 4.

Solution: There are 5! (= 120) 5-digit numbers made up of 0, 1, 2, 3, 4. In all these 120 numbers in unit's place all 0, 1, 2, 3, 4 appear 4! (= 24) times. Similarly in ten's, hundred's, thousand's and ten thousand's places too they appear 24 times.

Sum of 5-digit numbers made up of 0, 1, 2, 3, 4

$= 24(1 + 2 + 3 + 4) + 10 \times 24(1 + 2 + 3 + 4) + 100 \times 24(1 + 2 + 3 + 4)$
$+ 1000 \times 24\ (1 + 2 + 3 + 4) + 10000 \times 24(1 + 2 + 3 + 4)$

$= 240 + 2400 + 24000 + 240000 + 2400000 = 26,66,640.$

Required sum = Sum of 5-digit numbers made up of 0, 1, 2, 3, 4 – sum of 4 digit numbers made up of 1, 2, 3, 4

= 26,66,640 – 66660 {Obtained from previous example}

= 25,99,980.

Example 26 *Find the sum of all four digit numbers that can be formed using the digits* 0, 1, 2, 3, 4, *no digits being repeated in any number.*

Solution: Required sum of numbers = [Sum of four digit numbers using 0, 1, 2, 3, 4, allowing 0 in first place] – [Sum of three digit numbers using 1, 2, 3, 4].

$$= \frac{5!}{5}[0+1+2+3+4][1+10+10^2+10^3] - \frac{4!}{4}(1+2+3+4)(1+10+10^2)$$

$$= 24 \times 10 \times 1111 - 6 \times 10 \times 111 = 259980.$$

Example 27 *Let S be the set of natural numbers whose digits are chosen from* {1, 2, 3, 4} *such that*

(i) *When no digits are repeated, find $n(S)$ and the sum of all numbers in S.*
(ii) *When S is the set of up to 4-digit numbers where digits are repeated. Find $|S|$ and also find the sum of all the numbers in S.*

Solution:

(i) S consists of single-digit numbers, two-digit numbers, three-digit numbers and four-digit numbers.
Total number of single-digit numbers = 4
Total number of two-digit numbers = $4 \times 3 = 12$
(Since repetition is not allowed, there are four choices for tens place and three choices for units place.)
Total number of three-digit numbers = $4 \times 3 \times 2 = 24$
Total number of four-digit numbers = $4 \times 3 \times 2 \times 1 = 24$
$\therefore \quad n(S) = 4 + 12 + 24 + 24 = 64.$
Now, for the sum of these 64 numbers, sum of all the single-digit numbers = $1 + 2 + 3 + 4 = 10$.
(Since there are exactly 4 digits 1, 2, 3, 4 and their numbers are 1, 2, 3 and 4.)
Now,
The total number of two-digit numbers is 12.
The digits used in units place are 1, 2, 3 and 4.
In the 12 numbers, each of 1, 2, 3 and 4 occurs thrice in units digit $\left(\frac{12}{4} = 3\right)$.
Again in tens place, each of these digits occurs thrice.
So, sum of these 12 numbers
$= 30 \times (1 + 2 + 3 + 4) + 3 \times (1 + 2 + 3 + 4)$
$= 300 + 30 = 330.$
The number of three-digit numbers is 24.
So, the number of times each of 1, 2, 3, 4 occurs in each of units, tens and hundreds place is $\frac{24}{4} = 6$.
So, the sum of all these three-digit numbers is
$100 \times 6(1 + 2 + 3 + 4) + 10 \times 6(1 + 2 + 3 + 4) + 1 \times 6(1 + 2 + 3 + 4)$
$= 6{,}000 + 600 + 60 = 6{,}660.$
Similarly, for the four-digit numbers, the sum is computed as
$1000 \times 6(1 + 2 + 3 + 4) + 100 \times 6(1 + 2 + 3 + 4) + 10 \times 6(1 + 2 + 3 + 4)$
$+ 1 \times 6(1 + 2 + 3 + 4) = 60{,}000 + 6{,}000 + 600 + 60 = 66{,}660$
[Since there are 24 four-digit numbers, each of 1, 2, 3, 4 occurs in each of the four digits in $\frac{24}{4} = 6$ times.]

So, the sum of all the single-digit, two-digit, three-digit and four-digit numbers = $10 + 330 + 6{,}660 + 66{,}660 = 73{,}660$.

(ii) (a) There are just four single-digit numbers 1, 2, 3 and 4.
(b) There are $4 \times 4 = 16$ two-digit numbers, as digits can be repeated.
(c) There are $4 \times 4 \times 4 = 64$ three-digit numbers.
(d) There are $4 \times 4 \times 4 \times 4 = 256$ four-digit numbers.

So, total number of numbers up to four-digit numbers that could be formed using the digits 1, 2, 3 and 4 is $4 + 16 + 64 + 256 = 340$. Sum of the 4 single-digit numbers $= 1 + 2 + 3 + 4 = 10$. To find the sum of 16 two-digit numbers each of 1, 2, 3, 4 occur in each of units and tens place $= \frac{16}{4} = 4$ times.

So, the sum of all these 16 numbers is

$$= 10 \times 4(1 + 2 + 3 + 4) + 4(1 + 2 + 3 + 4)$$

$$= 400 + 40 = 440.$$

Similarly, the sum of all the 64 three-digit numbers

$$= 100 \times \frac{64}{4} \times (1+2+3+4) + 10 \times \frac{64}{4} \times (1+2+3+4) + 1 \times \frac{64}{4} \times (1+2+3+4)$$

$$= 16{,}000 + 1{,}600 + 160 = 17{,}760.$$

Again the sum of all the 256 four-digit numbers

$$= 1000 \times \frac{256}{4} \times (1+2+3+4) + 100 \times \frac{256}{4} \times (1+2+3+4)$$

$$+ 10 \times \frac{256}{4} \times (1+2+3+4) + 1 \times \frac{256}{4} \times (1+2+3+4)$$

$$= 6{,}40{,}000 + 64{,}000 + 6{,}400 + 640 = 7{,}11{,}040$$

Therefore, the sum of all the numbers is

$= 10 + 440 + 17{,}760 + 7{,}11{,}040 = 7{,}29{,}250$.

Build-up Your Understanding 1

1. How many four digit numbers can be made by using the digits 1, 2, 3, 7, 8, 9 when
   (i) repetition of a digit is allowed?
   (ii) repetition of a digit is not allowed?
2. Find the total number of 9-digit numbers of different digits.
3. Find the total number of 4 digit number that are greater than 3000, that can be formed by using the digits 1, 2, 3, 4, 5, 6 (no digit is being repeated in any number).
4. How many numbers greater than 1000 or equal to, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4, repetition of digits being allowed?
5. How many numbers between 400 and 1000 (both exclusive) which can be made with the digits 2,3,4,5,6,0 if
   (i) repetition of digits not allowed?
   (ii) repetition of digits is allowed?
6. A variable name in a certain computer language must be either an alphabet or a alphabet followed by a decimal digit. Find the total number of different variable names that can exist in that language.

7. Tanya typed a six-digit number, but the two 1's she typed did not show. What appeared was 2006. Find the number of different 6-digit numbers she would have typed.
8. A letter lock consists of three rings each marked with fifteen different letters. It is found that a man could open the lock only after he makes half the number of possible unsuccessful attempts to open the lock . If each attempt takes 10 seconds. Then find the minimum time he must have spent.
9. Find the number of 6-digit numbers that can be formed using 1, 2, 3, 4, 5, 6, 7 so that digits do not repeat and terminal digits are even.
10. Find the total number of numbers that can be formed by using all the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places.
11. Find the number of 6-digit numbers which have 3 digits even and 3 digits odd, if each digit is to be used atmost once.
12. Find the number of 4-digits numbers that can be made with the digits 1, 2, 3, 4 and 5 in which at least two digits are identical.
13. Find the number of 5-digit telephone numbers having atleast one of their digits is repeated.
14. Find the number of 3-digit numbers having only two consecutive digits identical.
15. Find the number of different matrices that can be formed with elements 0, 1, 2 or 3, each matrix having 4 elements.
16. Find the number of 6-digit numbers in which sum of the digits is even.
17. Find the number of 5-digit numbers divisible by 3 which can be formed using 0, 1, 2, 3, 4, 5 if repetition of digits is not allowed.
18. Find the number of 4-digit numbers divisible by 3 that can be formed by four different even digits.
19. Find the number of 5-digit numbers divisible by 6 which can be formed using 0, 1, 2, 3, 4, 5 if repetition of digits is not allowed.
20. Find the number of 5-digit numbers divisible by 4 which can be formed using 0, 1, 2, 3, 4, 5, when the repetition of digits is allowed
21. Natural numbers less than 10^4 and divisible by 4 and consisting of only the digits 0, 1, 2, 3, 4 and 5 (no repetition) are formed . Find the number of ways of formation of such number.
22. Find the number of natural numbers less than 1000 and divisible by 5 which can be formed with the ten digits, each digit not occurring more than once in each number.
23. Two numbers are chosen from 1, 3, 5, 7,…, 147, 149 and 151 and multiplied together. Find the number of ways which will give us the product a multiple of 5.
24. A 7-digit number divisible by 9 is to be formed by using 7 digits out of digits 1, 2, 3, 4, 5, 6, 7, 8, 9. Find the number of ways in which this can be done.
25. Find the number of 9-digits numbers divisible by nine using the digits from 0 to 9 if each digit is used atmost once.
26. Among 9! permutations of the digits 1, 2, 3,…, 9. Consider those arrangements which have the property that if we take any five consecutive positions, the product of the digits in those positions is divisible by 7. Find the number of such arrangements.
27. Find the number of distinct results which can be obtained when n distinct coins are tossed together.
28. Three distinct dice are rolled. Find the number of possible outcomes in which at least one die shows 5.
29. A telegraph has 'm' arms and each arm is capable of 'n' distinct positions including the position of rest. Find the total number of signals that can be made.
30. Find the number of possible outcomes in a throw of n distinct dice in which at least one of the dice shows an odd number.

31. Find the number of times the digit 5 will be written when listing integers from 1 to 1000.

32. Find the number of times of the digits 3 will be written when listing the integer from 1 to 1000.

33. If 33! is divisible by 2^n, then find the maximum value of n.

34. Let $E = \left[\frac{1}{3}+\frac{1}{50}\right]+\left[\frac{1}{3}+\frac{2}{50}\right]+\left[\frac{1}{3}+\frac{3}{50}\right]+\cdots$ upto 50 terms, then find the exponent of 2 in $(E)!$.

35. 3-digit numbers in which the middle one is a perfect square are formed using the digits 1 to 9 . Find their sum.

36. Find the sum of all the 4-digit even numbers which can be formed by using the digits 0, 1, 2, 3, 4 and 5 if repetition of digits is allowed.

37. Find sum of 5-digit numbers that can be formed using 0, 0, 1, 2, 3, 4.

38. Find sum of 5-digit numbers that can be formed using 0, 0, 1, 1, 2, 3.

39. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (that is 1, 16, 31, etc.) This process is continued until a number is reached which has already been marked, then find the all unmarked numbers.

40. Let S be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find the number of subsets A of S such that. $x \in A$ and $2x \in S \Rightarrow 2x \in A$.

7.3 Combinations

7.3.1 Definition of Combination

Let A, B, C, D be four distinct objects. The number of ways in which we can select two objects out of A, B, C and D is six and these are AB, AC, AD, BC, BD and CD.

These ways of selection of two objects from four different objects are also known as combinations of A, B, C and D taken two at a time or we can say grouping of A, B, C and D taken two at a time.

Similarly $\{a, b, c\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$ are *all* the selections of 3 objects from a, b, c, d. So we say that the number of ways of selecting 3 objects out of given 4 objects is 4 or the number of combinations of 3 objects out of given 4 objects is 4.

Note:
By changing the relative positions of objects, we do not get any new combinations. Combination (selection or group) of objects A, B is same as combination of objects B, A. Thus we treat AB and BA as same combination (selection or group). Formally

A combination of objects is merely a selection (*suppress order*) *from a given lot of objects, i.e., a combination is just a set, elements of which are not arranged in a particular way.*

7.3.2 Theorem

The number of selections of r objects at a time out of n distinct, is $\frac{n!}{r!(n-r)!}$.

This number is denoted as ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$.

Proof:

$\binom{n}{r-1}$ represents the number of selections of $r - 1$ objects out of n distinct objects.

Number of ways to select rth object from remaining $n-(r-1)$ objects is $n-(r-1)$.

By multiplication principle, the number of ways to select r objects out of n distinct objects is apparently $\binom{n}{r}\cdot(n-r+1)$.

However, each selection is counted r times. Note that we are aiming at counting the unordered selections.

For example, $\{a, b, c\}$ or $\{b, a, c\}$ or $\{c, a, b\}$ are to be considered as one selection (not 3 selections)

Therefore ${}^nC_r = {}^nC_{r-1}\dfrac{n-r+1}{r}$. (recurrence relation)

$${}^nC_{r-1} = {}^nC_{r-2}\cdot\frac{n-(r-1)+1}{r-1} = {}^nC_{r-2}\frac{n-r+2}{r-1}$$

$${}^nC_{r-2} = {}^nC_{r-3}\cdot\frac{n-r+3}{r-2}, \text{ etc.}$$

$$\therefore\ {}^nC_r = {}^nC_1\cdot\frac{(n-1)(n-2)\cdots(n-r+1)}{r(r-1)\cdots 2\cdot 1}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)\cdots 2\cdot 1} \quad (\text{Note that } {}^n\mathrm{C}_1 = n)$$

$$= \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 2\cdot 1}{(r(r-1)\cdots 2\cdot 1)((n-r)(n-r-1)\cdots 2\cdot 1)}$$

$${}^nC_r = \frac{n}{r!(n-r)!}$$

In general $$\binom{n}{r} = \begin{cases} \dfrac{n!}{r!(n-r)!}, & 0 \le r \le n;\ r, n \in \mathbb{N}_0 \\ 0 \quad , & \text{for } r < 0 \text{ or } r > n;\ n \in \mathbb{N}_0, r \in \mathbb{Z} \end{cases}$$

Note: $\binom{0}{0}$ is defined as 1.

7.3.3 Properties of $\binom{n}{r}$; $0 \le r \le n$; $r, n \in \mathbb{N}_0$

(i) $\binom{n}{0} = \binom{n}{n} = 1$

(ii) $\binom{n}{r} = \binom{n}{n-r}$

(iii) If $\binom{n}{r} = \binom{n}{k}$ then $r = k$ or $n - r = k$

(iv) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

(v) $\binom{n}{r} = \dfrac{n}{r}\binom{n-1}{r-1}$ or $r\binom{n}{r} = n\binom{n-1}{r-1}$

(vi) $\dfrac{1}{r+1}\binom{n}{r} = \dfrac{1}{n+1}\binom{n+1}{r+1}$

(vii) $\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$ or $\frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{n-r+1}{r}$

(viii) $\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r}$

(ix) (a) If n is even, $\binom{n}{r}$ is greatest for $r = \frac{n}{2}$.

(b) If n is odd, $\binom{n}{r}$ is greatest for $r = \frac{n-1}{2}, \frac{n+1}{2}$.

In general $\binom{n}{r}$ is maximum at $r = \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil$

Combinatorial proof of (v):

Consider a group of n people. A committee of r people is to be selected, out of these selected r people one chairperson is nominated.

This can be done in following two ways:

(i) Select r people from n people and select one person for chairperson from selected r people.

This can be done in $\binom{n}{r} \times \binom{r}{1}$ ways.

(ii) Another alternative is to select one person as the chairperson from n people and select remaining $(r-1)$ people from remaining $(n-1)$ people.

This can be done in $\binom{n}{1} \times \binom{n-1}{r-1}$ ways.

$$\Rightarrow r\binom{n}{r} = n\binom{n-1}{r-1}$$

Students are advised to develop the combinatorials proofs of the remaining properties.

Example 28 *If* ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ *and* ${}^nC_{r+1} = 126$, *then find r.*

Solution:

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{84}{36}$$

$$\Rightarrow \frac{n-r+1}{r} = \frac{7}{3} \qquad \left(\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right)$$

$$\Rightarrow 3n - 3r + 3 = 7r$$

$$\Rightarrow 10r - 3n = 3 \qquad (1)$$

and $$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-(r+1)+1}{(r+1)} = \frac{126}{84} \qquad \left(\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right)$$

$$\Rightarrow \frac{n-r}{r+1} = \frac{3}{2} \Rightarrow 2n - 2r = 3r + 3$$

$$\Rightarrow 5r - 2n = -3 \Rightarrow 10r - 4n = -6 \qquad (2)$$

Subtracting Eq. (2) from Eq. (1), we get $n = 9$

$$10r - 27 = 3\ 10 \Rightarrow r = 30 \Rightarrow r = 3$$

Example 29 *There were some men and two women participating in a chess tournament. Every participant played two games with every other participant. The number of games that the men played among themselves exceed by 66 that of the games which the men played with the two women. What was the total number of participants? How many games were played in all?*

Solution: Let the number of men participants be m.

The number of games which men have played among themselves is $2\binom{m}{2} = m(m-1)$.

The number of games which the men played with each of the two women is $2m \times 2$.

[m men played $2 \times m$ game with the first woman and another $2 \times m$ game with the second woman.]

According to the data given

$$\begin{aligned} m(m-1) - 2 \times 2m &= 66 \\ \Rightarrow m^2 - 5m - 66 &= 0 \\ \Rightarrow (m-11)(m+6) &= 0 \\ \Rightarrow m &= 11 (m = -6 \text{ is not acceptable}) \end{aligned}$$

So, there are totally $11 + 2 = 13$ players.

The number of games played is $2 \times {}^{13}C_2 = 2 \times \frac{13 \times 12}{1.2} = 156$.

7.3.4 Some Applications of Combinations

7.3.4.1 *Always Including p Particular Objects in the Selection*

The number of ways to select r objects from n distinct objects where p particular objects should always be included in the selection $= {}^{n-p}C_{r-p}$.

Logic:

We can select p particular objects in 1 way. Now from remaining $(n - p)$ objects we select remaining $(r - p)$ objects in ${}^{n-p}C_{r-p}$ ways.

Using fundamental principle of counting, number of ways to select r objects where p particular objects are always included

$$= 1 \times {}^{n-p}C_{r-p} = {}^{n-p}C_{r-p}.$$

Example 30 *In how many ways a team of* 11 *players be selected from a list of* 16 *players where two particular players should always be included in the team.*

Solution: Number of ways to make a team of 11 players from 16 players always including 2 particular players $= {}^{16-2}C_{11-2} = {}^{14}C_9$.

7.3.4.2 *Always Excluding p Particular Objects in the Selection*

The number of ways to select r objects from n different objects where p particular objects should never be included in the selection $= {}^{n-p}C_r$.

Logic:

As p particular objects are never to be selected, selection should be made from remaining $n - p$ objects. Therefore r objects can be selected from $(n - p)$ different objects in ${}^{n-p}C_r$ ways.

Example 31 *In how many ways a team of* 11 *players can be selected from a list of* 16 *players such that* 2 *particular players should never be included in the selection.*

Solution: The number of ways to select a team of 11 players from a list of 16 players, always excluding 2 particular players = ${}^{(16-2)}C_{11} = {}^{14}C_{11}$.

Example 32 *A mixed doubles tennis game is to be arranged from* 5 *married couples. In how many ways the game can be arranged if no husband and wife pair is included in the same game?*

Solution: To arrange the game we have to do the following operations.

(i) Select two men from from 5 men in ${}^5C_2$ ways.

(ii) Select two women from 3 women excluding the wives of the men already selected. This can be done in ${}^3C_2$ ways.

(iii) Arrange the 4 selected persons in two teams. If the selected men are M_1 and M_2 and the selected women are W_3 and W_4, this can be done in 2 ways :

M_1W_3 play against M_2W_4

M_2W_3 play against M_1W_4

Hence the number of ways to arrange the game

$$= {}^5C_2\,{}^3C_2\,(2) = 10 \times 3 \times 2 = 60.$$

7.3.4.3 Exactly or Atleast or Atmost Constraint in the Selection

There are problems in which constraints are to select exactly or minimum (atleast) or maximum (atmost) number of objects in the selection. In these problems, we should always make cases to select objects. If we do not make cases, we will get wrong answer. Following illustrations will show you how to make cases to solve problems of this type.

Example 33 *In how many ways can a cricket team be selected from a group of* 25 *players containing* 10 *batsmen,* 8 *bowlers,* 5 *all-rounders and* 2 *wicketkeepers? Assume that the team of* 11 *players requires* 5 *batsmen,* 3 *all-rounders,* 2*-bowlers and* 1 *wicketkeeper.*

Solution: Divide the selection of team into four operations.

(i) Selection of batsman can be done (5 from 10) in ${}^{10}C_5$ ways.

(ii) Selection of bowlers can be done (2 from 8) in ${}^8C_2$ ways.

(iii) Selection of all-rounders can be done (3 from 5) in ${}^5C_3$ ways.

(iv) Selection of wicketkeeper can be done (1 from 2) in ${}^2C_1$ ways.

$\Rightarrow$ The team can be selected in ${}^{10}C_5 \times {}^8C_2 \times {}^5C_3 \times {}^2C_1$ ways $= \dfrac{10! \times 8 \times 7 \times 10 \times 2}{5!5!2!} = 141120.$

Example 34 *In a group of* 80 *persons of an association, a chairman, a secretary and three members are to be elected for the executive committee. Find in how many ways this could be done.*

Solution: This would be done in:

Chairman can be elected in ${}^{80}C_1$ ways,

Secretary can be elected in ${}^{79}C_1$ ways and the three members can be elected in ${}^{78}C_3$ ways.

So, the total number of ways in which this executive committee can be selected is

$$\begin{aligned} {}^{80}C_1 \times {}^{79}C_1 \times {}^{78}C_3 &= 80 \times 79 \times \frac{78 \times 77 \times 76}{1 \times 2 \times 3} \\ &= 80 \times 79 \times 13 \times 77 \times 76 \\ &= 800{,}320 \text{ ways.} \end{aligned}$$

Example 35 *A box contains 5 distinct red and 6 distinct white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?*

Solution: The selection of balls from 5 red and 6 white balls will consist of any of the following possibilities.

Red Balls (out of 5)	2	3	4
White Balls (out of 6)	4	3	2

If the selection contains 2 red and 4 white balls, then it can be done in ${}^5C_2\,{}^6C_4$ ways.
If the selection contains 3 red and 3 white balls then it can be done in ${}^5C_3\,{}^6C_3$ ways.
If the selection contains 4 red and 2 white balls then it can be done in ${}^5C_4\,{}^6C_2$ ways.

Any one of the above three cases can occur. Hence the total number of ways to select the balls.

$$\begin{aligned}&= {}^5C_2\,{}^6C_4 + {}^5C_3\,{}^6C_3 + {}^5C_4\,{}^6C_2\\ &= 10(15) + 10(20) + 5(15)\\ &= 425.\end{aligned}$$

Example 36 *In how many ways a team of 5 members can be selected from 4 ladies and 8 gentlemen such that selection includes at least 2 ladies?*

Solution: As the selection includes 'atleast' constraint, we make cases to find total number of teams.

Ladies in the team	Gentlemen in the team	Number of ways to select team
2	3	${}^4C_2 \times {}^8C_3$
3	2	${}^4C_3 \times {}^8C_2$
4	1	${}^4C_4 \times {}^8C_1$

Combining all cases shown in the table, total number of ways to select a team of 5 members

$$= {}^4C_2 \times {}^8C_3 + {}^4C_3 \times {}^8C_2 + {}^4C_2 \times {}^8C_1 = 456.$$

Example 37 *In a company there are 12 job vacancies. Out of 12, 3 are reserved for 'reserved category' candidates and rest 9 are open for all. In how many ways these 12 vacancies can be filled by 5 from 'reserved category' and 10 from general category candidates?*

Solution: There are 12 vacancies. As 3 are reserved for 'reserved category' candidates, it means we have to select 12 candidates (to fill 12 vacancies) such that selection should include at least 3 candidates from 'reserved category'. As rest 9 vacancies are open for all, it means 'reserved category' candidates can also take these vacancies.

As selection includes atleast constraint, we need to make following cases:

Reserved category	General category candidates	Number of ways to select
3	9	${}^5C_3 \times {}^{10}C_9$
4	8	${}^5C_4 \times {}^{10}C_8$
5	7	${}^5C_5 \times {}^{10}C_7$

Combining all cases shown above, we get, number of ways to fill 2 vacancies

$$\begin{aligned}&= {}^5C_3 \times {}^{10}C_9 + {}^5C_4 \times {}^{10}C_8 + {}^5C_5 \times {}^{10}C_7\\ &= 100 + 225 + 120 = 445 \text{ ways.}\end{aligned}$$

Example 38 *A man has 7 relatives, 4 of them ladies and 3 gentlemen; his wife has 7 relatives, 3 of them are ladies and 4 gentlemen. In how many ways they can invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?*

Solution: The possible ways of selecting 3 ladies and 3 gentleman for the party can be analysed with the help of the following table.

Man's relative		Wife's relative		
Ladies (4)	Gentleman (3)	Ladies (3)	Gentleman (4)	Number of ways
3	0	0	3	${}^4C_3{}^3C_0{}^3C_0{}^4C_3 = 16$
2	1	1	2	${}^4C_2{}^3C_1{}^3C_1{}^4C_2 = 324$
1	2	2	1	${}^4C_1{}^3C_2{}^3C_2{}^4C_1 = 144$
0	3	3	0	${}^4C_0{}^3C_3{}^3C_3{}^4C_0 = 1$

Total number of ways to invite = 16 + 324 + 144 + 1 = 485.

7.3.4.4 Selection of One or More Objects

7.3.4.4.1 From n Distinct Objects

The number of ways to select one or more objects from n different objects or we can say, selection of at least one object from n different objects $= 2^n - 1$.

Logic:

The number of ways to select 1 object from n different objects $= {}^nC_1$
The number of ways to select 2 objects from n different objects $= {}^nC_2$

… … …
… … …
… … …

The number of ways to select n objects from n different objects $= {}^nC_n$
Combining all above cases, we get the number of ways to select at least one (one or more) object from n different objects

$= {}^nC_1 + {}^nC_2 + {}^nC_3 + {}^nC_4 + \cdots + {}^nC_n$

$= 2^n - 1$ [Using sum of binomial coefficients in the expansion of $(1 + x)^n = 2^n$]

Alternate logic:

Let us assume $a_1, a_2, a_3, \ldots, a_n$ be n distinct objects.

We have to make our selection from these n objects.

We can make out selection from a_1 object in 2 ways.

This is because either we will choose a_1 or we would not choose a_1. Similarly selection of $a_2, a_3, \ldots, a_n$ can be done in 2 ways each.

Objects	a_1	a_2	a_3	a_4	…	a_n
Ways	2	2	2	2	…	2

Using fundamental principle of counting, the total number of ways to make selection from $a_1, a_2, ..., a_n$

$$= 2 \times 2 \times 2 \times 2 \ldots n \text{ times}$$
$$= 2^n$$

But the above selection includes a case where we have not selected any object. On subtracting this case from 2^n we get, the number of ways to select atleast one (one or more) object from n different objects $= 2^n - 1$

Notes:

1. The number of ways to select 0 or more objects from n distinct objects $= 2^n$
2. The number of ways to select at least 2 objects from n distinct objects $= 2^n - 1 - {}^nC_1$
3. The number of ways to select at least r objects from n distinct objects $= 2^n - 1 - {}^nC_1 - {}^nC_2 - {}^nC_3 - \cdots - {}^nC_{r-1}$ or ${}^nC_r + {}^nC_{r+1} + {}^nC_{r+2} + \cdots + {}^nC_n$.

7.3.4.4.2 From n Identical Objects

The number of ways to select one or more objects (or at least one object) from n identical object $= n$.

Logic:

To select r objects from n identical objects, we cannot use $\binom{n}{r}$ formula here, as all objects are not distinct. In fact, all objects are identical. It means we cannot choose objects. It does not matter which r objects we take as all objects are identical.

The number of ways to select 1 object from n identical objects = 1

The number of ways to select 2 object from n identical objects = 1

… … …

… … …

… … …

The number of ways to select n objects from n identical objects = 1.

Combining all above cases, we get

Total number of ways to select 1 or more objects fron n identical objects

$$= 1 + 1 + \cdots n \text{ times} = n$$

Notes:

1. The number of ways to select r objects from n identical objects is 1.
2. The number of ways to select 0 or more objects from n identical objects $= n + 1$.
3. The number of ways to select at least 2 objects from n identical objects $= n - 1$.
4. The number of ways to select atleast r objects from n identical objects is $n - (r - 1) = n - r + 1$
5. The total number of selections of some or all out of $(p + q + r)$ objects where p are alike of one kind, q are alike of second kind and rest r are alike of third kind is $(p + 1)(q + 1)(r + 1) - 1$. [Using fundamental principle of counting]

7.3.4.4.3 From Objects Which are not All Distinct from Each Other

The number of ways to select one or more objects from $(p + q + r \cdots + n)$ objects where p objects are alike of one kind, q are alike of second kind, r are alike of third kind, … and remaining n are distinct from each other

$$= [(p + 1)(q + 1)(r + 1) \cdots 2^n] - 1.$$

Logic:

The numbers of ways to select 0 or more objects from p alike objects of one kind $= p + 1$

The number of ways to select 0 or more objects from q alike objects of second kind $= q + 1$

The number of ways to select 0 or more objects from r alike objects of third kind $= r + 1$

… … …

… … …

… … …

The number of ways to select 0 or more objects from n distinct objects $= 2^n$

Combining all cases and using fundamental principle of counting, we get:

Total number of ways to select 0 or more objects

$$= [(p + 1)(q + 1)(r + 1) \ldots]2^n \quad (1)$$

But above selection includes a case where we have not selected any object. So we need to subtract 1 from the above result if we want to select at least one object.

Therefore, the total number of ways to select one or more objects (at least one) from p alike of one kind, q alike of another kind, r alike to third kind … and n distinct objects

$$= [(p + 1)(q + 1)(r + 1) \ldots]2^n - 1$$

Notes:

1. The number of ways to select 0 or more objects from p alike of one kind, q alike of second kind, r alike of third kind and n distinct objects $= (p + 1)(q + 1)(r + 1)\, 2^n$.
2. The number of ways to select objects from p alike of one kind, q alike of second kind and r alike of third kind and n distinct objects such that selection includes at least one object each of first, second, and third kind and atleast one object from n different kind $= pqr(2^n - 1)$.
3. The number of ways to select objects from p alike of one kind, q alike of second kind and r alike of third kind and n distinct objects such that selection includes at least one object of each kind $= pqr$.

Example 39 *A man has 5 friends. In how many ways can he invite one or more of them to a party?*

Solution: If he invites one person to the party, number of ways $= {}^5C_1$

If he invites two persons to the party, number of ways $= {}^5C_2$

Proceeding on the similar pattern, total number of ways to invite

$$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$$
$$= 5 + 10 + 10 + 5 + 1 = 31$$

Alternate Method:

To invite one or more friends to the party, he has to take 5 decisions, one for every friend.

Each decision can be taken in two ways, invited or not invited.

Hence, the number of ways to invite one or more

$$= (\text{number of ways to make 5 decisions - 1})$$
$$= 2 \times 2 \times 2 \times 2 \times 2 - 1 = 2^5 - 1 = 31$$

Note that we have to subtract 1 to exclude the case, when all are not invited.

Example 40 *Prove that there are* $2(2^{n-1} - 1)$ *ways of dealing n distinct cards to two persons.* (*The players may receive unequal number of cards and each one receiving at least one card*).

Solution: Let us number the cards for the moment. Let us accept the case where all the cards go to one of the two players, also with just two cards, we have the possibilities,

$$\text{AA} \quad \text{AB} \quad \text{BA} \quad \text{BB} \quad (1)$$

Here, AA means A gets card 1 and also card 2,

AB means A gets card 1 and B gets card 2,

BA means B gets card 1 and A gets card 2, BB means B gets card 1 and also card 2.

Thus, for two cards we have four possibilities.

For three cards

$$\text{AAA, ABA, BAA, BBA, AAB, ABB, BAB, BBB} \tag{2}$$

That is, for three cards there are $2^3 = 8$ possibilities. Here, if the third card goes to A, then, in Eq. (1) annex A at the end, thus getting

AAA, ABA, BAA, BBA.

Thus, the possibilities doubled, when a new card (third card) is included.

In fact just with one card it may either go to A or B.

By annexing the second card, it may give

AA	BA	giving (1)
AB	BB	

Thus, every new card doubles the existing number of possibilities of distributing the cards.

Hence, the number of possibilities with n cards is 2^n. But this includes the 2 distributions where one of them gets all the cards, and the other none.

So, total number of possibilities is $2^n - 2 = 2(2^{n-1} - 1)$.

Note: We can look at the same problem in the following way. The above distribution of cards is the same as number of possible n-digit numbers where only two digits 1 and 2 are used, and each digit must be used at least once. This is $2^n - 2 = 2(2^{n-1} - 1)$.

Aliter: Since n cards are dealt with and each player must get at least one card, player 1 can get r cards and player 2 get $(n - r)$ cards where $1 \le r \le n - 1$. Now, player 1 can get r cards in $C(n, r)$ ways. Total number of ways of dealing cards to players 1 and 2

$$= \sum_{r=1}^{n-1} C(n, r) = \sum_{r=0}^{n} C(n, r) - C(n, 0) - C(n, n) = 2^n - 2.$$

Example 41 *Find the number of ways in which one or more letters can be selected from the letters:*

A A A A B B B C D E

Solution: The given letters can be divided into five following categories: (AAAA), (BBB), C, D, E

To select at least one letter, we have to take five decisions—one for every category. Selections from (AAAA) can be made in 5 ways: include no A, include one A, include AA, include AAA, include AAAA.

Similarly, selections from (BBB) can be made in 4 ways, and selections from C, D, E can be made in $2 \times 2 \times 2$ ways.

$\Rightarrow$ Total number of selections $= 5 \times 4 \times (2 \times 2 \times 2) - 1 = 159$

(excluding the case when no letter is selected).

Example 42 *The question paper in the examination contains three sections: A, B, C. There are 6, 4, 3 questions in sections A, B, C respectively. A student has the freedom to answer any number of questions attempting at least one from each section. In how many ways can the paper be attempted by a student?*

Solution: There are three possible cases:

Case 1: Section A contains 6 questions. The student can select at least one from these in $2^6 - 1$ ways.

Case 2: Section B contains 4 questions. The student can select at least one from these in $2^4 - 1$ ways.

Case 3: Section C can similarly be attempted in $2^3 - 1$ ways.

Hence, total number of ways to attempt the paper

$$= (2^6 - 1)(2^4 - 1)(2^3 - 1)$$
$$= 63 \times 15 \times 7 = 6615.$$

Example 43 *Find the number of factors (excluding 1 and the expression itself) of the product of $a^7 b^4 c^3 d e f$ where a, b, c, d, e, f are all prime numbers.*

Solution: A factor of expression $a^7 b^4 c^3 d e f$ is simply the result of selecting none or one or more letters from 7 a's, 4 b's, 3 c's, d, e, f

The collection of letters can be observed as a collection of 17 objects out of which 7 are alike of one kind (a's), 4 are of second kind (b's), 3 are of third kind (c's) and 3 are distinct (d, e, f).

The number of selections $= (1 + 7)(1 + 4)(1 + 3)\, 2^3 = 8 \times 5 \times 4 \times 8 = 1280$.

But we have to exclude two cases :

(i) When no letter is selected, (ii) When all letters are selected.

Hence the number of factors $= 1280 - 2 = 1278$.

Example 44 *Find the number of positive divisors of $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r}$, where p_1, $p_2, \ldots, p_r$ are distinct prime numbers and $k_1, k_2, \ldots, k_r$ are positive integers.*

Solution: A divisor d of n is of the form

$$d = p_1^{l_1} \cdot p_2^{l_2} \cdots p_r^{l_r} \text{ where } 0 \le l_i \le k_i,\ i = 1, 2, \ldots, r.$$

Associate each divisor d of n with an r tuple $(l_1, l_2, \ldots, l_r)$ such that $0 \le l_i \le k_i$. Therefore, the number of divisors is the same as the number of r tuples $(l_1, l_2, \ldots, l_r)$, $0 \le l_i \le k_i$, $i = 1, 2, \ldots, r$.

Since l_1, can have $k_1 + 1$ possible values $0, 1, 2, \ldots, k_1$ similarly l_2, can have $k_2 + 1$ values and so on. The number of r-triples $(l_1, l_2, \ldots, l_r)$ is

$$(k_1 + 1) \times (k_2 + 1) \times (k_3 + 1) \times \cdots \times (k_r + 1) = \prod_{i=1}^{r} (k_i + 1)$$

That is the total number of divisors of

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r} \text{ is } (k_1 + 1)(k_2 + 1)\cdots(k_r + 1) = \prod_{i=1}^{r} (k_i + 1).$$

Note: Also refer article 6.6 on page 6.13 of number theory chapter.

7.3.4.5 Selection of r Objects from n Objects when All n Objects are not Distinct

In this problem type we will discuss how to select r objects from n objects when all n objects are not distinct.

For example, selection of 3 letters from letters AABBBC.

To find number of ways to select, it is possible to derive a formula that can be applied in all such cases.

Instead of formula, we will discuss a method (procedure) that should be applied to find selections.

The method involves making cases based on alike items in the selection. You should go through the following examples to learn how to apply this 'method of cases' to find selections of r objects from n objects when all n objects are not distinct.

Example 45 *In how many ways* 3 *letters can be selected from letters AABBBC.*

Solution: The given letters include AA, BBB, C, *i.e.*, 2A letters, 3B letters and 1C letter.

To find number of selections, we will make the following cases based on alike letters we choose in the selection.

Case 1: All 3 letters are alike

3 alike letters can be selected from given letters in only 1 way, *i.e.*, BBB.

$\Rightarrow$ The number of selections with all 3 letters alike = 1 (1)

Case 2: 2 alike and 1 distinct letter

2 alike letters can be selected from 2 sets of alike letters (AA, BB) in ${}^2C_1$ ways.

1 distinct letter (distinct from selected alike letters) can be selected from remaining letters in ${}^2C_1$ ways. (either A or B).

Using fundamental principle of counting, total number of selections with 2 alike and 1 distinct letter

$$= {}^2C_1 \times {}^2C_1 = 4 \text{ ways} \quad (2)$$

Case 3: All letters distinct

All 3 letters distinct can be selected from 3 distinct letters (A, B, C) in 1 way.

$\Rightarrow$ Total number of ways to select 3 distinct letters is 1 way (3)

Combining (1), (2) and (3).

Total number of ways to select 3 letters from given letters = 1 + 4 + 1 = 6.

Example 46 *In how many ways* 4 *letters can be selected from the letters of the word INEFFECTIVE?*

Solution: INEFFECTIVE contains 11 letters: EEE, FF, II, C, T, N, V

We will make following cases to select 4 letters.

Case 1: 3 alike and 1 distinct

3 alike letters can be selected from 1 set of 3 alike letters (EEE) in 1 way.

$\Rightarrow$ The number of ways to select 3 alike letters = 1

$\Rightarrow$ The number of ways to select 1 distinct letters = 6

$\Rightarrow$ Total ways = 6 × 1 = 6 (1)

Case 2: 2 alike and 2 alike

'2 alike and 2 alike' means we have to select 2 groups of 2 alike letters (EE, FF, II) in ${}^3C_2$ ways.

$\Rightarrow$ The number of ways to select '2 alike and 2 alike' letters = ${}^3C_2 = 3$. (2)

Case 3: 2 alike and 2 distinct

1 group of 2 alike letters can be selected from 3 sets of 2 alike letters (EE, FF, II) in ${}^3C_1$ ways.

2 distinct letters can be selected from 6 distinct letters (C, T, N, V, remaining 2 sets of two letters alike) in ${}^6C_2$ ways.

The number of ways to select '2 alike and 2 distinct letters'

$${}^3C_1 \times {}^6C_2 = 3 \times 15 = 45 \quad (3)$$

Case 4: All distinct letters

All distinct letters can be selected from 7 distinct letters (I, E, F, N, C, T, V) in ${}^7C_4$ ways.

$\Rightarrow$ The number of ways to select all distinct letters = ${}^7C_4 = 35$ (4)

Combining (1), (2), (3), and (4), we get,

Total number of ways to select 4 letters from the letter of the word 'INEFFECTIVE' = 6 + 3 + 45 + 35 = 89.

Example 47 *In how many ways a child can select 5 balls from 5 red, 4 black, 3 white, 2 green, 1 yellow balls? (Assume balls of the same colour are identical)*

Solution: It is given that child can select 5 balls from RRRRR BBBB WWW GG Y balls. We will make following cases:

(i) **All alike:**
There is one group of all alike balls (5 red balls)
$\Rightarrow$ Number of ways to choose 1 group = ${}^1C_1 = 1$

(ii) **4 alike and 1 distinct:**
There are 2 groups of 4 alike balls (4 red balls, 4 black balls) and after selecting one group, there are 4 distinct balls left from where we require to choose one ball.
$\Rightarrow$ Number of ways to select **'4 alike and 1 distinct'** = ${}^2C_1 \times {}^4C_1 = 8$

(iii) **3 alike and 2 alike:**
Select 3 alike balls from 3 groups of 3 alike balls (RRR, BBB, WWW) in ${}^3C_1$ ways.
Then select 2 alike balls from remaining 3 groups of 2 alike balls in ${}^3C_1$ ways.
$\Rightarrow$ Number of ways to select **'3 alike and 2 alike'**
$= {}^3C_1 \times {}^3C_1 = 9$

(iv) **3 alike and 2 distinct:**
Select one group of 3-alike balls from 3 groups of 3-alike balls in ${}^3C_1$ ways. Select 2 balls from remaining 4 distinct balls in ${}^4C_2$ ways.
$\Rightarrow$ Number of ways to select **'3 alike and 2 distinct'**
$= {}^3C_1 \times {}^4C_2 = 18$

(v) **2 alike, 2 alike and 1 distinct:**
Select 2 groups of 2-alike balls from 4 groups of 2-alike balls in ${}^4C_2$ ways. Further select 1 ball from remaining 3 distinct balls in ${}^3C_1$ ways.
$\Rightarrow$ Number of way to select **'2 alike, 2 alike and 1 distinct'**
$= {}^4C_2 \times {}^3C_1 = 18$

(vi) **2 alike and 3 distinct:**
Select one group of 2-alike balls from 4 groups of 2-alike balls in ${}^4C_1$ ways. Then select 3 balls from remaining 4 distinct balls in ${}^4C_3$ ways.
$\Rightarrow$ Number of ways to select **'2 alike and 3 distinct'**
$= {}^4C_1 \times {}^4C_3 = 16$

(vii) **All distinct:**
Select 5 distinct balls from 5 distinct balls (R, B, W, G, Y) in ${}^5C_5$ ways.
$\Rightarrow$ Number of ways to select **'All distinct'**
$= {}^5C_5 = 1$.
Combining all above cases, total number of ways in which child can select 5 balls = 1 + 8 + 9 + 18 + 18 + 16 + 1 = 71 ways.

7.3.4.6 Occurrence of Order in Selection

If n objects are chosen as 'first $(n-1)$ objects are chosen and then nth object' or 'n objects are chosen one by one' then always ordered selections are made and hence the repetitions. So in the final count, these repetitions are to be deleted.

Example 48 *In how many ways we can select two unit square on an ordinary chess board such that both square neither in same row nor in same column.*

Solution: First square is selected in 64 ways.
After selection of first, we can't select any of the remaining 7 squares which are in the same row with first square and similarly we cannot select any of remaining 7 squares which are in the same column with first square. So number of choices for second square is $64 - 1 - 7 - 7 = 49$.

Hence, apparently, by multiplication principle, number of ways $= 64 \times 49$.

But in this count, repetitions occurred. In fact, each selection is counted twice.

So final answer $= \dfrac{64 \times 49}{2} = 1568$ ways.

Example 49 *Find the number of pairings of a set of* $2n$ *elements* [*e.g.,* {(1, 2), (3, 4), (5, 6)} {(1, 3), (2, 4), (5, 6)} *are two pairings of the set* {1, 2, 3, 4, 5, 6}].

Solution: Let $A = \{1, 2, 3, 4, \ldots, 2n - 1, 2n\}$.
A pair having 1 as one element (out of the two elements) can be obtained in $(2n - 1)$ ways. Say, selected element is k (Assuming $k \neq 2$). Similarly a pair having 2 as one element (out of two elements), can be obtained in $2n - 3$ ways, etc.

Number of pairings $= (2n - 1)(2n - 3)(2n - 5) \cdots 3 \cdot 1$

Aliter: First pair can be obtained in ${}^{2n}C_2$ ways. Second pair can be obtained in ${}^{2n-2}C_2$ ways.

Third pair can be obtained in ${}^{2n-4}C_2$ ways.

⋮

nth pair can be obtained in ${}^{2}C_2$ ways.

Apparently, by multiplication principle,
number of pairings $= {}^{2n}C_2 \cdot {}^{2n-2}C_2 \cdots {}^{2}C_2$.

But in this count, too many repetitions have been counted. In fact, each pairing is counted $n!$ times.

Required number $= \dfrac{{}^{2n}C_2 \cdot {}^{2n-2}C_2 \cdots {}^{4}C_2 \cdot {}^{2}C_2}{n!}$

(Verify this number is same as $(2n - 1)(2n - 3)(2n - 5) \cdots 3 \cdot 1$)

7.3.4.7 Points of Intersection between Geometrical Figures

We can use ${}^{n}C_r$ (number of ways to select r objects from n different objects) to find points of intersection between geometrical figures.

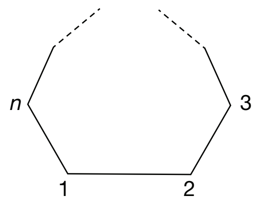
For example:

1. Number of points of intersection of 'n' non-concurrent and non parallel lines is ${}^{n}C_2$.
   Logic: When two lines intersect, we get a point of intersection. Two lines from n distinct lines can be selected in ${}^{n}C_2$ ways. Therefore, number of points of intersection is ${}^{n}C_2$.
2. Number of lines that can be drawn, passing though any 2 points out of n given points in which no three of them are collinear, is ${}^{n}C_2$.
   Logic: A line can be drawn through two points. Two points can be selected from n distinct points in ${}^{n}C_2$ ways. Therefore, number of lines that can be drawn is ${}^{n}C_2$.
3. Number of triangles that can formed, by joining any three points out of n given points in which no three of them are collinear is ${}^{n}C_3$.
   Logic: A triangle is formed using 3 different points. Three points can be selected from n distinct points in ${}^{n}C_3$ ways. Therefore, we can form ${}^{n}C_3$ triangles using n distinct points.

4. Number of diagonals that can be drawn in a 'n' sided polygon is $\dfrac{n(n-3)}{2}$.

   Logic: There are n vertices in a n sided polygon. When two vertices are joined (excluding the adjacent vertices), we get a diagonal. The number of ways to select 2 vertices from n vertices is ${}^nC_2$. But this also includes n sides (when adjacent vertices are selected). Therefore number of diagonals

$$= {}^nC_2 - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$



Aliter: $n-3$ diagonals emerge from each vertex. For example, from vertex named 1, $n-3$ diagonals emerge whose other ends are vertices 3, 4, …, $n-1$.

Number of diagonals apparently, by multiplication principle, is $n(n-3)$ but each diagonal is counted twice.

Required number $= \dfrac{n(n-3)}{2}$.

$\left(\text{Verify that, } {}^nC_2 - n \text{ is same as } \dfrac{n(n-3)}{2}.\right)$

Example 50 *How many triangles can be formed by joining the vertices of a hexagon?*

Solution: Let $A_1, A_2, A_3, \ldots, A_6$ be the vertices of the hexagon. One triangle is formed by selecting a group of 3 points from 6 given vertices.
Number of triangles = Number of groups of 3 each from 6 points.

$$= {}^6C_3 = \frac{6!}{3!\,3!} = 20.$$

Example 51 *There are* 10 *points in a plane, no three of which are in the same straight line, except* 4 *points, which are collinear. Find the*

(i) *number of straight lines obtained from the pairs of these points;*
(ii) *number of triangles that can be formed with the vertices as these points.*

Solution:

(i) Number of straight lines formed joining the 10 points, taking 2 at a time

$$= {}^{10}C_2 = \frac{10!}{2!8!} = 45$$

Number of straight lines formed by joining the four points (which are collinear), taking 2 at a time $= {}^4C_2 = \dfrac{4!}{2!2!} = 6$

But, 4 collinear points, when joined pairwise give only one line.
So, required number of straight lines = 45 – 6 + 1 = 40.

(ii) Number of triangles formed by joining the points, taking 3 at a time

$$= {}^{10}C_3 = \frac{10!}{3!7!} = 120$$

Number of triangles formed by joining the 4 points (which are collinear), taken 3 at a time $= {}^4C_3 = 4$.
But, 4 collinear points cannot form a triangle when taken 3 at a time.
So, required number of triangles = 120 – 4 = 116.

Example 52 *There are* 12 *points in a plane,* 5 *of which are concyclic and out of remaining* 7 *points, no three are collinear and none concylic with previous* 5 *points. Find the number of circles passing through at least* 3 *points out of* 12 *given points.*

Solution: Consider Set A consists of 5 concyclic points. Set B consists of remaining 7 points.

Case 1: Circle passes through 3 points of set B
Number of circles = ${}^7C_3$

Case 2: Circle passes through 2 points of set B and one point of set A
Number of circles = ${}^7C_2 \cdot {}^5C_1$

Case 3: Circle passes through 1 point of set B and two points of set A
Number of circles = ${}^7C_1 \cdot {}^5C_2$

Case 4: Circle passes through no point from set B.
Number of circles = 1

All 4 cases are ***exhaustive and mutually exclusive***.

So, total number of circles

$$= {}^7C_3 + {}^7C_2 \cdot {}^5C_1 + {}^7C_1 \cdot {}^5C_2 + 1$$

$$= \frac{7!}{3!4!} + \frac{7!}{2!5!}.5 + 7.\frac{5!}{2!3!} + 1$$

$$= \frac{7.6.5}{1.2.3} + \frac{7.6}{1.2}.5 + 7.\frac{5.4}{1.2} + 1$$

$$= 35 + 105 + 70 + 1$$

$$= 211.$$

Aliter: Select three points out of 12 in ${}^{12}C_3$ ways. This number includes the number of circles obtained from 3 points out of 5 concyclic points. Note that we get the same circle by selecting any three points out of 5 concyclic points but we count it ${}^5C_3$ times.

Required number = ${}^{12}C_3 - {}^5C_3 + 1$
= 211.

Example 53 *In a plane there are* 37 *straight lines, of which* 13 *pass through the point A and* 11 *pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.*

Solution: The number of points of intersection of 37 straight lines is ${}^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore instead of getting ${}^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore instead of getting ${}^{11}C_2$ points, we get only one point B. Hence, the number of intersection points of the lines is ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535$.

Example 54 *l_1 and l_2 are two parallel lines; m and n are the points on l_1 and l_2, respectively. Find the number of triangles that could be constructed using these points as vertices.*

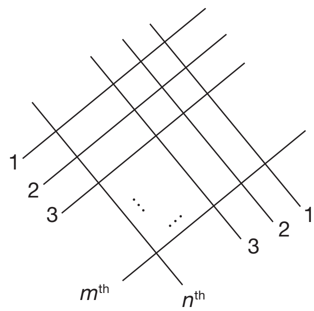
Solution: Any two points on l_1 and a point on l_2 form a triangle; again any two points on l_2 and a point on l_1 also form a triangle.
2 points can be chosen in ${}^mC_2$ ways from m points of l_1 and we have n choices for a point on l_2 and similarly, 2 points can be chosen in ${}^nC_2$ ways from n points of l_2 and in m ways we can choose a point on l_1,

Therefore, the number of triangles formed is given by

$$^{m}C_2 \times n + {}^{n}C_2 \times m = n \times \frac{m(m-1)}{2} + m \times \frac{n(n-1)}{2} = \frac{mn}{2}(m+n-2).$$

Example 55 *If m parallel lines in plane are intersected by a family of n parallel lines. Find the number of parallelogram formed.*

Solution: A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.
Two straight lines from the set of m parallel lines can be chosen in $^{m}C_2$ ways and two straight lines from the set of n parallel lines can be chosen in $^{n}C_2$ ways. Hence, the number of parallelograms formed.



$$= {}^{m}C_2 \times {}^{n}C_2 = \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn(m-1)(n-1)}{4}$$

Example 56 *In a plane, a set of* 8 *parallel lines intersects a set of n other parallel lines, giving rise to* 420 *parallelograms* (*many of them overlap with one another*). *Find the value of n.*

Solution: If two lines which are parallel to one another (in one direction) intersect another two lines which are parallel, we get one parallelogram. Thus, we can choose $C(8, 2)$ pairs of parallel lines in one direction and the number of parallel lines intersecting there will be $C(n, 2)$ pairs.

So, the number of parallelograms thus obtained is

$$C(n, 2) \times C(8, 2) = 420$$

$$\Rightarrow \quad \frac{n(n-1)}{1.2} \times \frac{8 \times 7}{1.2} = 420$$

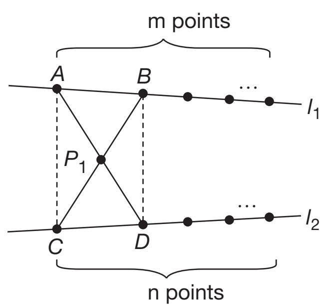
$$\Rightarrow \quad n(n-1) = 30$$

$$\Rightarrow \quad n = 6 \text{ (or } n = -5\text{, which is not admissible)}$$

Thus $n = 6$ is the solution.

Example 57 *Prove that, if each of the m points in one straight line be joined to each of the n points by straight lines terminated by the points then excluding the given points, these lines will intersect in* $\frac{1}{4}mn\,(m-1)(n-1)$ *points.*

Solution: Two straight lines intersect in one point.



So to get one point of intersection, we require two points on the first line (l_1) and two points on the second line (l_2).
For joining A of l_1 to C and D of l_2, they intersect in A, which is not counted as the required point. However, AD and CB intersect at the point P_1, AC and BD intersect only when extended which is also not counted as the required point. Thus to get an intersection, other than the points in l_1 and l_2, we should take two points from each of l_1 and l_2 and joined them in cross pattern.

The number of ways we can choose two points from l_1 in which m points are plotted, is $^{m}C_2$. Similarly, we can choose two points from l_2 in $^{n}C_2$ ways. For each pair of points from l_1 and l_2, we get one point of intersection.

So, the total number of points when there are $^{m}C_2$ pairs from l_1 and $^{n}C_2$ pairs from l_2 is

$$^mC_2 \times {}^nC_2 = \frac{m(m-1)}{1.2} \times \frac{n(n-1)}{1.2}$$

$$= \frac{1}{4} mn(m-1)(n-1).$$

Example 58 *Let there be n concurrent lines and another line parallel to one of them. Find the number of different triangles that will be formed by the* $(n + 1)$ *lines.*

Solution: The number of triangles = Number of selections of 2 lines from the $(n - 1)$ lines which are cut by the last line

$$= {}^{n-1}C_2 = \frac{(n-1)!}{2!(n-3)!} = \frac{(n-1)(n-2)}{2}.$$

Example 59 *Out of 18 points in a plane no three are in the same straight line except five points which are collinear. Find the number of straight lines that can be formed by joining any two of them.*

Solution: The number of straight lines = $^{18}C_2 - ({}^5C_2 - 1) = 144$.

Example 60 *There are p points in a plane, no three of which are in the same straight line with the exception of q, which are all in the same straight line. Find the number of*

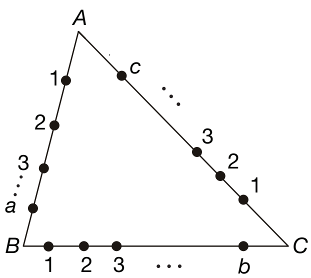
(i) *straight lines*
(ii) *triangles which can be formed by joining them.*

Solution:

(i) If no three of the p points were collinear, the number of straight lines = Number of groups of two that can be formed from p points = $^pC_2$.
Due to the q points being collinear, there is a loss of $^qC_2$ lines that could be formed from these points.
But these points are giving exactly one straight line passing through all of them.
Hence, the number of straight lines = $^pC_2 - {}^qC_2 + 1$.

(ii) If no three points were collinear, the number of triangles = $^pC_3$
But there is a loss of $^qC_3$ triangles that could be formed from the group of collinear points.
Hence the number of triangles formed = $^pC_3 - {}^qC_3$.

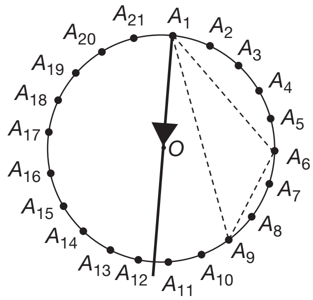
Example 61 *The sides AB, BC and CA of a triangle ABC have a, b and c interior points on them respectively then find the number of triangles that can be constructed using these interior points as vertices.*



Solution: Required number of triangles
= Total number of ways choosing 3 points
– Number of ways of choosing all the 3 points either from AB or BC or CA
$= {}^{a+b+c}C_3 - ({}^aC_3 + {}^bC_3 + {}^cC_3)$

Example 62 *Let* A_i, $i = 1, 2, \ldots, 21$ *be the vertices of a 21-sided regular polygon inscribed in a circle with centre O. Triangles are formed by joining the vertices of the 21-sided polygon. How many of them are acute-angled triangles? How many of them are right-angled triangles? How many of them are obtuse-angled triangles? How many of them are equilateral? How many of them are isosceles?*

Solution: Since this is a regular polygon with odd number of vertices, no two of the vertices are placed diagonally opposite, so there is no right-angled triangle. Hence number of right-angled triangle is zero. Let A be the number of acute-angled triangles. To form a triangle we need to choose 3 vertices out of the 21 vertices which can be done in $C(21,3)=\dfrac{21\times 20\times 19}{6}=1330$ ways. Since the triangles are either acute or obtuse, we get $A+O=1330$.



Let us find O, the number of obtuse angled triangles first.
Draw one diameter say passing through A_1. Now let us count all obtuse angle triangle on right side of the diameter and having one verfex at A_1. For these triangles we need two more vertex out of A_2 to A_{11}. Which can be seleted in $\binom{10}{2}$ ways.

Hence total number of obtuse angle triangles is $21\cdot\binom{10}{2}=945$

Now acute angle triangles

$A=1330-945$
$=385$

A triangle $A_iA_jA_k$ is equilateral if A_i, A_j, A_k are equally spaced.

Out of $A_1,\ldots,A_{21}$, we have only 7 such triplets $A_1A_8A_{15}$, $A_2A_9A_{16},\ldots, A_7A_{14}A_{21}$. Therefore, there are only 7 equilateral triangles.

Consider the diameter A_1OB where B is the point where A_1O meets the circle. If we have an isosceles triangle A_1 as its vertex then A_1B is the altitude and the base is bisected by A_1B. This means that the other two vertices, A_j and A_k, are equally spaced from B.

We have 10 such pairs, so we have 10 isosceles triangles with A_1 as vertex of which one is equilateral.

Because proper isosceles triangles with A_1, as vertex (non-equilateral) are 9, with each vertex A_i, $i=1, 2, \ldots, 21$ we have 9 such isosceles triangles.

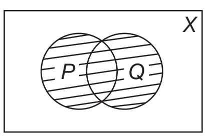
So, total number of isosceles but non-equilateral triangles are $9\times 21=189$. But the 7 equilateral triangles are also to be considered as isosceles.

$\therefore$ The total number of isosceles triangles is $189+7=196$.

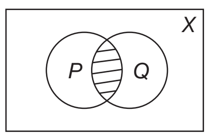
Note: This problem can be generalized to a regular polygon having n vertices. Find the number of acute, obtuse, right, isosceles, equilateral and scalene triangles.

7.3.4.8 Formation of Subsets

In these type of problems, we select elements from a given set to form subsets. We are supposed to form subsets under constraints. For example, two subsets P and Q are to be formed such that $P\cup Q$ has all elements, $P\cap Q$ has no elements, etc. To understand the problems based on this type, read the following examples carefully.

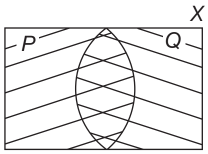


$P\cup Q$



$P\cap Q$

Example 63 *Let X be a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random then find the number of ways to form sets such that $P\cup Q=X$.*



Solution: As $P \cup Q = X$, it means every element would be either included in P or in Q or both so for every element, there are 3 choices.
$\Rightarrow$ Number of ways to select P and Q such that $(P \cup Q = X) = 3^n$.

Example 64 *Let X be a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to choose P and Q such that $P \cup Q$ contains exactly r elements.*

Solution: $P \cup Q$ has r elements. It means r elements out of n elements should be present in either P or in Q or in both. r elements out of n elements can be selected in ${}^nC_r$ ways.
Each of these r elements has 3 choices

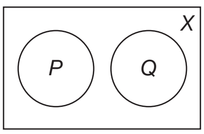
$\Rightarrow$ Number of ways to select elements of P and $Q = 3^r$

Each of remaining $(n - \text{r})$ elements has 1 choice, *i.e.*, neither belongs to P nor belongs to $Q \Rightarrow$ Number of ways $= 1^{n-r}$.

$\Rightarrow$ Number of ways to select P and Q such that $P \cup Q$ has exactly r elements

$$= {}^nC_r 3^r (1)^{n-r} = {}^nC_r 3^r.$$

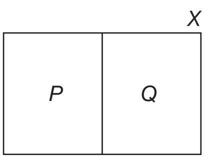
Example 65 *Let X be a set containing n elements. A subset P of set X is chosen at random . The set X then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P \cap Q$ is empty, i.e., $P \cap Q = \phi$.*



Solution: $P \cap Q = \phi$. It means P and Q should be disjoint sets. That is there is no element common in P and Q.
$\Rightarrow$ For every elements in set X there are 3 choices. Either it is selected in P but not in Q or selected in Q but not in P or not selected in both P and Q.

$\Rightarrow$ Number of ways to select P and Q such that $P \cap Q$ is $\phi = 3^n$.

Example 66 *Let X be a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P = \bar{Q}$.*



Solution: $P = \bar{Q}$ or Q^C. It means P and Q are complementary sets, *i.e.*, every element present in X is either present in P or Q.
$\Rightarrow$ For every element there are 2 choices to select. Either it will be selected for P or it will be selected for Q.

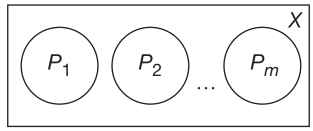
$\Rightarrow$ Number of ways to select $= 2^n$

Example 67 *Let X be a set containing n elements. A subset P_1 is chosen at random and then set X is reconstructed by replacing the elements of set P_1. A subset P_2 of X is now chosen at random and again set X is reconstructed by replacing the elements of P_2. This process is continued to choose subsets $P_3, P_4, P_5, \ldots, P_m$ where $m \geq 2$. Find numbers of ways to select sets such that:*

(i) $P_i \cap P_j = \phi$ *for $i \neq j$ and $i, j = 1, 2, \ldots, m$.*

(ii) $P_1 \cap P_2 \cap P_3 \cap \ldots \cap P_m = \phi$.

Solution:



(i) $P_i \cap P_j = \phi \ \forall \ i \neq j$
Every element in X has $(m + 1)$ choices because either it can be selected for P_1 or P_2 or P_3 or … or P_m or not get selected in any of the sets.
$\Rightarrow$ Number of favourable ways $= (m + 1)\ (m + 1) \ldots n$ times $= (m + 1)^n$

(ii) $P_1 \cap P_2 \cap P_3 \ldots \cap P_m = \phi$.
This means there is no element to be common to all sets $P_1, P_2, P_3 \ldots P_m$.
For each element out of $a_1, a_2, \ldots a_n$ there are $(2^m - 1)$ choices to get selected. It can be selected in any sets but not for all sets together so we subtract 1 from 2^m.
Total ways to select $P_1, P_2, P_3, \ldots, P_m$ such that $P_1 \cap P_2 \ldots \cap P_m = \phi$ is $(2^m - 1)^n$.

7.4 The Bijection Principle

Let A $= \{ a_1, a_2, \ldots, a_n \}$ and $B = \{b_1, b_2, \ldots, b_m \}$.

If $f : A \to B$ is an injective function then $n \leq m$.

If $f : A \to B$ is a surjective function then $n \geq m$.

If $f : A \to B$ is injective and surjective then f is known to be a bijective function. For a bijective function, $n = m$.

Example 68 *What is the total number of subsets of a set containing exactly n elements?*

Solution: It is a well known result, number of subsets $= 2^n$.
Let $S = \{a_1, a_2, a_3, \ldots, a_n \}$ be a set of exactly n elements.
Let P be the set of all subsets of S and Q be the set of all binary sequences of n elements.

Let $A \in P$. Let $f : P \to Q$ be a function that associates a binary sequence with A as follows:

$a_i \in A$, iff ith term of the sequence is 1.

For example, subset $\{a_2, a_4, a_{n-1} \}$ corresponds to binary sequence

$$\begin{array}{ccccccccc} 0 & \underline{1} & 0 & \underline{1} & 0 & 0 \ldots 0 & \underline{1} & 0 \\ & \text{2nd} & & \text{4th} & & & (n-1)\text{th} & \\ & \text{place} & & \text{place} & & & \text{place.} & \end{array}$$

Observe that, for every subset A, there is a binary sequence of n terms and for every binary sequence of n terms as stated above, there is a subset A of S.

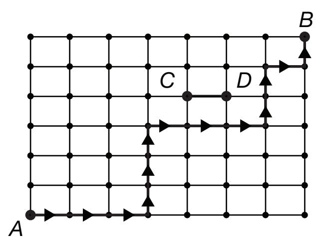
Therefore f is a bijection between P and Q.

Hence, the number of subsets = number of binary sequences $= 2^n$.

Example 69 *Consider a network as shown in the figure. Paths from* A to B consists of the horizontal or vertical line segments.

No diagonal movement is allowed. We can only move left to right or down to up. One sample path from A to B is shown.

(i) *How many paths are there from A to B?*
(ii) *How many paths go via C?*
(iii) *How many paths go via CD?*



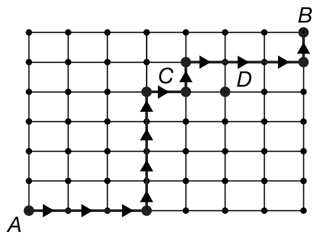
Solution: Assign 0 for horizontal line segment of one unit. Assign 1 for vertical line segment of one unit. For example, corresponding to the path shown in the figure, we can write one binary sequence as 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 1.

Note that there are 7 horizontal and 6 vertical line segments, of one unit each, in every path from A to B.

(i) Since, for every path between A and B, there is a binary sequence of 7, 0's and 6, 1's and for every sequence we can have corresponding one path made up of horizontal and vertical lines. Therefore there is bijection between the set of all paths from A to B and the set of all binary sequences of 7, 0's and 6, 1's.
$\Rightarrow$Number of paths between A and B = Number of binary sequences

= Number of ways to select 7 places to put 0 out of 13 different places $= \binom{13}{7}$

$= \dfrac{13!}{7!6!}$

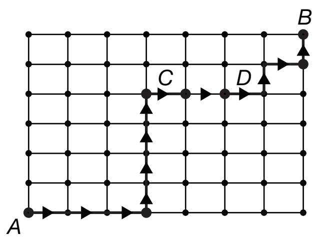


(ii) Number of paths through C
= (Number of paths from A to C) × (Number of paths from C to B)
= Number of ways to select 4 places to put 0 out of first 8 different places × Number of ways to select 3 places to put 0 out of next 5 different places

$= \binom{8}{4} \times \binom{5}{3}$

$= \dfrac{8!}{4!4!} \times \dfrac{5!}{3!\,2!}$

(Note that there are 4 horizontal and 4 vertical line segments of one unit each, in every path from A to C. There are 3 horizontal and 2 vertical line segments of one unit each in every path between C and B.)



(iii) Similarly number of paths from D to B $= \dfrac{4!}{2! \times 2!}$

(as there are 2 horizontal and 2 vertical line segments of one unit each in every path between B and D.)

Number of paths containing CD $= \dfrac{8!}{4! \times 4!} \times \dfrac{4!}{2! \times 2!}$.

Note: If a problem, similar to street network, but in three dimensions, is to be solved, we define *ternary sequences* consisting of 0's, 1's and 2's.
For example, number of paths between (0, 0, 0) and (3, 4, 6), consisting of line segments of one unit each in positive directions of the co-ordinate axes $= \dfrac{13!}{3!4!6!}$.

7.5 Combinations with Repetitions Allowed

Here we will discuss combinations of n different objects taken r at a time when each object can be repeated any number of times in a combination.

Suppose three different objects A, B, C are given. We have to select two objects from A, B, C and in our selection we can include A, B, C repeatedly any number of times. This selection can be done in following ways.

AA, BB, CC, AB, AC, BC, *i.e.*, 6 ways.

This number 6 cannot be obtained using formula ${}^nC_r$ as here repetition of objects is allowed. To find answer to this type of problem, where repetition of objects is allowed, we use the following formula:

Number of ways to select r objects from n different objects where each object can be selected any number of times is ${}^nH_r$.

$$^nH_r = \binom{n+r-1}{r}$$

Logic:

Let n different objects be numbered as 1, 2, 3, …, n.

And selected numbers be $a_1, a_2, a_3, \ldots, a_r$, such that

$$1 \le a_1 \le a_2 \le a_3 \le \cdots \le a_r \le n \qquad (1)$$

Here we allowed weak inequalities between a_i's, as numbers may be repeated which will correspond to repetition of objects.

Now consider another sequence,

$$a_1, a_2+1, a_3+2, \cdots, a_r+r-1 \qquad (2)$$

We can observe following properties in sequence (2):

1. Sequence is strictly increasing
2. Minimum and Maximum element in the sequence can be 1 and $n + r$ -1 respectively.
3. There are $\binom{n+r-1}{r}$ such sequence

(As any r numbers can be selected from 1 to $n + r - 1$)

Now there is a Bijection between sequence (1) and sequence (2)

Hence total number of sequence (1) is also $\binom{n+r-1}{r}$.

Example 70 *In how many ways a person can buy 5 icecreams from a shop in which four different flavours of icecreams are available.*

Solution: Here person can buy all five icecreams of same flavour or in any other combination, *i.e.*, any flavour can be taken 0 or 1 or 2 … or 5 times.

Hence our current problem is selection of 5 icecreams from 4 flavours with repetition allowed, so answer is

$$^4\mathrm{H}_5 = \binom{4+5-1}{5} = \binom{8}{5} = 56.$$

Build-up Your Understanding 2

1. **(a)** Find 'n' if (i) $^{2n}C_3 : {}^nC_2 = 12 : 1$ (ii) $^{25}C_{n+5} = {}^{25}C_{2n-1}$
   (b) Prove that $^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$ if $n > 7$.
2. Find the number of positive integers satisfying the inequality
   $^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \le 100$.
3. There are 20 questions in a questions paper. If no two students solve the same combination of questions but solve equal number of questions then find the maximum number of students who appeared in the examination.
4. In how many ways can 5 colours be selected out of 8 different colours including red, blue, and green
   (i) if blue and green are always to be included,
   (ii) if red is always excluded,
   (iii) if red and blue are always included but green excluded?

5. The kindergarten teacher has 25 kids in her class. She takes 5 of them at a time, to zoological garden as often as she can, without taking the same 5 kids more than once. Find the number of visits, the teacher makes to the garden and also the number of of visits every kid makes.
6. A teacher takes 3 children from her class to the zoo at a time as often as she can, but does not take the same three children to the zoo more than once. She finds that she goes to the zoo 84 more than a particular child goes to the zoo. Find the number of children in her class.
7. A team of four students is to be selected from a total of 12 students. Find the total number of ways in which team can be selected such that two particular students refuse to be together and other two particular students wish to be together only.
8. A women has 11 close friends. Find the number of ways in which she can invite 5 of them to dinner, if two particular of them are not on speaking terms and will not attend together.
9. Four couples (husband and wife) decide to form a committee of four members. Find the number of different committees that can be formed in which no couple finds a place.
10. Find the number of ways in which a mixed double tennis game can be arranged from amongst 9 married couple if no husband and wife plays in the same game.
11. Find the number of ways of choosing a committee of 2 women and 3 men from 5 women and 6 men, if Mr. A refuses to serve on the committee if Mr. B is a member and Mr. B can only serve, if Miss C is the member of the committee.
12. Find the number of ways in which we can choose 3 squares on a chess board such that one of the squares has its two sides common to other two squares.
13. Find the number of ways of selecting three squares on a chessboard so that all the three be on a diagonal line of the board or parallel to it.
14. 5 Indian and 5 American couples meet at a party and shake hands. If no wife shakes hands with her husband and no Indian wife shakes hands with a male, then find the number of hand shakes that takes place in the party.
15. A person predicts the outcome of 20 cricket matches of his home team. Each match can result either in a win, loss or tie for the home team. Find the total number of ways in which he can make the predictions so that exactly 10 predictions are correct.
16. A forecast is to be made of the results of five cricket matches, each of which can be a win, a draw or a loss for Indian team. Find
    (i) the number of different possible forecasts.
    (ii) the number of forecasts containing 0, 1, 2, 3, 4 and 5 errors respectively.
17. A forecast is to be made of the results of five cricket matches, each of which can be a win or a draw or a loss for Indian team.
    Let p = Number of forecasts with exactly 1 error
    q = Number of forecasts with exactly 3 errors and
    r = Number of forecasts with all five errors
    then prove that $2q = 5r$, $8p = q$, and $2(p + r) > q$.
18. In a club election the number of contestants is one more than the number of maximum candidates for which a voter can vote. If the total number of ways is which a voter can vote be 62, then find the number of candidates.
19. Every one of the 10 available lamps can be switched on to illuminate certain Hall. Find the total number of ways in which the hall can be illuminated.
20. In a unique hockey series between India and Pakistan, they decide to play on till a team wins 5 matches . Find the number of ways in which the series can be won by India, if no match ends in a draw.

21. There are n different books and p copies of each in a library. Find the number of ways in which one or more books can be selected.

22. A class has n students. We have to form a team of the students by including atleast two students and also by excluding atleast two students. Find the number of ways of forming the team.

23. If the $(n + 1)$ numbers $a_1, a_2, a_3, \ldots, a_{n+1}$, be all different and each of them is a prime number, then find the number of different factors (other than 1) of $a_1^m \cdot a_2 \cdot a_3 \cdots a_{n+1}$.

24. In a certain algebraical exercise book there are 4 examples on arithmetical progressions, 5 examples on permutation-combination and 6 examples on binomial theorem. Find the number of ways a teacher can select for his pupils atleast one but not more than 2 examples from each of these sets.

25. Find the number of straight lines that can be drawn through any two points out of 10 points, of which 7 are collinear.

26. n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. Find the number of different points at which these lines will cut each other.

27. Eight straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Find The number of parts into which these lines divides the plane.

28. In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon be 70 then find the number of diagonals of the polygon.

29. In a plane there are two families of lines $y = x + r$, $y = -x + r$, where $r \in \{0, 1, 2, 3, 4\}$. Find the number of squares of diagonals of the length 2 formed by the lines.

30. Find the number of triangles whose vertices are at the vertices of an octagon but none of whose side happen to come from the sides of the octagon.

31. Let there be 9 fixed points on the circumference of a circle . Each of these points is joined to every one of the remaining 8 points by a straight line and the points are so positioned on the circumference that atmost 2 straight lines meet in any interior point of the circle. Find the number of such interior intersection points.

32. A bag contains 2 Apples, 3 Oranges and 4 Bananas. Find the number of ways in which 3 fruits can be selected if atleast one banana is always in the combination (Assume fruit of same species to be alike).

33. Find the number of selections of four letters from the letters of the word ASSASSINATION.

34. Find the number of ways to select 2 numbers from $\{0, 1, 2, 3, 4\}$ such that the sum of the squares of the selected numbers is divisible by 5 (repetition of numbers is allowed).

35. Find the number of ways in which we can choose 2 distinct integers from 1 to 100 such that difference between them is at most 10.

36. If a set A has m elements and another set B has n elements then find the number of functions from A to B.

37. Let $A = \{x : x \text{ is a prime number and } x < 30\}$. Find the number of different rational numbers whose numerator and denominator belongs to A.

38. Find the number of all three elements subsets of the set $\{a_1, a_2, a_3, \ldots, a_n\}$ which contain a_3.

39. If the total number of m-element subsets of the set $A = \{a_1, a_2, a_3, \ldots, a_n\}$ is k times the number of m-elements subsets containing a_4, then find n.

40. A set contains $(2n + 1)$ elements. Find the number of subsets of the set which contains at most n elements.

41. Find the number of subsets of the set $A = \{a_1, a_2, \ldots, a_n\}$ which contain even number of elements.

42. 'A' is a set containing 'n' distinct elements. A subset P of 'A' is chosen. The set 'A' is reconstructed by replacing the elements of P. A subset 'Q' of 'A' is again chosen. Find the number of ways of choosing P and Q so that $P \cap Q$ contains exactly two elements.

43. Find the number of ways of choosing triplets (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, \ldots, n, n+1\}$.

44. Find the number of ways in which the number 94864 can be resolved as a product of two factors.

45. Find the sum of the divisors of $2^5 \cdot 3^4 \cdot 5^2$.

46. In the decimal system of numeration, find the number of 6-digits numbers in which the digit in any place is greater than the digit to the left to it.

47. Find the number of 3-digit numbers of the form xyz such that $x < y$ and $z \leq y$.

48. Find the total number of 6-digit numbers $x_1\, x_2\, x_3\, x_4\, x_5\, x_6$ having the property $x_1 < x_2 \leq x_3 < x_4 < x_5 \leq x_6$.

49. The streets of a city are arranged like the lines of a chess board. There are m streets running North to South and 'n' streets running East to West. Find the number of ways in which a man can travel from NW to SE corner going the shortest possible distance.

50. Let there be $n \geq 3$ circles in a plane. Find the value of n for which the number of radical centres, is equal to the number of radical axes. (Assume that all radical axes and radical centre exist and are different)

51. Rajdhani express going from Bombay to Delhi stops at 4 intermediate stations. 10 passengers enter the train during the journey (including Bombay and 4 intermediate stations) with ten distinct tickets of two classes. Find the number of different sets of tickets they may have.

52. Find the number of functions f from the set $A = \{0, 1, 2\}$ into the set $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \leq f(j)$ for $i < j$ and, $i, j \in A$.

53. Show that the number of ways of selecting n-objects out of $3n$-objects, n of which are alike and rest different is $2^{2n-1} + \binom{2n-1}{n-1}$.

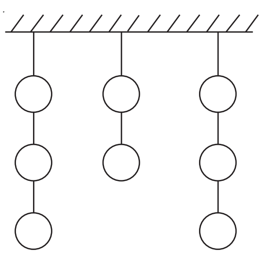
54. Use a combinatorial argument to prove that:
(i) ${}^{2n}C_2 = 2 \cdot {}^nC_2 + n^2$ (ii) $r \cdot {}^nC_r = n\, {}^{n-1}C_{r-1}$

55. Prove (combinatorially) that
${}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \cdots + n\,{}^nC_n = n\, 2^{n-1}$.

56. Prove (combinatorially) that
${}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^nC_r = {}^{n+1}C_{r+1}$, $r \leq n$.

57. In a chess tournament, each participant was supposed to play exactly one game with each of the others. However, two participants withdraw after having played exactly 3 games each, but not with each other. The total number of games played in the tournament was 84. How many participants were there in all?

58. A positive integer n is called strictly ascending if its digits are in the increasing order. For example, 2368 and 147 are strictly ascending but 43679 is not. Find the number of strictly ascending numbers $< 10^9$.



59. The given figure shows 8 clay targets, arranged in 3 columns, to be shot by 8 bullets. Find the number of ways in which they can be shot, such that no target is shot before all the targets below it, if any, are first shot.

60. How many hexagons can be constructed by joining the vertices of a quindecagon (15 sides) if none of the sides of the hexagon is also the side of the 15-gon.

7.6 Definition of Permutation (Arrangements)

A permutation of given objects is an arrangement of the objects in a line or row, unless specified otherwise. These arrangements can be generated by changing the relative positions of objects in the row. Every possible relative order between the objects is taken into account.

For example, if 3 objects are represented as A, B, C, then permutations (arrangements or orders) of A, B, C in a row can be done in the following ways:

ABC, BAC, CAB, ACB, BCA, CBA

It can be observed that these permutations of A, B, C in a row are made by changing relative positions of A, B, C among themselves.

The permutations of A, B, C can also be made by taking not all A, B, C at a time but by just taking 2 objects at a time. This can be done in the following ways;

AB, BA, BC, CB, CA, AC

It can be observed that first, 2 objects are selected and then they are permutated (ordered or arranged) in the row by changing their relative positions among themselves.

Similarly (2, 1, 3, 4, 5), (5, 2, 1, 4, 3), (1, 2, 5, 4, 3), etc. are permutations of 1, 2, 3, 4, 5.

7.6.1 Theorem I

(Number of Permutations (arrangements, order) of n distinct objects taken all at a time)

The total number of permutations of n **distinct** objects = $n!$

Proof:

Let us consider that we have n distinct objects say $a_1, a_2, a_3, \ldots, a_n$. We have to find total number of different permutations (arrangements or orders) of these objects along a row.

Every permutation of n objects is equivalent to fill n boxes (which are in a line) with these objects.

Let us consider n boxes

	1	2	3	4	5		$n-1$	n
Boxes:	$\square$	$\square$	$\square$	$\square$	$\square$	…	$\square$	$\square$
Ways:	n	$n-1$	$n-2$	$n-3$	$n-4$		2	1

Box-1 can be filled in n ways by any of the n objects $a_1, a_2, a_3 \ldots, a_n$.

Box-2 can be filled in $(n-1)$ ways by any of the remaining $(n-1)$ objects (excluding the object that has been used to fill Box-1).

Similarly, Box-3, Box-4, …, Box-n can be filled in $(n-2)$, $(n-3)$, …, 1 ways respectively.

Using fundamental principle of counting, total number of different ways to fill n boxes

$$= n\,(n-1)\,(n-2) \cdots 3 \cdot 2 \cdot 1$$
$$= n!$$

Hence, total number of permutation of n distinct objects is $n!$

Example 71 *Find number of different words which can be formed using all the letters of the word 'HISTORY'.*

Solution: Every way of arranging letters of the HISTORY will give us a word.
Therefore total number of ways to permutate letters H, I, S, T, O, R, Y, in a row
= Total number of words that can be formed using all letters together = 7!
$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
= 5040.

Example 72 *In how many way 5 distinct red balls, 3 distinct black balls and 2 distinct white balls can be arranged along a row?*

Solution: Total number of ways to arrange 10 balls along a row
= Number of permutations of 10 distinct objects in a row
$= \underline{|10}$.

Example 73 *In how many ways can the letters of the word 'DELHI' be arranged so that the vowels occupy only even places?*

Solution: All the letters in the word 'DELHI' are distinct with 2 vowels (E, I) and 3 consonants (D, L, H).
In five letter words, two even places can occupy 'E' and 'I' in 2! ways and remaining 3 places can occupy consonants D, L, H in 3! ways. So, number of words = (3!) × (2!) = 12.

Example 74

(i) *How many words can be made by using letters of the word COMBINE all at a time?*
(ii) *How many of these words begin and end with a vowel?*
(iii) *In how many of these words do the vowels and the consonants occupy the same relative positions as in COMBINE?*

Solution:

(i) The total number of words = arrangements of seven letters taken all at a time = 7! = 5040.
(ii) The corresponding choices for all the places are as follows:

Place	vowel						vowel
Number of choices	3	5	4	3	2	1	2

As there are three vowels (O, I, E), first place can be filled in three ways and the last place can be filled in two ways. The rest of the places can be filled in 5! ways by five remaining letters.
Number of words = 3 × 5! × 2 = 720.
(iii) Vowels should be at second, fifth and seventh positions.
They can be arranged in 3! ways.
Consonants should be at first, third, fourth and sixth positions.
They can be arranged here in 4! ways.
Total number of words = 3! × 4! = 144.

Example 75

(i) *How many words can be formed using letters of the word EQUATION taken all at a time?*
(ii) *How many of these begin with E and end with N?*
(iii) *How many of these end and begin with a consonants?*
(iv) *In how many of these, vowels occupy the first, third, fourth, sixth and seventh positions?*

Solution:

(i) Number of arrangements taken all at a time = 8! = 40320
$\Rightarrow$ 40320 words can be formed.

(ii) **Places:** **E _ _ _ _ _ _ N**
Choices: **1 6 5 4 3 2 1 1**
Number of words = $1 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times 1$
$= 6! = 720$ words can be formed.

(iii) There are three consonants and five vowels.
Places: **_ _ _ _ _ _ _ _**
Choices: **3 6 5 4 3 2 1 2**

- First place can be filled in three ways, using any of the three consonants (T, Q, N).
- Last place can be filled in two ways, using any of the remaining two consonants.
- Remaining places can be filled by using remaining six letters

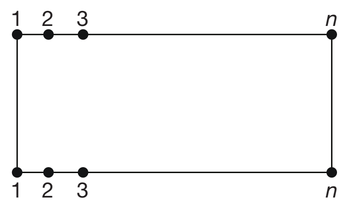
Number of words = $3 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times 2$
$= 3 \times (6!) \times 2 = 4320$ words.

(iv) Let v: vowels and c: constants
Places: **v c v v c v v c**
Choices: **5 3 4 3 2 2 1 1**

- First, put the vowels in the corresponding places in $5 \times 4 \times 3 \times 2 \times 1 = 5!$ ways
- Put the consonants in remaining three places in $3 \times 2 \times 1 = 3!$ ways

$\Rightarrow$ Number of words = $5!\ 3! = 120 \times 6 = 720$.

Example 76 *$2n$ people (including A and B) are to be seated across a table, n people on each side (as shown in the figure). Find the number of arrangements so that A, B are neither next to each other nor directly opposite each other.*



Solution:

Case 1: 'A' at a corner seat
Options available for A = 4
Options available for B = $2n - 3$
Number of arrangements = $4 \times (2n - 3) \times (2n - 2)$!
(Note that remaining $2n - 2$ people in the remaining seats can be seated in $(2n - 2)!$ ways)

Case 2: 'A' not in a corner seat
Options available for A = $2n - 4$
Options available for B = $2n - 4$
Number of arrangements = $(2n - 4) \times (2n - 4) \times (2n - 2)!$
Using addition principle, total number of arrangements
$= 4 \times (2n - 3) \times (2n - 2)! + (2n - 4)^2\,(2n - 2)!$
$= (4n^2 - 8n + 4)\,(2n - 2)!$
$= 4(n - 1)^2\,(2n - 2)!$

7.6.2 Theorem 2

(Number of Permutations (arrangements, order) of n distinct objects taken r at a time)

The total numbers of permutations of r objects, out of n distinct objects, is $\dfrac{n!}{(n-r)!}$, $1 \le r \le n$.

This number is denoted as ${}^nP_r$ or $P(n, r)$ or ${}^nA_r$ or $A(n, r)$

Proof:

Let us consider that we have n different objects say $a_1, a_2, a_3, \ldots, a_n$. We have to find number of different permutations (arrangements or orders) of these objects taken only r at a time. (*i.e.*, we have to select r objects and arrange them).

Every arrangement of n objects taken r at a time is equivalent to fill r boxes.

Let us consider r boxes as shown in the figure:

	1	2	3	4	5		r
Boxes:	☐	☐	☐	☐	☐	…	☐
Ways:	n	$n-1$	$n-2$	$n-3$	$n-4$		$n-(r-1)$

Box-1 can be filled in n ways by any of the n objects $a_1, a_2, a_3, \ldots, a_n$.

Box-2 can be filled in $(n-1)$ ways by any of the remaining $(n-1)$ objects (excluding the one that is used to fill Box-1).

Similarly, boxes 3, 4, 5, …, rth can be filled in $(n-2), (n-3), \ldots, n-(r-1)$ ways respectively.

Using fundamental principle of counting, total number of ways to fill r boxes

$= n(n-1)(n-2)(n-3)\ldots(n-r+1)$

Multiply and divide by $\underline{|n-r}$ to get,

Number of ways to permutate n things taken r at a time

$$= \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1))\underline{|n-r}}{\underline{|n-r}}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{\underline{|n-r}}$$

$$= \frac{\underline{|n}}{\underline{|n-r}} \qquad \{\text{Using}: \underline{|n-r} = (n-r)\ (n-r-1)\cdots 3\cdot 2\cdot 1\}$$

$$= {}^nP_r \qquad \text{[read it as ‘}n\,P\,r\text{’]}$$

Alternatively, number of permutation of r objects, out of n distinct objects is equivalent to selecting r objects first out of n distinct which can be selected in $\binom{n}{r}$ ways and then arranging them in *a* line in $r!$ ways so total ways is $\binom{n}{r}\times r!$

$$\Rightarrow {}^np_r = \binom{n}{r}\times r!$$

$$= \frac{n!}{r!(n-r)!}\times r!$$

$$= \frac{n!}{(n-r)!}$$

Example 77 *If* ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, *find* ${}^rP_2$.

Solution: We have

$$\frac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}} = \frac{30800}{1}$$

$$\Rightarrow \frac{56!}{(50-r)!} \times \frac{(51-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow 56.55(51-r) = 30800 \;\Rightarrow\; r = 41$$

$$\Rightarrow \quad {}^{r}P_2 = {}^{41}P_2 = 41 \cdot 40 = 1640.$$

Example 78 *Prove that* ${}^{n}P_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$.

Solution: RHS $= {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-1-r)!} + r\frac{(n-1)!}{(n-1-r+1)!} = \frac{(n-1)!}{(n-r-1)!} + \frac{r(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r)!}[n-r+r] = \frac{n!}{(n-r)!}$$

$$= {}^{n}P_r = \text{LHS}.$$

Aliter (Combinatorial): ${}^{n}P_r$ denotes the number of ways of arranging r-objects out of n-objects, in a line. This work can be done in the following way also. Suppose the objects are $a_1, a_2, \ldots, a_n$. First we find the number of permutations, in which a_1 does not appear. Number of such permutations is ${}^{n-1}P_r$. Further we consider those arrangements, in which a_1 necessarily appears. Number of such permutation is $r \cdot {}^{n-1}P_{r-1}$, (as we can arrange $(r-1)$ objects out of $(n-1)$ objects in ${}^{n-1}P_{r-1}$ ways, and then in any such permutation we can fix the position of a_1 in r ways). Now using the principle of addition, the required number is ${}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$.

Example 79 *Find number of different 4 letter words which can be formed using the letters of the word 'HISTORY'.*

Solution: Making a 4-letter word is equivalent to permutation of letters of the word 'HISTORY' taken 4 at a time.

$\Rightarrow$ Number of 4-letter words using letters of the word 'HISTORY'

= Number of permutation of letters H, I, S, T, O, R, Y taken only 4 at a time

$$= {}^{7}P_4 = \frac{\lfloor 7}{\lfloor 7-4} = \frac{\lfloor 7}{\lfloor 3}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times \lfloor 3}{\lfloor 3} = 7 \times 6 \times 5 \times 4 = 840.$$

Example 80 *In how many ways 5 distinct red balls, 3 distinct black balls and 2 distinct white balls can be placed in 3 distinct boxes such that each box contains only 1 ball.*

Solution: Total number of balls = 10. All balls are distinct.

The placement of 10 balls in 3 distinct boxes is equivalent to permutations of 10 distinct balls taken 3 at a time. This is because every arrangement of 3 balls will give a different way of placing 3 balls in 3 distinct boxes.

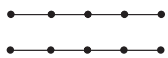
Therefore, total number of ways to place 10 distinct balls in 3 distinct boxes

= Number of permutations of 10 distinct balls taken 3 at a time

$$= {}^{10}P_3 = \frac{\lfloor 10}{\lfloor 10-3} = \frac{\lfloor 10}{\lfloor 7} = \frac{10 \times 9 \times 8 \times 7}{\lfloor 7}$$

$= 10 \times 9 \times 8 = 720$ ways.

Example 81 *In a railway train compartment there are two rows of facing seats, five in each. Out of* 10 *passengers,* 4 *wish to sit looking forward and* 3 *looking towards rear of the train. The other three are indifferent. In how many ways can the passengers take seats?*

Solution:  (Forward) (Row A, say)
(Rear) (Row B, say)

4 people, in row A, can sit in $^5P_4$ ways $= \dfrac{5!}{(5-4)!} = 5 \times 4 \times 3 \times 2$ ways

3 people, in row B, can sit in $^5P_3$ ways $= \dfrac{5!}{(5-3)!} = 5 \times 4 \times 3$ ways

3 (indifferent) people in remaining 3 seats can sit in $^3P_3$ ways
$= 3! = 3 \times 2 \times 1$

By multiplication principle, the total number of ways in which 10 people can sit in rows A and B
$= (5 \times 4 \times 3 \times 2) \times (5 \times 4 \times 3) \times (3 \times 2 \times 1)$
$= (5!)^2 \times 3$
$= 43,200$ ways

Example 82 *A tea party is arranged for* 16 *people along two sides of a long table with* 8 *chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated?*

Solution: Let $A_1, A_2, A_3, \ldots, A_{16}$ be the sixteen persons. Assume that A_1, A_2, A_3, A_4 want to sit on side 1 and A_5, A_6 want to sit on side 2.
The persons can be made to sit if we complete the following operations:

(i) Select 4 chairs from the side 1 in $^8C_4$ ways and allot these chairs to A_1, A_2, A_3, A_4 in 4! ways.

(ii) Select two chairs from side 2 in $^8C_2$ ways and allot these two chairs to A_5, A_6 in 2! ways.

(iii) Arrange the remaining 10 persons in remaining 10 chairs in 10! ways.
$\Rightarrow$ Hence the total number of ways in which the persons can be arranged

$$= (^8C_4 4!)(^8C_2 2!)(10!)$$

$$= \frac{8!}{4!\,4!}4! \times \frac{8!\,2!}{2!\,6!}10! = \frac{8!\,8!\,10!}{4!\,6!}.$$

Note: It is advised to use $\dbinom{n}{r} \times r!$ instead of ^nP_r directly as after selecting r objects you can always decide that whether you have to arrange them or not!

7.6.3 Theorem 3

(Permutation of Objects when not all objects are distinct)
Let there be n_1 A_1s, n_2 A_2s, …, n_k A_k's. Then the number of permutations $= \dfrac{(n_1 + n_2 + \cdots + n_k)!}{n_1!\,n_2!\ldots n_k!}$ (This number is known as a ***multinomial coefficient.***)

Numerator of the above formula is factorial of total number of items. Each terms in denominator is factorial of number of objects which are of same type and identical to each other. In earlier sections, we discussed how to permutate n different objects either taking all at a time or just r at a time. In this section, we will discuss how to arrange objects taken all at a time when all object are not distinct from each other.

For example, if we have to permutate A, A, B (Two A letters are identical) then number of permutations would not be same as permutations of 3 distinct objects say A, B, C. This is because two A letters cannot be permutated among themselves. Following are the ways to permutate A, A, B.

AAB, ABA, BAA, *i.e.*, 3 ways. This is not equal to $\underline{|3}$.

So we need to re define the formula we use to arrange n distinct objects.

For a case when all objects are not distinct. The redefined formula is given in theorem 3.

Proof: Total places we need to arrange all A_i's is $n_1 + n_2 + n_3 + \cdots + n_k = n$ (say)

Let us first select n_1 place out of n places to arrange n_1 A_1's this can be done in $\binom{n}{n_1}$ ways and there is only one way to arrange A_1 on these places. Now select n_2 places for A_2's out of remaining $n - n_1$ places.

This can be done in $\binom{n-n_1}{n_2}$ ways and arrange A_2's at these places in 1 way only and so on

$$\Rightarrow \quad \text{Total ways} = \binom{n}{n_1}\cdot 1 \binom{n-n_1}{n_2}\cdot 1 \cdots \binom{n_k}{n_k}\cdot 1$$

$$= \frac{n!}{n_1!\,n_2!\cdots n_k!}$$

Example 83 *How many different words can be formed by permuting all the letters of the word MATHEMATICS.*

Solution: In the word MATHEMATICS, total letters are 11

Number of 'M' letters = 2

Number of 'A' letters = 2

Number of 'T' letters = 2

Number of different letters = 5 (H, E, I, C, S)

Number of ways to arrange letters of the word 'MATHEMATICS'

$= \dfrac{\underline{|11}}{\underline{|2}\,\underline{|2}\,\underline{|2}}$ [using the formula given in Theorem 3]

Example 84 *How many different words can be formed by permuting all the letters of the word MISSISSIPPI?*

Solution: The word MISSISSIPPI is formed by 4S's, 4I's' 2P's and 1 M. Required number of different words $= \dfrac{11!}{4!\,4!\,2!\,1!}$ (using theorem 3).

Example 85 *How many n-term binary sequences can be formed of r 0's and* $(n - r)$ *1's?*

Solution: Number of binary sequences having n terms (r 0's, $(n - r)$ 1's) $= \dfrac{n!}{r!(n-r)!}$

This number known as a **binomial coefficient**.

Example 86 *How many 9-letter words can be formed by using the letters of the words*
(i) *EQUATIONS* (ii) *ALLAHABAD?*

Solution:
(i) All 9-letters in the word EQUATIONS are different.
Hence number of words = 9! = 362880.
(ii) ALLAHABAD contains LL, AAAA, H, B, D

$$\text{Number of words} = \frac{9!}{2!\,4!} = \frac{9\times 8\times 7\times 6\times 5}{2} = 7560.$$

Example 87 *How many anagrams (rearrangements) can be formed of the word 'PRIYANKA'?*

Solution: Here total letters are 8, in which 2 A's, but the rest are different. Hence the number of words formed $= \frac{8!}{2!} = 20160$.

As we have to count rearrangements, so remove one word that is 'PRIYANKA'
Hence number of anagrams =20160 − 1= 20159.

Example 88 *Find the number of permutations of* 1, 2, …, 6, *in which*
(i) *1 occurs before 2,*
(ii) *3 occurs before 4,*
(iii) *5 occurs before 6.*

For example, 3 5 1 4 2 6

Solution: Let us use the following terms.
A permutation has property P_1 if 1 occurs before 2. A permutation has property P_2 if 3 occurs before 4. A permutation has property P_3 if 5 occurs before 6.

$$P_1^{\,C} \Leftrightarrow \text{not } P_1$$
$$P_2^{\,C} \Leftrightarrow \text{not } P_2$$
$$P_3^{\,C} \Leftrightarrow \text{not } P_3.$$

So there are 8 possibilities, *e.g.*, $P_1\, P_2^{\,C}\, P_3$, $P_1^{\,C}\, P_2\, P_3^{\,C}$, *etc.*

Number of $P_1\, P_2\, P_3$ = Number of $P_1^{\,C}\, P_2\, P_3 = \ldots =$ Number of $P_1\, P_2\, P_3^{\,C}$

$\Rightarrow$ Number of permutations having $P_1\, P_2\, P_3 = \frac{6!}{8} = 90.$

Aliter 1: Assume 1 and 2 as a, a, 3, 4 as b, b, 5, 6 as c,c now arrange a, a, b, b, c, c in a line. This can be done in $\frac{6!}{2!2!2!}$ ways = 90.

Now starting from left first a replaced by 1 and second a replaced by 2, similarly b and c, we will get the desired permutation.

Aliter 2: Arrange 1 and 2 in 6 places in ${}^6C_2$ ways.
Now, to arrange 3 and 4 we have ${}^4C_2$ ways and to arrange 5, 6 we have only one way.

Finally by Multiplication Principle total number of ways ${}^6C_2\,{}^4C_2 = \frac{6!}{8}. = 90.$

7.6.3.1 Permutations of n Objects Taken r at a Time when All n Objects are not Distinct

In this section we will discuss how to arrange (permutate) n objects taken r at a time where all n objects are not distinct.

For example, arrangements of letters AABBBC taken 3 at a time.

To find such arrangements, it is not possible to derive a formula that can be applied in all such cases.

So, we will discuss a method (or procedure) that should be applied to find arrangements. The method involves making cases based on alike items that we choose in the arrangement. You should read the following examples to learn how to apply this 'method of cases' to find arrangements of n objects taken r at a time when all objects are not different.

Example 89 *Find the number of 4-letter words, that can be formed from the letters of the word 'ALLAHABAD'.*

Solution: We have four A, two L, and one each of H, B and D.
Four letters from the letters of the word ALLAHABAD would be one of the following types; (i) all same (ii) three same, one distinct (iii) two same, two same (iv) two same, two distinct and (v) all four distinct

Now number of words of type (i) is 1

Number of words of type (ii) is ${}^4C_1 \times \dfrac{4!}{3!} = 16$

Number of words of type (iii) is $\dfrac{4!}{2!2!} = 6$

Number of words of type (iv) is ${}^2C_1 {}^4C_2 \times \dfrac{4!}{2!} = 144$

Number of words of type (v) is ${}^5C_4\, 4! = 120$

Thus the required number = 1 + 16 + 6 + 144 + 120 = 287.

Example 90 *Find in how many ways we can arrange letters AABBBC taken 3 at a time.*

Solution: The given letters include AA, BBB, C, *i.e.*, 2 A letters, 3 B letters and 1 C letters.
To find arrangements of 3 letters, we will make following cases based on alike letters we choose in the arrangement.

Case 1: All 3 letters are alike
3 alike letters can be selected from given letters in only 1 way, *i.e.*, BBB.

Further 3 selected letters can be arranged among themselves in $\dfrac{\lfloor 3}{\lfloor 3} = 1$ way.

$\Rightarrow$ Total number of arrangement with all letters alike = 1 (1)

Case 2: 2 alike and 1 distinct
2 alike letters can be selected from 2 sets of alike letters (AA, BB) in ${}^2C_1$ ways.

1 distinct letter (distinct from selected alike letters) can be selected from remaining letters in ${}^2C_1$ ways. (C, A or B either).

Further 2 alike and 1 distinct selected letters can be arranged among themselves in $\dfrac{\lfloor 3}{\lfloor 2}$ ways.

$\Rightarrow$ Total number of arrangements with '2 alike and 1 distinct letter'

$$= {}^2C_1 \times {}^2C_1 \times \frac{\lfloor 3}{\lfloor 2} = 2 \times 2 \times 3 = 12 \qquad (2)$$

Case 3: All distinct letters

All 3 letters distinct can be selected from 3 distinct letters (A, B, C) in 1 way.

Further 3 distinct letters can be arranged among themselves in $\lfloor 3$ ways.

$\Rightarrow$ Total number of arrangements with all 3 letters distinct $= 1 \times \lfloor 3 = \lfloor 3 = 6$ (3)

Combining (1), (2) and (3)

Total number of permutations of AABBBC taken 3 at a time = 1 + 12 + 6 = 19.

Example 91 *How many 4-letters words can be formed using the letters of the word INEFFECTIVE?*

Solution: INEFFECTIVE contains 11 letters: EEE, FF, II, C, T, N, V.

As all letters are not distinct, we cannot use ${}^nP_r$. The 4-letter words will be among any one of the following cases:

1. 3 alike letters, 1 distinct letter.
2. 2 alike letters, 2 alike letters.
3. 2 alike letters, 2 distinct letters.
4. All distinct letters.

Case 1: 3 alike, 1 distinct

3 alike can be selected in one way, *i.e.*, EEE.

Distinct letters can be selected from F, I, T, N, V, C in ${}^6C_1$ ways.

$\Rightarrow$ Number of groups $= 1 \times {}^6C_1 = 6 \Rightarrow$ Number of words $= 6 \times \dfrac{4!}{3! \times 1!} = 24$.

Case 2: 2 alike, 2 alike

Two sets of 2 alike can be selected from 3 sets (EE, II, FF) in ${}^3C_2$ ways.

$\Rightarrow$ Number of words $= {}^3C_2 \times \dfrac{4!}{2! \times 2!} = 18$

Case 3: 2 alike, 2 distinct

$\Rightarrow$ Number of groups $= ({}^3C_1) \times ({}^6C_2) = 45 \Rightarrow$ Number of words $= 45 \times \dfrac{4!}{2!} = 540$

Case 4: All distinct

$\Rightarrow$ Number of groups $= {}^7C_4$ (out of E, F, I, T, N, V, C)

$\Rightarrow$ Number of words $= {}^7C_4 \times 4! = 840$

Hence total 4-letter words = 24 + 18 + 540 + 840 = 1422.

7.6.4 Theorem 4

(Arrangement of n distinct objects with repetition of objects)

Total number of ways to permutate n distinct things taken r at a time when objects can be repeated any number of times is n^r.

Proof:

Here we have to arrange n distinct objects in a row taken only r at a time when objects can be repeated any number of times, *i.e.*, repetition of objects is allowed.

Permutation of n objects in a row taken r at a time is equivalent to filling r boxes. Let us consider r boxes as shown in the figure:

	1	2	3	4		r
Boxes:	☐	☐	☐	☐	…	☐
Ways:	n	n	n	n		n

Box-1 can be filled in n ways by any of the n objects.

Box-2 can also be filled in n ways as any of the n objects can be used to fill Box-2. This is because, we can reuse the object used to fill Box-1 to fill Box-2 as repetition of objects is allowed.

Similarly Box-3, Box-4, …, Box-r each one can be fill in n ways each.

Using fundamental principle of counting, total number of way to fill n boxes
$= n \times n \times n \ldots r$ times $= n^r$.

Example 92 *A child has four pockets and three different marbles. In how many ways can the child put the marbles in his pockets?*

Solution: The first marble can be put into the pocket in 4 ways, so the second can also be put in the pocket in 4 ways so can the third . Thus, the number of ways in which the child can put the marbles $= 4 \times 4 \times 4 = 64$ ways.

Example 93 *In how many ways can 5 letters be posted in 4 letter boxes?*

Solution: Since each letter can be posted in any one of the four letter boxes. So, a letter can be posted in 4 ways. Since there are 5 letters and each letter can be posted in 4 ways. So, total number of ways in which all the five letters can be posted $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

Example 94 *Five person entered the lift cabin on the ground floor of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin*

(i) *at any one of the 7 floors* (ii) *at different floors.*

Solution: Suppose A_1, A_2, A_3, A_4, A_5 are five persons.

(i) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7 \times 7 \times 7 \times 7 \times 7 = 7^5$.

(ii) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4 and A_5 can leave the cabin in 5, 4 and 3 ways respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is $7 \times 6 \times 5 \times 4 \times 3 = 2520$.

Example 95 *There are 6 single choice questions in an examination. How many sequence of answers are possible, if the first three questions have 4 choices each and the next three have 5 each?*

Solution: Here we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences $= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$.

Example 96 *Three tourist want to stay in five different hotels. In how many ways can they do so if:*

(i) *each hotel can not accommodate more than one tourist?*
(ii) *each hotel can accommodate any number of tourists?*

Solution:

(i) Three tourists are to be placed in 3 different hotels out of 5. This can be done as:
Place first tourist in 5 ways
Place second in 4 ways
Place third in 3 ways
$\Rightarrow$ Required number of placements $= 5 \times 4 \times 3 = 60$

(ii) To place the tourists we have to do following three operations.
(a) Place first tourist in any of the hotels in 5 ways.
(b) Place second tourist in any of the hotels in 5 ways.
(c) Place third tourist in any of the hotels in 5 ways.
$\Rightarrow$ the required number of placements $= 5 \times 5 \times 5 = 125$.

7.6.5 Some Miscellaneous Applications of Permutations

7.6.5.1 *Always Including p Particular Objects in the Arrangement*

The number of ways to select and arrange (permutate) r objects from n distinct objects such that arrangement should always include p particular objects $= {}^{n-p}C_{r-p} \times r!$.

Logic: First select p particular objects which should always be included in 1 way (1)

Then select remaining $(r - p)$ objects from remaining $(n - p)$ objects in ${}^{n-p}C_{r-p}$ ways. (2)

Finally arrange r selected objects in $r!$ ways (3)

Using fundamental principle of counting, operations (1), (2) and (3) can be performed together in ways

$= 1 \times {}^{n-p}C_{r-p} \times r!$ ways.

7.6.5.2 *Always Excluding p Particular Objects in the Arrangement*

The number of ways to select and arrange r objects from n distinct objects such that p particular objects are always excluded in the selection $= {}^{n-p}C_r \times r!$.

Logic: First exclude p particular objects from n different objects.

Then select r objects from $(n - p)$ different objects in ${}^{n-p}C_r$ ways. (1)

Then permutate r selected objects in r! ways. (2)

Using fundamental principle of counting, operations (1) and (2)can be performed together in ${}^{n-p}C_r \times r!$ ways.

Example 97 *How many three letter words can be made using the letters of the words SOCIETY, so that*

(i) *S is included in each word?* (ii) *S is not included in any word?*

Solution:

(i) To include S in every word, we will use following steps.
Step 1: Select the remaining two letters from remaining 6 letters, *i.e.*, O, C, I, E, T, Y in ${}^6C_2$ ways.

Step 2: Include S in each group and then arrange each group of three in 3! ways.
$\Rightarrow$ Number of words = ${}^6C_2\ 3! = 90$.

(ii) If S is not to be included, then we have to make all the three words from the remaining 6.
$\Rightarrow$ Number of words = ${}^6C_3\ 3! = 120$.

7.6.5.3 'p' Particular Objects Always Together in the Arrangement

The number of ways to arrange n distinct objects such that p particular objects remain together in the arrangement $(n-p+1)!\ p!$

Logic: Make a group of p particular objects that should remain together. Arrange this group of p particular objects and remaining $(n-p)$ objects in $(n-p+1)!$ ways. (1)

Finally arrange p particular objects among themselves in $p!$ ways. (2)

Using fundamental principle of counting operations (1) and (2) can be performed together in $(n-p+1)! \times p!$ ways

Example 98 *How many words can be formed using the letters of the word TRIANGLE so that*

(i) *A and N are always together?* (ii) *T, R, I are always together?*

Solution:

(i) Assume (AN) as a single letter. Now there are seven letters in all:
(AN), T, R, I, G, L, E
Seven letters can be arranged in 7! ways.
All these 7! words will contain A and N together. A and N can now be arranged among themselves in 2! ways (AN and NA).
Hence total number of words = 7! 2! = 10080.

(ii) Assume (TRI) as a single letter.
The letters: (TRI), A, N, G, L, E can be rearranged in 6! ways.
TRI can be arranged among themselves in 3! ways.
Total number of words = 6! 3! = 4320.

Example 99 *How many 5-letter words containing* 3 *vowels and* 2 *consonants can be formed using the letters of the word EQUATIONS so that the two consonants occur together in every word?*

Solution: There are 5 vowels and 3 consonants in EQUATION. To form the words we will use following steps:

Step 1: Select vowels (3 from 5) in ${}^5C_3$ ways.

Step 2: Select consonants (2 from 3) in ${}^3C_2$ ways.

Step 3: Arrange the selected letters (3 vowels and 2 consonants (always together)) in $4! \times 2!$ ways.
Hence the number of words = ${}^5C_3\ {}^3C_2\ 4!\ 2! = 10 \times 3 \times 24 \times 2 = 1440$.

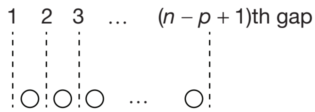
7.6.5.4 'p' Particular Objects Always Separated in the Arrangement

The number of ways to arrange n different objects such that p particular objects are always separated

$$= {}^{n-p+1}C_p \times (n-p)! \times p!$$

Logic: First arrange $n - p$ objects in $(n - p)!$ ways. Now we have to place p particular objects between $(n - p)$ remaining objects so that all p particular objects must be separated from each other.

From figure we can see there are $(n - p + 1)$ gaps (including before and after) between $(n - p)$ objects where we can place p particular objects such that p objects are separated from each other.



(n – p) remaining objects

Select p gaps from $(n - p + 1)$ gaps for p particular objects in ${}^{n-p+1}C_p$ ways. Now place and arrange p objects in p selected gaps in $p!$ ways. Using fundamental principle of counting, all operations can be performed together in ${}^{n-p+1}C_p \times (n-p)! \times p!$ ways.

Example 100 *There are 9 candidates for an examination out of which 3 are appearing in Mathematics and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two Mathematics candidates are together?*

Solution: Divide the work in two steps.

Step 1: First, arrange the remaining candidates in 6! ways.

Step 2: Place the three Mathematics candidates in the row of six other candidates so that no two of them are together.

***x*:** Places available for Mathematics candidates.

***o*:** Others.

x	o	x	o	x	o	x	o	x	o	x	o	x

In any arrangement of 6 other candidates (o), there are seven places available for Mathematics candidate so that they are not together. Now 3 Mathematics candidates can be placed in these 7 places in $\binom{7}{3} 3!$ ways.

Hence total number of arrangements

$$= 6!\binom{7}{3}3! = 720 \times \frac{7!}{4!} = 151200.$$

Example 101 *In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus (–) signs are together?*

Solution:

Step 1: The plus signs can be arranged in one way (because all are identical).

	+		+		+		+		+		+		+	

A blank box shows available spaces for the minus signs.

Step 2: The 5 minus (–) signs are now to be placed in the 8 available spaces so that no two of them are together.

(i) Select 5 places for minus signs in ${}^8C_5$ ways.

(ii) Arrange the minus signs in the selected places in 1 way (all signs being identical).

Hence number of possible arrangements $= 1 \times {}^8C_5 \times 1 = 56$.

Example 102 *There are 20 stations between stations A and B. In how many ways a train moving from station A to station B can stop at 3 stations between A and B such that no two stopping stations are together?*

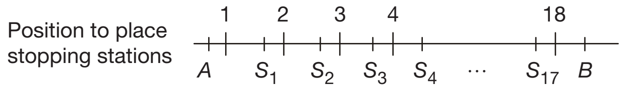
Solution: We have to select 3 stations from 20 stations between A and B so that train can stop at these stations.

According to the question:

There are 3 stopping stations that should be separated from each other, *i.e.*, even no two of them are together.

First separate out 3 stations to the selected from 20 stations, *i.e.*, 17 station left.

Now, we select 3 positions between 17 stations so that we can place 3 stopping stations. There are 18 positions between 17 stations where we can place 3 stopping stations.



Therefore, number of ways to select 3 stations where train can stop

= number of ways to place 3 stopping stations between remaining 17 stations

= $^{18}C_3$.

7.6.5.5 Rank of a Word in the Dictionary

In these type of problems, dictionary of words is formed by using all the arrangement of all letters at a time of the given word. The dictionary format means words are arranged in the alphabetical order. You will be supposed to find the rank (position) of the given word or some other word in the dictionary.

Following examples will help you learn how to find the rank in the dictionary.

Example 103 *Find the rank of the word MOTHER in the dictionary order of the words formed by M, T, H, O, E, R.*

Solution: Number of words starting with E, having other letters M, T, H, O, R = 5 ! = 120

Number of words starting with H, having other letters M, T, E, O, R = 5 ! = 120

Number of words having first two letters M,E and other letters O, T, H, R = 4 ! = 24

Number of words having first two letters M,H and other letters T, E, O, R = 4 ! = 24

Number of words having first three letters M,O,E and other letters H, T, R = 3 ! = 6

Number of words having first three letters M,O,H and other letters T, E, R = 3 ! = 6

Number of words having first three letters M,O,R and other letters T, H, E = 3 ! = 6

Number of words having first four letters M,O,T,E and other letters H, R = 2 ! = 2

Total number of words, before MOTHER, in the dictionary order made up of M, O, E, T, H, R = 120 + 120 + 24 + 24 + 6 + 6 + 6 + 2 = 308

$\therefore$ Rank of the word MOTHER = 309.

Example 104 *If all the letters of the word RANDOM are written in all possible orders and these words are written out as in a dictionary, then find the rank of the word RANDOM in the dictionary.*

Solution: In a dictionary the words at each stage are arranged in alphabetical order. In the given problem, we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a time, *i.e.*, A will occur 5! times. D, M, N, O will occur in the first place the same number of times.

Number of words starting with A = 5! = 120

Number of words starting with D = 5! = 120

Number of word starting with M = 5! = 120

Number of words starting with N = 5! = 120

Numbers of words starting with O = 5! = 120
After this, words beginning with RA must follow.
Number of words beginning with RAD or RAM = 3!
Now the words beginning with RAN must follow.
First one is RANDMO and the next one is RANDOM.
∴ Rank of RANDOM = 5(5!) + 2(3!) + 2 = 614.

Example 105 *Find the rank of the word 'TTEERL' in the dictionary of words formed by using the letters of the word 'LETTER'.*

Solution: In the dictionary of words formed, we need to count words before the word 'TTEERL' in the dictionary. To count such words, we need to first count words starting with E, L, R, TE, TL, TR and then add 2 to the count for words 'TTEELR' and 'TTEERL'.

Number of words starting with E = Arrangement of letter E, T, T, R, L $= \dfrac{|\underline{5}}{|\underline{2}}$

Number of words starting with L = Arrangement of letters E, T, T, E, R $= \dfrac{|\underline{5}}{|\underline{2}\,|\underline{2}}$

Number of words starting with R = Arrangement of letters E, T, T, E, L $= \dfrac{|\underline{5}}{|\underline{2}\,|\underline{2}}$

Number of words starting with TE = Arrangement of letters T, E, R, L $= |\underline{4}$

Number of words starting with TL = Arrangement of letters E, T, E, R $= \dfrac{|\underline{4}}{|\underline{2}}$

Number of words starting with TR = Arrangement of letters T, E, E, L $= \dfrac{|\underline{4}}{|\underline{2}}$

Rank of TTEERL $= \dfrac{|\underline{5}}{|\underline{2}} + \dfrac{|\underline{5}}{|\underline{2}\,|\underline{2}} + \dfrac{|\underline{5}}{|\underline{2}\,|\underline{2}} + |\underline{4} + \dfrac{|\underline{4}}{|\underline{2}} + \dfrac{|\underline{4}}{|\underline{2}} + 2 = 170$

(Now, try to find the rank of the word COCHIN, in the list, in the dictionary order, of the words made up of C, C, H, I, O, N. Your answer should be 97).

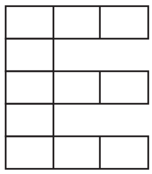
Build-up Your Understanding 3

1. Find the value of r in following equations:
   (i) ${}^{5}P_r = {}^{6}P_{r-1}$ **(ii)** ${}^{10}P_r = 720$ **(iii)** ${}^{20}P_r = 13 \times {}^{20}P_{r-1}$
2. In a railway compartment 6 seats are vacant on a berth. Find the number of ways in which 3 passengers sit on them.
3. Three men have 6 different trousers, 5 different shirts and 4 different caps. Find the number of different ways in which they can wear them.
4. Find the number of words of four letters containing equal number of vowels and consonants (repetition not allowed).
5. Find the number of words that can be formed using 6 consonants and 3 vowels out of 10 consonants and 4 vowels.
6. Find the number of ways in which the letters of the word ARRANGE can be made such that both R's do not come together.
7. Find the number of arrangements of the letters of the word BANANA is which the two 'N's do not appear adjacently.
8. We are required to form different words with the help of the letters of the word INTEGER. Let m_1 be the number of words in which I and N are never together and m_2 be the number of words which begin with I and end with R, then find m_1/m_2.

9. Find the number of arrangements that can be made with the letters of the word MATHEMATICS and also find the number of them, in which the vowels occur together.
10. Find the number of ways in which letters of the word VALEDICTORY be arranged so that the vowels may never be separated.
11. Find the number of different words which can be formed from the letters of the word LUCKNOW when
    (i) all the letters are taken.
    (ii) all the letters are taken and words begin with L.
    (iii) all the letters are taken and the letters L and W respectively occupy the first and last places.
    (iv) all the letters are taken and the vowels are always together.
12. Find the number of permutations of the word AUROBIND in which vowels appear in an alphabetical order.
13. If as many more words as possible be formed out of the letters of the word DOGMATIC then find the number of words in which the relative position of vowels and consonants remain unchanged.
14. Find the number of words which can be formed using all letters of the word 'Pataliputra' without changing the relative order of the vowels and consonants.
15. Find the total numbers of words that can be made by writing all letters of the word PARAMETER so that no vowel is between two consonants.
16. Find the total number of permutation of $n(n > 1)$ distinct things taken not more than r at a time and atleast 1, when each thing may be repeated any number of times.
17. Find the number of permutations of n distinct objects taken
    (i) atleast r objects at a time
    (ii) atmost r objects at a time
    (Where repetition of the objects is allowed)
18. If the number of arrangements of $n - 1$ things from n distinct things is k times the number of arrangements of $n - 1$ things taken from n things in which two things are identical then find the value of k.
19. Find the number of different 7-digit numbers that can be written using only the three digits 1, 2 and 3 with the condition that the digit 2 occurs twice in each number.
20. Six identical coins are arranged in a row. Find the total number of ways in which the number of heads is equal to the number of tails.
21. There are n distinct white and n distinct black balls. Find the number of ways in which we can arrange these balls in a row so that neighboring balls are of different colours.
22. Find number of ways in which 6 girls and 6 boys can be arranged in a line if no two boys or no two girls are together.
23. Find the number of ways in which 3 boys and 3 girls (all are of different heights) can be arranged in a line so that boys as well as girls among themselves are in decreasing order of height (from left to right).
24. Find the number of ways in which 10 candidates $A_1, A_2, \ldots, A_{10}$ can be ranked so that A_1 is always above A_2.
25. Let A be a set of n (≥ 3) distinct elements. Find the number of triples (x, y, z) of the elements of A in which atleast two coordinates are equal.
26. Find the number of ways of arranging m numbers out of 1, 2, 3, …, n so that maximum is $(n - 2)$ and minimum is 2 (repetitions of numbers is allowed) such that maximum and minimum both occur exactly once, $(n > 5, m > 3)$.
27. Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4, and then the men select the chairs from amongst the remaining. Find the number of possible arrangements.

28. There are 10 numbered seats in a double decker bus, 6 in the lower deck and 4 on the upper deck. Ten passengers board the bus, of them 3 refuse to go to the upper deck and 2 insist on going up. Find the number of ways in which the passengers can be accommodated.

29. In how many different ways a grandfather along with two of his grandsons and four grand daughters can be seated in a line for a photograph so that he is always in the middle and the two grandsons are never adjacent to each other.



30. Find the number of ways in which A A A B B B can be placed in the squares of the figure as shown, so that no row remains empty.

31. The tamer of wild animals has to bring one by one 5 lions and 4 tigers to the circus arena. Find the number of ways this can be done if no two tigers immediately follow each other.

32. In a conference 10 speakers are present. If S_1 wants to speak before S_2 and S_2 wants to speak after S_3, then find the number of ways all the 10 speakers can give their speeches with the above restriction if the remaining seven speakers have no objection to speak at any number.

33. Find the total number of flags with three horizontal strips, in order, that can be formed using 2 identical red, 2 identical green and 2 identical white strips.

34. Messages are conveyed by arranging 4 white, 1 blue and 3 red flags on a pole. Flags of the same colour are alike. If a message is transmitted by the order in which the colours are arranged then the find the total number of messages that can be transmitted if exactly 6 flags are used.

35. Find number of arrangements of 4-letters taken from the word EXAMINATION.

36. Find number of ways in which an arrangement of 4-letters can be made from the letters of the word PROPORTION.

37. Find the number of permutations of the word ASSASSINATION taken 4 at a time.

38. The letters of the word TOUGH are written in all possible orders and these words are written out as in a dictionary, then find the rank of the word TOUGH.

39. The letters of the word SURITI are written in all possible orders and these words are written out as in a dictionary. What is the rank of the word SURITI?

40. There are 720 permutations of the digits 1, 2, 3, 4, 5, 6. Suppose these permutations are arranged from smallest to largest numerical values, beginning from 1 2 3 4 5 6 and ending with 6 5 4 3 2 1.
(a) What number falls on the 124th position?
(b) What is the position of the number 321546?

41. All the five digits number in which each successive digit exceeds its predecessor are arranged in the increasing order of their magnitude. Find the 97th number in the list.

42. All the 5 digit numbers, formed by permuting the digits 1, 2, 3, 4 and 5 are arranged in the increasing order. Find:
(i) the rank of 35421 (ii) the 100th number.

43. There are 11 seats in a row. Five people are to be seated. Find the number of seating arrangements, if
(i) the central seat is to be kept vacant;
(ii) for every pair of seats symmetric with respect to the central seat, one seat is vacant.

44. Find the number of ways in which six children of different heights can line up in a single row so that none of them is standing between the two children taller than him.

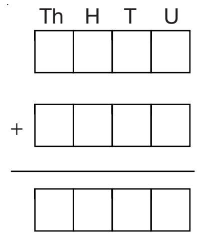
45. Define a 'good word' as a sequence of letters that consists only of the letters A, B and C and in which A never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. If the number of n-letter good words are 384, then find the value of n.

46. There are 2 identical white balls, 3 identical red balls and 4 green balls of different shades. Find the number of ways in which they can be arranged in a row so that atleast one ball is separated from the balls of the same colour.

47. Eight identical rooks are to be placed on an 8×8 chess-board. Find the number of ways of doing this, so that no two rooks are in attacking positions.

48. How many arrangements of the 9 letters $a, b, c, p, q, r, x, y, z$ are there such that y is between x and z? (Any two, or all three, of the letters x, y, z, may not be consecutive.)

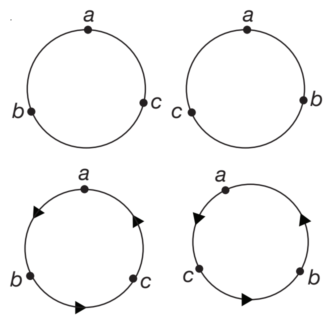
49. In the figure, two 4-digit numbers are to be formed by filling the place with digits. Find the number of different ways in which these places can be filled by digits so that the sum of the numbers formed is also a 4-digit number and in no place the addition is with carrying.



50. Two n-digit integers (leading 0 allowed) are said to be equivalent if one is a permutation of the other. Thus 10075 and 01057 are equivalent. Find the number of 5-digit integers such that no two are equivalent.

7.7 Introduction to Circular Permutation

When objects are to be arranged (ordered) in a circle instead of a row, it is known as Circular Permutation. For example, three objects a, b, c can be permutated in a circle as shown in figure:



Number of ways to arrange a, b, c in circle is not same as number of ways to arrange a, b, c in a row.

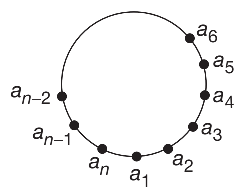
This is because arrangements abc, bca, cab in a row are same in circle as shown in the figure.

Similarly arrangements acb, cba, bac in a row are same in circle as shown in the figure.

7.7.1 Theorem

The number of *circular permutations* of n distinct objects is $(n - 1)!$

Proof: Let $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ be n distinct objects. Let the total number of circular permutations be x. Consider one of these x permutations as shown in Figure.



Clearly, this circular permutation provides n linear permutations as given below:

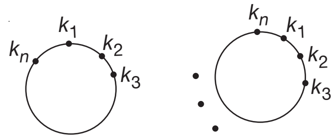
$$a_1, a_2, a_3 \ldots a_{n-1}, a_n$$
$$a_2, a_3, a_4, \ldots a_n, a_1$$
$$a_3, a_4, a_5, \ldots a_n, a_1, a_2$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$a_n, a_1, a_2, a_3, \ldots, a_{n-1}$$

Thus, each circular permutation gives n linear permutations. As there are x circular permutations, the number of linear permutations is xn. But the number of linear permutations of n distinct objects is $n!$.

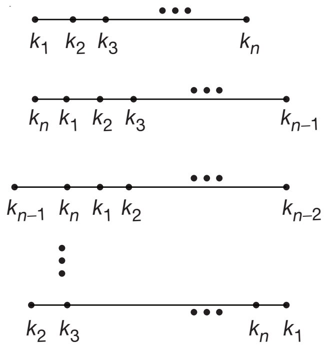
$$\therefore xn = n! \Rightarrow x = \frac{n!}{n} = (n-1)!$$

Aliter 1: Number of linear permutations of n distinct objects $= n!$. Consider two linear permutation of n distinct objects $k_1, k_2, k_3, \ldots, k_n$ and $k_n, k_1, k_2, \ldots, k_{n-1}$.

Consider a corresponding circular permutation as shown in the following figure.



(For example, think of two thread each having n beads)



In fact, both the circular arrangements are same. Not only that, there are more similar looking circular permutations. There are n linear permutations as shown, which give the same circular permutation.

So while counting the number of circular permutations from the number of linear permutations, one circular permutation is counted n times.

$\therefore$ Number of circular permutations $= \dfrac{n!}{n} = (n-1)!$

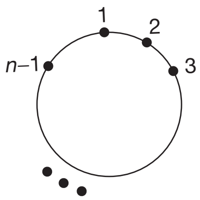
Aliter 2: Let P_n denote the number of circular permutations of n distinct objects.

Note that $P_1 = 1$.

Let $(n - 1)$ objects (out of these n objects) be placed on a circle.

This can be done in P_{n-1} ways.

These $n - 1$ objects break the circle into $n - 1$ arcs. Now the nth object is to be kept some where on these $(n - 1)$ arcs. This can be done in $(n - 1)$ ways.



$\therefore P_n = (n-1)\,P_{n-1}$ (recurrence relation)

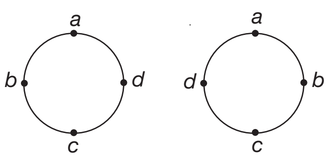
$= (n-1)(n-2)\,P_{n-2}$

$= (n-1)(n-2)(n-3)\,P_{n-3}$ and so on

$= (n-1)(n-2)(n-3)\ldots 3.\ 2.\ 1.\ P_1$

$= (n-1)!$

7.7.2 Difference between Clockwise and Anti-clockwise



In some of the problems we need to consider clockwise and anti-clockwise arrangements of objects as same arrangements. See the adjacent circular permutations. There is a difference of just the cyclic order. In first arrangement a, b, c, d are arranged in anti-clockwise order where as in second they are arranged clockwise order.

If we have to consider these arrangements same (for example, arrangement of flowers in garland or arrangement of beads in a necklace), then we need to divide total circular permutation by 2.

Therefore,

Number of circular permutations of n distinct objects such that clockwise and anti-clockwise arrangements of objects are same $= \dfrac{(n-1)!}{2},\ n \geq 3.$

Notes:

1. Number of circular permutations of 'n' distinct things taken 'r' at a time $= \dbinom{n}{r}(r-1)!$ (when clockwise and anticlockwise orders are taken as different)
2. If clockwise and anticlockwise orders are taken as same, then the required number of circular permutations $= \dfrac{\dbinom{n}{r}(r-1)!}{2},\ r \geq 3.$

Example 106 *In how many ways can* 13 *persons out of* 24 *persons be seated around a table.*

Solution: In case of circular table the clockwise and anti-clockwise orders are different, thus the required number of circular permutations $= \dfrac{\binom{24}{13}13!}{13} = \dfrac{24!}{13 \times 11!}$.

Example 107 *Out of ten people,* 5 *are to be seated around a round table and* 5 *are to be seated across a rectangular table. Find the number of ways to do so.*

Solution: First select 5 people out of 10, those who sit around the table. This can be done in ${}^{10}C_5$ ways.

Number of ways in which these 5 people sit around the round table = 4!

Remaining 5 people sit across a rectangular table in 5! ways.

Total number of arrangements

$$= {}^{10}C_5 \times 4! \times 5!$$

$$= \frac{10!}{5!5!} \times 4! \times 5!$$

$$= 10! \times \frac{1}{5} = 9! \times 2.$$

Example 108 *There are* 20 *persons among whom are two brothers. Find the number of ways in which we can arrange them around a circle so that there is exactly one person between the two brothers.*

Solution: Let B_1 and B_2 be two brothers among 20 persons and let M be a person that will sit between B_1 and B_2. The person M can be chosen from 18 person (excluding B_1 and B_2) in 18 ways. Considering the two brothers B_1 and B_2 and person M as one person and remaining 17 persons separately, we have 18 persons in all. These 18 persons can be arranged around a circle in (18 – 1)! = 17! ways. But B_1 and B_2 can be arranged among themselves in 2! ways.

Hence, the total number of ways = 18 × 17! × 2! = 2 × 18!

Example 109 *In how many ways can a party of* 4 *men and* 4 *women be seated at a circular table so that no two women are adjacent?*

Solution: The 4 men can be seated at the circular table such that there is a vacant seat between every pair of men in (4 – 1)! = 3! ways. Now, 4 vacant seats can be occupied by 4 women in 4! ways.

Hence, the required number of seating arrangements = 3! × 4! = 144.

Example 110 *A round table conference is to be held between* 20 *delegates of* 2 *countries. In how many ways can they be seated if two particular delegates are* (i) *always together?* (ii) *never together?*

Solution:

(i) Let D_1 and D_2 be two particular delegates. Considering D_1 and D_2 as one delegate, we have 19 delegates in all. These 19 delegates can be seated round a circular table in (19 – 1)! = 18! ways. But two particular delegates can arrange among themselves in 2! ways (D_1 D_2 and D_2 D_1).

Hence, the total number of ways = 18! × 2! = 2 (18!).

(ii) To find the number of ways in which two particular delegates never sit together, we subtract the number of ways in which they sit together from the total number of seating arrangements of 20 persons around the round table. Clearly 20 persons can be seated around a circular table in $(20 - 1)! = 19!$ ways.

Hence, the required number of seating arrangements $= 19! - 2 \times 18! = 17\,(18!)$.

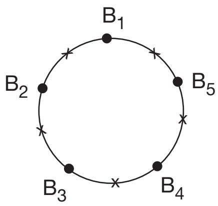
Alternate Solution:

First arrange remaining 18 persons in $(18 - 1)! = 17!$ ways.

Then select two gaps out of 18 gaps between 18 persons on the circle in $^{18}C_2$ ways and arrange the two in 2! ways.

Number of ways $= 17! \times {}^{18}C_2 \times 2!$
$= 17\,(18!)$.

Example 111 *In how many different ways can five boys and five girls form a circle such that the boys and girls are alternate?*



Solution: After fixing up one boy on the table the remaining can be arranged in 4! ways. There will be 5 places, one place each between two boys which can be filled by 5 girls in 5! ways.

Hence by the principle of multiplication, the required number of ways

$$= 4! \times 5! = 2880.$$

Example 112 *Find the number of ways to seat 5 boys and 5 girls around a table so that boy B_1 and girl G_1 are not adjacent.*

Solution: Number of ways of arranging 5 boys and 5 girls around a table is

$$(10 - 1)! = 9!.$$

Among these, we have to discard the arrangements where B_1 and G_1 sit together. Consider B_1G_1 as a single entity. There all 9 people can be arranged around a circle in $(9 - 1)! = 8!$ ways.

But the boy B_1 and girl G_1 can either be arranged in B_1G_1 or in G_1B_1 position. So, the number of ways in which boy B_1 and girl G_1 are together is $2 \times 8!$.

Therefore, the number of ways in which boy B_1 and girl G_1 are not together is $9! - 2 \times 8! = 8!(9 - 2) = 7 \times 8! = 2{,}82{,}240$.

Aliter: Exclude G_1 initially. The remaining 9 can be arranged in $(9 - 1)! = 8!$ ways around a circle. Now, there are 9 in-between positions among the 9 people seated around a circle. Of these 9, the two sides of B_1, *i.e.*, his left and right are not suited for G_1 (as B_1 and G_1 must not come together). Hence, there are 7 choices in each of the circular permutations for G_1.

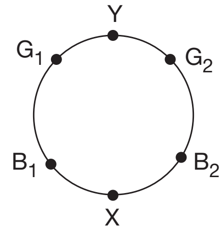
$\therefore$ The total number of ways of arranging the person is 7(8!) ways.

Example 113 *There are 5 gentlemen and 4 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together?*

Solution: Five gentlemen can be seated at a round table in $(5 - 1)! = 4!$ ways. Now, 5 places are created in which 4 ladies are to be seated. Select 4 seats for 4 ladies from 5 seats in $^5C_4$ ways. Now 4 ladies can be arranged on the 4 selected seats in 4! ways. Hence, the total number of ways in which no two ladies sit together

$$= 4! \times {}^5C_4 \times \underline{|4} = (4!)\,5(4!) = 2880.$$

Example 114 *Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible?*



Solution: Let B_1, B_2 and X be three boys and G_1, G_2 and Y be three girls. Since the boy X does not want any girl neighbour. Therefore boy X will have his neighbours as boys B_1 and B_2 as shown in in the figure. Similarly, girl Y has her neighbour as girls G_1 and G_2 as shown figure. But the boys B_1 and B_2 can be arranged among themselves in 2! ways and the girls G_1 and G_2 can be arranged among themselves in 2! ways.
Hence, the required number of arrangements = $2! \times 2! = 4$.

Example 115 *Find the number of ways in which* 8 *distinct flowers can be strung to form a garland so that* 4 *particular flowers are never separated.*

Solution: Considering 4 particular flowers as one group of flower, we have five flowers (one group of flowers and remaining four flowers) which can be strung to form a garland in $\frac{4!}{2}$ ways. But 4 particular flowers can be arranged themselves in 4! ways.
Thus, the required number of ways $= \frac{4! \times 4!}{2} = 288$.

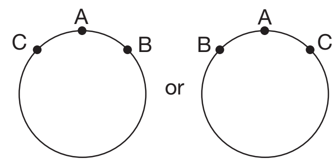
Example 116 *Find the number of arrangements in which g girls and b boys are to be seated around a table, $b \le g$, so that no two boys are together.*

Solution: g girls can be seated around a table in $(g - 1)!$
This positioning of g girls create g gaps for b boys to be seated. b boys in those g gaps can be seated in $\binom{g}{b} b!$ ways.

$$\text{Total number of arrangements} = (g - 1)! \times \binom{g}{b} b!.$$

Example 117 *Find the number of arrangements of* 10 *people including A, B, C such that B and C occupy the chairs next to A on a circular arrangement.*

Solution: 'A' occupies his chair in 1 way. B and C occupy their chairs in 2 ways.
Remaining 7 people occupy their chairs in 7 ! ways.
Total number of arrangements = $1 \times 2 \times 7$!



Aliter: Consider A, B, C as one person so there are 8 person and we can arrange them in 7! ways. Now B and C can interchange their position in 2! ways. So total ways = $2 \times 7!$.

Example 118 *Find the number of ways in which* 12 *distinct beads can be arranged to form a necklace.*

Solution: 12 distinct beads can be arranged among themselves in a circular order in $(12 - 1)! = 11!$ ways. Now in the case of necklace there is no distinction between clockwise and anti-clockwise arrangements. So the required number of arrangements $= \frac{1}{2}(11!)$.

Example 119 *How many necklace of* 12 *beads each can be made from* 18 *beads of various colours?*

Solution: In the case of necklace there is no distinction between the clockwise and anticlockwise arrangements, thus the required number of circular permutations

$$=\frac{\binom{18}{12}12!}{2\times 12}=\frac{18!}{6!\times 24}=\frac{18\times 17\times 16\times 15\times 14\times 13!}{6\times 5\times 4\times 3\times 2\times 1\times 24}=\frac{119\times 13!}{2}.$$

Example 120 *In how many ways can seven persons sit around a table so that all shall not have the same neighbours in any two arrangements?*

Solution: Clearly, 7 persons can sit at a round table in $(7-1)! = 6!$ ways. But, in clockwise and anti-clockwise arrangements, each person will have the same neighbours.

So, the required number of ways $=\frac{1}{2}(6!)=360$

Example 121 *If n distinct objects are arranged in a circle, show that the number of ways of selecting three of these things so that no two of them are next to each other is* $\frac{n}{6}(n-4)(n-5)$.

Solution: Let $a_1, a_2, a_3, \ldots, a_n$ be the n distinct objects.
Number of ways to select three objects so that no two of them are consecutive = Total number of ways to select three objects − Number of ways to select three consecutive objects – Number of ways to select three objects in which two are consecutive and one is separated. (1)

Total number of ways to select 3 objects from n distinct objects = ${}^nC_3$ (2)

Select three consecutive objects:
The three consecutive objects can be selected in the following manner.

Select from: $a_1\, a_2\, a_3,\ a_2\, a_3\, a_4,\ a_3\, a_4\, a_5, \ldots, a_{n-1}\, a_n\, a_1,\ a_n\, a_1\, a_2$

So, number of ways in which 3 consecutive objects can be selected from n objects arranged in a circle is n. (3)

Select two consecutive (together) and 1 separated:
The three objects so that 2 are consecutive and 1 is separated can be selected in the following manner:

Take $a_1\, a_2$ and select third object from $a_4, a_5, \ldots, a_{n-1}$, *i.e.*, take $a_1\, a_2$ and select third object in $(n-4)$ ways or in general we can say that select one pair from n available pairs, *i.e.,* $a_1\, a_2\; a_2\, a_3 \ldots a_n\, a_1$ and third object in $(n-4)$ ways.

So, number of ways to select 3 objects so that 2 are consecutive and 1 is separated $= n\,(n-4)$ (4)

Using (1), (2), (3) and (4), we get:

Number of ways to select 3 objects so that all are separated $= {}^nC_3 - n - n\,(n-4)$

$$=\frac{n(n-1)(n-2)}{6}-n-n(n-4)=n\left[\frac{n^2-3n+2-6(n-3)}{6}\right]$$

$$=\frac{n}{6}(n^2-9n+20)=\frac{n}{6}(n-4)(n-5).$$

Build-up Your Understanding 4

1. A cabinet of ministers consists of 11 ministers, one minister being the chief minister. A meeting is to be held in a room having a round table and 11 chairs round it, one of them being meant for the chairman. Find the number of ways in which the ministers can take their chairs such that the chief minister occupying the chairman's place.
2. 20 persons were invited for a party. In how many ways can they and the host be seated at a circular table? In how many of these ways will two particular persons be seated on either side of the host?
3. In how many ways can 7 boys be seated at a round table so that two particular boys are
   (i) next to each other (ii) separated.
4. A round table conference is to be held between 20 delegates of 2 countries. In how many ways can they be seated if two particular delegates
   (i) always sit together (ii) never sit together.
5. There are 20 persons including two brothers. In how many ways can they be arranged on a round table if:
   (i) There is exactly one person between the two brothers.
   (ii) The two brothers are always separated.
   (iii) What will be the corresponding answers if the two brothers were twins (alike in all respects)?
6. $2n$ chairs are arranged symmetrically around a table. There are $2n$ people, including A and B, who wish to occupy the chairs. Find the number of seating arrangements, if:
   (i) A and B are next to each other;
   (ii) A and B are diametrically opposite.
7. The 10 students of Batch B feel they have some conceptual doubt on circular permutation. Mr. Tiwari called them in discussion room and asked them to sit down around a circular table which is surrounded by 13 chairs. Mr. Tiwari told that his adjacent seat should not remain empty. Then find the number of ways, in which the students can sit around a round table if Mr. Tiwari also sit on a chair.
8. Find the number of ways in which 5 boys and 4 girls can be arranged on a circular table such that no two girls sit together and two particular boys are always together.
9. A person invites a party of 10 friends at dinner and place them
   (i) 5 at one round table, 5 at the other round table.
   (ii) 4 at one round table and 6 at other round table.
   Find the ratio of number of circular permutation of case (i) to case (ii).
10. Six persons A, B, C, D, E and F are to be seated at a circular table. Find the number of ways this can be done if A must have either B or C on his right B must have either C or D on his right.
11. Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated.
12. Find the number of different garlands, that can be formed using 3 flowers of one kind and 3 flowers of other kind.
13. Find the number of seating arrangements of 6 persons at three identical round tables if every table must be occupied.
14. Let $1 \le n \le r$. The Stirling number of the first kind, $S(m, n)$, is defined as the number of arrangements of m distinct objects around n identical circular tables so that each table contains atleast one object. Show that:
    (i) $S(m, 1) = (m - 1)!$;
    (ii) $S(m, m - 1) = {}^{m}C_2,\ m \ge 2$.

15. Find the number of different ways of painting a cube by using a different colour for each face from six available colours.

(Any two colour schemes are called different if one cannot coincide with the other by a rotation of the cube.)

16. Find number of ways in which n things of which r alike and the rest distinct can be arranged in a circle distinguishing between clockwise and anti-clockwise arrangement.

7.8 Division and Distribution of Non-identical Items in Fixed Size

7.8.1 Unequal Division and Distribution of Non-identical Objects

In this section we will discuss ways to divide non-identical objects into groups. For example, if we have to divide three different balls (b_1, b_2, b_3) among 2 boys (B_1 and B_2) such that B_1 gets 2 balls and B_2 gets 1 ball, then

Number of ways to divide balls among boys is 3 ways as shown in the following table.

B_1	B_2
b_1, b_2	b_3
b_2, b_3	b_1
b_3, b_1	b_2

Instead of writing all ways and counting them, we can make a formula to find number of ways.

First select 2 balls for B_1 in ${}^3C_2$ and then remaining 1 ball for B_2 in ${}^1C_1$ ways.

Total number of ways, using fundamental principle of counting, is

$= {}^3C_2 \times {}^1C_1 = 3 \times 1 = 3$ ways.

If we have to divide 3 non-identical balls among 2 boys such that one boy should get 2 and other boy should get 1, then following are the ways:

B_1	B_2
b_1, b_2	b_3
b_2, b_3	b_1
b_3, b_1	b_2
b_3	b_1, b_2
b_1,	b_2, b_3
b_2	b_3, b_1

Distribution of above 3 ways among 2 boys you can observe that entries are interchanged,between B_1 and B_2

$\Rightarrow$ Total ways to distribute = 6.

Instead of writing all ways and counting them, we can just find number of ways using fundamental principle of counting.

First select 2 balls for B_1 in ${}^3C_2$ ways, then select 1 remaining ball for B_2 in ${}^1C_1$ ways, finally distribute among 2 boys in $\lfloor 2$ ways (ball given to B_1 and B_2 are interchanged) because any boy can get 2 balls and the other 1 ball.

Using fundamental principle of counting, total number of ways

$= {}^3C_2 \times {}^1C_1 \times \lfloor 2 = 3 \times 1 \times 2 = 6$ ways.

Now generalising the above cases, we can write the following formula:

1. Number of ways in which $(m + n + p)$ distinct objects can be divided into 3 unequal (groups contain unequal number of objects) **unnumbered groups** containing m, n, p objects $= {}^{m+n+p}C_m \, {}^{n+p}C_n \, {}^{p}C_p = \dfrac{(m+n+p)!}{m!n!p!}$ (Here among m, n, p no two are equal)
2. Number of ways in which $(m + n + p)$ distinct objects can be divided and distribute into 3 unequal **numbered groups** (Here among m, n, p no two are equal) containing m, n, p objects
   = Number of ways to divide $(m+n+p)$ objects in 3 groups $\times$ Number of ways to distribute 'division-ways' among groups
   = Number of ways to divide $(m+n+p)$ objects in 3 groups $\times$ (Number of groups)!
   $= \dfrac{(m+n+p)!}{m!n!p!} \times 3!$

Above formulae are written for dividing objects into 3 groups but in case groups are more, then also we follow the same approach. For example,

Number of ways to divide 10 non-identical objects in 4 groups (G_1, G_2, G_3, G_4) such that groups G_1, G_2, G_3, G_4 gets 1, 2, 3, 4 objects respectively $= \dfrac{\lfloor 10}{\lfloor 1 \lfloor 2 \lfloor 3 \lfloor 4}$

Number of ways to divide 10 non-identical objects in 4 groups (G_1, G_2, G_3, G_4) such that groups get objects in number 1, 2, 3, 4 (*i.e.*, any group can get 1 object or 2 objects or 3 objects or 4 objects).

= Number of ways to divide and distribute 10 objects in 4 groups containing 1, 2, 3, 4 objects

$$= \frac{\lfloor 10}{\lfloor 1 \lfloor 2 \lfloor 3 \lfloor 4} \times \lfloor 4.$$

7.8.2 Equal Division and Distribution of Non-identical objects

Here we will see formulae to divide and distribute non-identical objects equally in groups, *i.e.*, each group get equal numbers of objects.

1. Number of ways to divide (mn) distinct objects equally in m unnumbered group (each group get n objects)
   $$\binom{mn}{n} \cdot \binom{mn-n}{n} \binom{mn-2n}{n} \cdots \binom{n}{n} \cdot \frac{1}{m!} = \frac{(mn)!}{(n!)^m \, m!}$$
2. Number of ways to divide (mn) objects equally in m numbered group (each group gets n objects)
   $$= \frac{(mn)!}{(n!)^m \, m!} \times m! = \frac{mn!}{(n!)^m}$$

Example 122 *In how many ways,* 12 *distinct objects can be distributed equally in* 3 *groups?*

Solution: Let the groups be labelled as A, B, C. (For our convenience)
Select 4 objects out of 12 to be given to group A in ${}^{12}C_4$ ways. Select 4 objects out of remaining 8 to be given to group B in ${}^{8}C_4$ ways. Rest 4 objects are to be given to group C in one way. (*i.e.,* ${}^{4}C_4$ ways)

Apparently, by multiplication principle, the total number of ways is ${}^{12}C_4 \cdot {}^{8}C_4 \cdot {}^{4}C_4$ but each grouping is counted 3! times ! ${}^{12}C_4 \cdot {}^{8}C_4 \cdot {}^{4}C_4$ is the number of **ordered** grouping.

Understand that, if objects are named as $a_1, a_2, a_3, \ldots, a_{12}$ then the grouping 12 elements as

$(a_1\, a_2\, a_3\, a_4)\,(a_5\, a_6\, a_7\, a_8)\,(a_9\, a_{10}\, a_{11}\, a_{12})$ is same as $(a_1\, a_2\, a_3\, a_4)\,(a_9\, a_{10}\, a_{11}\, a_{12})\,(a_5\, a_6\, a_7\, a_8)$ or same as $(a_9\, a_{10}\, a_{11}\, a_{12})\,(a_1\, a_2\, a_3\, a_4)\,(a_5\, a_6\, a_7\, a_8)$, etc.

$$\therefore \text{ Required number } = \frac{{}^{12}C_4 \cdot {}^{8}C_4 \cdot {}^{4}C_4}{3!} = \frac{12!}{4!8!} \cdot \frac{8!}{4!4!} \cdot \frac{1}{3!} = \frac{12!}{3!(4!)^3}.$$

Example 123 *In how many ways can* 12 *books be equally distributed among* 3 *students?*

Solution: In this question we have to divide books equally among 3 students. So we will use formulae (2) given in section 7.8.2. Where we divided non-identical objects equally among numbered groups as all students are distinct.

Therefore, number of ways to divide and distribute 12 non-identical objects among 3 students equally $= \dfrac{\lfloor 12}{(\lfloor 4)^3}$.

Example 124 *In how many ways we can divide 52 playing cards*

(i) *among 4 players equally?* (ii) *in 4 equal parts?*

Solution:

(i) 52 cards is to be divided equally among 4 players. Each player will get 13 cards. It means we should apply distribution formula. Using formula (2) given in section 7.8.2, we get:

Number of ways to divide playing cards $= \dfrac{\lfloor 52}{(\lfloor 13)^4}$

(ii) As we have to make 4 equal parts, each part consist of 13 cards. We will apply division formula (not distribution). Using formula (1) used in section 7.8.2 we get:

Number of ways to divide 52 cards in 4 parts $= \dfrac{\lfloor 52}{(\lfloor 13)^4}\dfrac{1}{\lfloor 4}$.

7.8.3 Equal as well as Unequal Division and Distribution of Non-identical Objects

Here we will see formulae to divide and distribute non-identical objects into groups such that not all groups contain equal or unequal number of objects, *i.e.*, some groups get equal and some get unequal number of objects.

1. Number of ways to divide $(ma + nb + nc)$ distinct (Out of a, b, c no two numbers are equal) objects in $(m + n + p)$ unnumbered groups such that m groups contains a objects each, n groups contains b objects each, p group contains c objects each

$$\frac{(ma+nb+nc)!}{(a!)^m (b!)^n (c!)^p \; m!\; n!\; p!}$$

Note: We divided by $m!$ because there are m groups containing a objects each (equal number of objects).
We divided by $n!$ also because there are n groups containing b objects each (equal number of objects). We also divided by $p!$ as p groups of are equal size.

2. Number of ways to divide and distribute $(ma + nb + pc)$ distinct objects (out of a, b, c no two numbers are equal) in $(m + n + p)$ numbered groups such that m groups contains a objects each, n groups contains b objects each, p groups contains c object each

$$= \frac{(ma+nb+pc)!}{(a!)^m (b!)^n (c!)^p \; m!\, n!\, p!} \times (m+n+p)!$$

We can make similar formulae for other cases.

Illustration Number of ways to divide 10 objects in 4 groups containing 3, 3, 2, 2 objects

$$\frac{\binom{10}{3}\binom{7}{3}}{2!}\frac{\binom{4}{2}\binom{2}{2}}{2!} = \frac{\lfloor 10}{(\lfloor 2)^2}\frac{1}{(\lfloor 3)^2}\frac{1}{\lfloor 2}\frac{1}{\lfloor 2}$$

Number of ways to divide and distribute completely 10 objects in 4 groups containing 3, 3, 2, 2 objects

$$\frac{\binom{10}{3}\binom{7}{3}}{2!}\frac{\binom{4}{2}\binom{2}{2}}{2!} \times 4! = \left[\frac{\lfloor 10}{(\lfloor 2)^2}\frac{1}{(\lfloor 3)^2}\frac{1}{\lfloor 2}\frac{1}{\lfloor 2}\right] \times \lfloor 4$$

Number of ways to divide and distribute $(m + 2n + 3p)$ distinct in 6 numbered groups such that 3 particular groups get p objects each, 2 particular gets n objects each, one one get m objects

$$= \frac{\lfloor m+2n+3p}{\lfloor m (\lfloor n)^2 (\lfloor p)^3}$$

Example 125 10 *different toys are to be distributed among* 10 *children. Find the total number of ways of distributing these toys so that exactly* 2 *children do not get any toy.*

Solution: It is possible in two mutually exclusive cases;

Case 1: 2 children get none, one child gets three and all remaining 7 children get one each.

Case 2: 2 children get none, 2 children get 2 each and all remaining 6 children get one each.
Using formula (2) given in section 7.8.3, we get:

Case 1: Number of ways $= \left(\frac{10!}{(0!)^2 \; 2!\,3!(1!)^7\,7!}\right)10!$

Case 2: Number of ways $= \left(\frac{10!}{(0!)^2\,2!\,(2!)^2\,2!\,(1!)^6\,6!}\right)10!$

Thus total ways $= (10!)^2\left(\frac{1}{3!7!2!} + \frac{1}{(2!)^4 6!}\right).$

Example 126 *In how many ways can 7 departments be divided among 3 ministers such that every minister gets at least one and atmost 4 departments to control?*

Solution: Let 3 minister be M_1, M_2, M_3.
Following are the ways in which we can divide 7 departments among 3 ministers such that each minister gets at least one and atmost 4.

S.No.	M_1	M_2	M_3
1	4	2	1
2	2	2	3
3	3	3	1

Note: If we have a case (2, 2, 3), then there is no need to make cases (3, 2, 2) or (2, 3, 2) because we will include them when we apply distribution formula to distribute ways of division among ministers.

Case 1: We divide 7 departments among 3 ministers in number 4, 2, 1, *i.e.*, unequal division. As any minister can get 4 departments, any one can get 2 any one can get 1 department, we should apply distribution formula. Using formula (2) given in section 7.8.1, we get:
Number of ways to divide and distribute departments in number 4, 2, 1

$$= \left[\frac{\lfloor 7}{\lfloor 4 \lfloor 2 \lfloor 1}\right] \times 3! = 630 \qquad (1)$$

Case 2: It is 'equal as well as unequal' division. As any minister can get any number of departments, we use complete distribution formula. Using formula (2) given in section 7.8.3 we get:
Number of ways to divide and distribute departments in number 2, 2, 3.

$$= \left[\frac{\lfloor 7}{\lfloor 2 \lfloor 2 \lfloor 3} \frac{1}{\lfloor 2}\right] \times 3! = 630 \qquad (2)$$

Case 3: It is also 'equal as well as unequal' division. As any minister can get any number of departments, we use complete distribution formula. Using formula (2) given in section 7.8.3 we get:
Number of ways to divide and distribute departments in number 3, 3, 1

$$= \left[\frac{\lfloor 7}{(\lfloor 3)^2 (\lfloor 1)} \frac{1}{\lfloor 2}\right] \times \lfloor 3 = 420 \qquad (3)$$

Combining (1), (2) and (3), we get number of ways to divide 7 departments among 3 minister = 630 + 630 + 420 = 1680 ways.

Build-up Your Understanding 5

1. Find the total number of ways of dividing 15 different things into groups of 8, 4 and 3 respectively.
2. Find the number of ways of distributing 50 identical things among 8 persons in such a way that three of them get 8 things each, two of them get 7 things each and remaining 3 get 4 things each.
3. Find the number of ways in which 14 men be partitioned into 6 committees where two of the committees contain 3 men each, and the others contain 2 men each.
4. If $3n$ different things can be equally distributed among 3 persons in k ways then find the number of ways to divide the $3n$ things in 3 equal groups.

5. Find the number of ways to give 16 different things to three persons A, B, C so that B gets 1 more than A and C gets 2 more than B.
6. Find the number of ways of distributing 10 different books among 4 students S_1, S_2, S_3 and S_4 such that S_1 and S_2 get 2 books each and S_3 and S_4 get 3 books each.
7. Find the number of different ways in which 8 different books can be distributed among 3 students, if each student receives at least 2 books.
8. Find the number of ways in which n different prizes can be distributed amongst m $(<n)$ persons if each is entitled to receive at most $n - 1$ prizes.
9. In a school there are two prizes for excellence in physics (Ist and IInd) two in Chemistry (Ist and IInd) and only 1 in Mathematics (Ist). In how many ways can these prizes be awarded to 20 students.
10. In an election three districts are to be canvassed by 2, 3 and 5 men respectively . If 10 men volunteer, then find the number of ways they can be allotted to the different districts.
11. A train time-table must be compiled for various days of the week so that two trains a day depart for three days, one train a day for two days and three trains a day for two days. Assuming all trains are identical how many different time-tables can be compiled?
12. In how many ways can 3 persons stay in 5 hotels? In how many of these each person stays in a different hotel.
13. 'n' different toys have to be distributed among 'n' children. Find the total number of ways in which these toys can be distributed so that exactly one child gets no toy.
14. Find the number of ways in which 7 different books can be given to 5 students if each can receive none, one or more books.
15. There are $(p + q)$ different books on different topics in Mathematics, where $p \neq q$. If L = The number of ways in which these books are distributed between two students X and Y such that X get p books and Y gets q books.
 M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books and another gets q books.
 N = The number of ways in which these books are divided into two groups of p books and q books.
 Then prove that $2L = M = 2N$.

7.9 Number of Integral Solutions

7.9.1 Number of Non-negative Integral Solutions of a Linear Equation

Let the given equation be

$$x_1 + x_2 + x_3 + \cdots + x_r = n$$

Let A be the set of all non-negative integral solutions of the given equation and B be the set of all $(n + r - 1)$ term binary sequences containing n, 1's and $(r - 1)$, 0's. Here number of 1's before the first zero is value of x_1, number of 1's between first zero and second zero is value of x_2 and so on, number of 1's after the $r - 1$th zero is the value of x_r.

So for every non-negative integral solution of the equation there is a binary sequence of n, 1's and $(r - 1)$, 0's. And for every binary sequence of n 1's and $(r - 1)$ 0's, we can write a non-negative integral solution. Therefore there is bijection between the sets A and B.

$\Rightarrow$ Number of non negative integral solutions of the equation is same as the number of binary sequences.

Number of non-negative integral solutions $= \dfrac{(n+r-1)!}{n!(r-1)!} = \dbinom{n+r-1}{r-1}$

Example 127 *Find the number of non-negative integral solutions of*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8$$

Solution: Take a sample solution,

$$\underset{x_1}{\boxed{2}} + \underset{x_2}{\boxed{0}} + \underset{x_3}{\boxed{3}} + \underset{x_4}{\boxed{2}} + \underset{x_5}{\boxed{1}} = 8 \tag{1}$$

Take a binary sequence of 8, 1's and 4, 0's as

$$110011101101 \tag{2}$$

which corresponds to the sample solution.

(2) is an arrangement of 12 objects, 8 of which are of one type and 4 of which are of another type.

Total number of such arrangements $= \dfrac{12!}{8!4!}$

= Total number of binary sequences of 8, 1's and 4, 0's.

$$\text{Number of non-negative integral solutions } = \frac{12!}{8!4!} = \frac{12\times 11\times 10\times 9}{4.3.2.1} = 495.$$

Observe that:

1. 0's we have used as demakers or separators. Since there are 4 gaps between the x_is, therefore we need 4 0's.
2. Pocket of x_2 is filled in the sample solution by 0 (that is the value of the variable; students are advised not to get confused between the value zero of a variable and a 0 used in the binary sequence) and the corresponding binary sequence shows a 0 followed by another 0.

Example 128 *Find the number of positive integral solutions of* $x_1 + x_2 + x_3 + x_4 + x_5 = 8$

Solution: Since we are interested in finding the number of positive solutions, therefore each x_i must have minimum value 1. So we take 8 identical coins (*i.e.*, similar to taking 8, 1's basically 8 identical objects to be taken) and 5 pockets of x_i' s.

$$\underset{x_1}{\boxed{}} + \underset{x_2}{\boxed{}} + \underset{x_3}{\boxed{}} + \underset{x_4}{\boxed{}} + \underset{x_5}{\boxed{}} = 8$$

Fill each pocket by one coin. So 3 coins are left, which are now to be filled in the pockets of x_i's.

Now this problem is similar to finding number of binary sequences of 3, 1's and 4, 0's. This number is

$$\frac{7!}{3!4!} = \frac{7\times 6\times 5}{3!}$$
$$= 35$$
$$= \text{Number of positive integral solutions.}$$

7.9.2 Number of Non-negative Integral Solutions of a Linear Inequation

Consider the given inequation as

$$x_1 + x_2 + x_3 + \cdots + x_r \le n \tag{1}$$

Add a non-negative integer x_{r+1} to get

$$x_1 + x_2 + x_3 + \cdots + x_r + x_{r+1} = n. \tag{2}$$

Number of solutions of Eq. (2)

$$= \binom{n+r}{r} = \frac{(n+r)!}{r!n!}$$

7.9.3 Number of Integral Solutions of a Linear Equation in $x_1, x_2, \ldots, x_r$ when x_i's are Constrained

Consider

$$x_1 + x_2 + x_3 + \cdots + x_r = n \tag{1}$$

where $x_1 \ge a_1, x_2 \ge a_2, \ldots, x_r \ge a_r,$ all a_i's are integers.

Take $x_1 = a_1 + x_1'$

$x_2 = a_2 + x_2'$, etc.,

where $x_1' \ge 0, x_2' \ge 0, \ldots, x_r' \ge 0$

Eq. (1) reduces to

$$(a_1 + a_2 + \ldots + a_r) + x_1' + x_2' + \cdots + x_r' = n$$

$$\Leftrightarrow x_1' + x_2' + \cdots + x_r' = n - (a_1 + a_2 + \cdots + a_r) \tag{2}$$

For every solution of Eq. (1), we can write a corresponding solution of Eq. (2) and for every solution of Eq. (2), we can write a corresponding solution of Eq. (1). Therefore there is a bijection between the sets of solutions of Eqs. (1) and (2).

Number of solutions of Eq. (1) = Number of non-negative integral solutions of Eq. (2)

$$= \frac{(n + r - 1 - (a_1 + a_2 + \cdots + a_r))!}{(r-1)!(n - (a_1 + a_2 + \cdots + a_r))!}$$

Example 129 *Find the number of integral solutions of* $x_1 + x_2 + x_3 + x_4 = 14$, *where* $x_1 \ge -2, x_2 \ge 1, x_3 \ge 2$ *and* $x_4 \ge 0$.

Solution: Let $x_1 = -2 + x_1', x_2 = 1 + x_2', x_3 = 2 + x_3'$,

Then given equation can be written as

$$x_1' + x_2' + x_3' + x_4 = 13, \quad x_1', x_2', x_3', x_4 \ge 0 \tag{1}$$

Number of non-negative integral solutions of Eq. (1)

$$= \frac{16!}{3!13!}$$

$$= \frac{16 \times 15 \times 14}{1 \times 2 \times 3}$$

$= 560$

= Number of integral solutions of the given equation.

Example 130 *How many integral solutions are there to $x + y + z + t = 29$, when $x \geq 1$, $y \geq 2$, $z \geq 3$ and $t \geq 0$?*

Solution: We have,

$$x \geq 1, y \geq 2, z \geq 3 \text{ and } t \geq 0, \text{ where } x, y, z, t \text{ are integers}$$

Let $u = x - 1$, $v = y - 2$, $w = z - 3$.

Then, $x \geq 1 \Rightarrow u \geq 0$; $y \geq 2 \Rightarrow v \geq 0$; $z \geq 3 \Rightarrow w \geq 0$

Thus, we have

$u + 1 + v + 2 + w + 3 + t = 29 \Rightarrow u + v + w + t = 23$ [where $u \geq 0$; $v \geq 0$; $w \geq 0$]

$\Rightarrow$ The total number of solutions of this equation is

$$^{23+4-1}C_{4-1} = {}^{26}C_3 = 2600.$$

Example 131 *How many integral solutions are there to the system of equations $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$ when $x_k \geq 0$?*

Solution: We have: $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$

These two equations reduce to

$$x_4 + x_5 = 15 \tag{1}$$

$$\text{and } x_1 + x_2 + x_3 = 5 \tag{2}$$

Since corresponding to each solution of Eq. (1) there are solutions of Eq. (2). So, total number of solutions of the given system of equations.

= Number of solutions of Eq. (1) $\times$ Number of solutions of Eq. (2)

$= ({}^{15+2-1}C_1)\,({}^{5+3-1}C_2) = {}^{16}C_1 \times {}^{7}C_2 = 336.$

7.10 Binomial, Multinomial and Generating Function

7.10.1 Binomial Theorem

Blaise Pascal

19 Jun 1623–19 Aug 1662

Nationality: French

Given $n, r \in N$, $0 \leq r \leq n$, the number $\binom{n}{r}$ or ^nC_r is defined to be the number of r elements subsets of an n elements set. These are also called the binomial coefficients as these occur as the coefficients in the expansion of

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n$$

Some important results related to summation of binomial coefficients:

1. $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r} = \binom{n}{m-r}\binom{n-m+r}{r}$
2. $\sum_{r=0}^{n}\binom{n}{r} = 2^n$; $\quad \sum_{r\geq 0}\binom{n}{2r} = \sum_{r\geq 0}\binom{n}{2r+1} = 2^{n-1}$
3. $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0$
4. $\sum_{r=0}^{n} r\binom{n}{r} = n \cdot 2^{n-1}$

5. $\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}=\binom{m}{0}\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\cdots+\binom{m}{r}\binom{n}{0}=\binom{m+n}{r}$

(Vandermonde Identity)

6. $\sum_{i+0}^{n}\binom{n}{i}^2=\binom{2n}{n}$

7. $\binom{r}{r}+\binom{r+1}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1}; n,\ r\in N, n\geq r$ (Hockey stick Identity)

8. $\binom{r}{0}+\binom{r+1}{r}+\cdots+\binom{r+k}{k}=\binom{r+k+1}{k}; r,k\in N$

Alexandre-Théophile van der Monde

28 Feb 1735–1 Jan 1796
Nationality: French

7.10.2 Binomial Theorem for Negative Integer Index

Given $n\in\mathbb{N}, x\in(-1,1)$

then $(1+x)^{-n}=\sum_{r\geq 0}^{\infty}\binom{n+r-1}{r}x^r$

7.10.3 Multinomial Coefficients

Like binomial coefficients, if we consider the expansion of $(x_1+\cdots+x_m)^n$, then we get the following expansion:

$(x_1+x_2+\cdots+x_m)^n=\sum\binom{n}{n_1,n_2,\ldots,n_m}x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$, where the sum is taken over all sequences $(n_1, n_2, \ldots, n_m)$ of non-negative integers with $\sum_{i=1}^{m}n_i=n$.

Here $\binom{n}{n_1,n_2,\ldots,n_m}=\frac{n!}{n_1!\cdot n_2!\cdots n_m!}$ is called multinomial coefficient.

7.10.3 Application of Generating Function

For large number of selection of objects we use 'Integral Equation Method followed by generating function'. In this method we group alike objects together and with each group we define a variable representing number of objects selected from the group. Then we add all variables and equate the sum to the total objects to be selected.

For example, if we have to select 3 objects from AAAAABBBBCCC objects, then we make groups of identical objects, group of all A objects, group of all B objects and group of all C objects. Let x_1, x_2, x_3 be the number of A, B, C objects selected respectively.

As total number of objects to be selected is 3, we can make following integral equation:

$$x_1+x_2+x_3=3 \qquad [\text{where } 0\leq x_i\leq 3,\ i=1,2,3]$$

Number of solutions of the above integral equation is same as the number of ways to select 3 objects from the given objects. This is because every solution of the equation is a way to select 3 objects.

Number of solutions of the equation

$$=\text{Coefficient of } x^{\text{Sum}} \text{ in } \left[x^{\min(x_1)}+x^{\min(x_1)+1}+\cdots+x^{\max(x_1)}\right]$$
$$\times\left[x^{\min(x_2)}+x^{\min(x_2)+1}+\cdots+x^{\max(x_2)}\right]\times\left[x^{\min(x_3)}+x^{\min(x_3)+1}+\cdots+x^{\max(x_3)}\right]$$

Note: Sum represents right hand side of the equation. For each variable x_1, x_2, x_3 a bracket is formed using the values the variable can take.

$\Rightarrow$ Number of solutions

$$= \text{Coefficient of } x^3 \text{ in } (x^0 + x^1 + x^2 + x^3)^3$$
$$= \text{Coefficient of } x^3 \text{ in } \left[\frac{1-x^4}{1-x}\right]^3$$
$$= \text{Coefficient of } x^3 \text{ in } (1 - x^4)^3 (1 - x)^{-3}$$
$$= \text{Coefficient of } x^3 \text{ in } ({}^3C_0 - {}^3C_1 x^4 + {}^3C_2 x^8 - {}^3C_3 x^{12}) (1 - x)^{-3}$$
$$= \text{Coefficient of } x^3 \text{ in } (1 - x)^{-3} \qquad \text{[as other terms cannot generate } x^3 \text{ term]}$$
$$= {}^{3+3-1}C_3 = {}^5C_3 = 10 \qquad \text{[using: coefficient of } x^r \text{ in } (1 - x)^{-n} = {}^{n+r-1}C_r]$$

Example 132 *In a box there are* 10 *balls,* 4 *red,* 3 *black,* 2 *white and* 1 *yellow. In how many ways can a child select* 4 *balls out of these* 10 *balls? (Assume that the balls of the same colour are identical)*

Solution: Let x_1, x_2, x_3 and x_4 be the number of red, black, white, yellow balls selected respectively.

Number of ways to select 4 balls

$= $ Number of integral solutions of the equation $(x_1 + x_2 + x_3 + x_4) = 4$

Conditions on x_1, x_2, x_3 and x_4:

The total number of red, black, white and yellow balls in the box are 4, 3, 2 and 1 respectively.

So we can take: Max $(x_1) = 4$, Max $(x_2) = 3$, Max $(x_3) = 2$, Max $(x_1) = 1$

There is no condition on minimum number of red, black, white and yellow balls selected, so take:

Min $(x_i) = 0$ for $i = 1, 2, 3, 4$

Number of ways to select 4 balls

$$= \text{Coefficient of } x^4 \text{ in } (1 + x + x^2 + x^3 + x^4) \times (1 + x + x^2 + x^3) \times (1 + x + x^2) \times (1 + x)$$
$$= \text{Coefficient of } x^4 \text{ in } (1 - x^5)(1 - x^4)(1 - x^3)(1 - x^2)(1 - x)^{-4}$$
$$= \text{Coefficient of } x^4 \text{ in } (1 - x^2 - x^3 - x^4)(1 - x)^{-4}$$
$$= \text{Coefficient of } x^4 \text{ in } (1 - x)^{-4} - \text{Coefficient of } x^2 \text{ in } (1 - x)^{-4} - \text{coeff of } x^1 \text{ in } (1 - x)^{-4} - \text{Coefficient of } x^0 \text{ in } (1 - x)^{-4}$$
$$= {}^7C_4 - {}^5C_2 - {}^4C_1 - {}^3C_0 = \frac{7 \times 6 \times 5}{3!} - 10 - 4 - 1 = 35 - 15 = 20$$

Thus, number of ways of selecting 4 balls from the box subjected to the given conditions is 20.

Alternate solution (Using 'case' method):

The 10 balls are RRRR BBB WW Y (where R, B, W, Y represent red, black, white and yellow balls respectively).

The work of selection of the balls from the box can be divided into following categories.

Case 1: All alike

Number of ways of selecting all alike balls $= {}^1C_1 = 1$

Case 2: 3 alike and 1 distinct

Number of ways of selecting 3 alike and 1 distinct balls $= {}^2C_1 \times {}^3C_1 = 6$

Case 3: 2 alike and 2 alike
Number of ways of selecting 2 alike and 2 alike balls = ${}^3C_2 = 3$

Case 4: 2 alike and 2 distinct
Number of ways of selecting 2 alike and 2 distinct balls = ${}^3C_1 \times {}^3C_2 = 9$

Case 5: All distinct
Number of ways of selecting all distinct balls = ${}^4C_4 = 1$

Total number of ways to select 4 balls = 1 + 6 + 3 + 9 + 1 = 20.

Example 133 *There are three papers of* 100 *marks each in an examination. Then find the number of ways in which a student can get* 150 *marks such that he gets atleast* 60% *in two papers.*

Solution: Suppose the student gets atleast 60% marks in first two papers, then he just get atmost 30% marks in the third paper to make a total of 150 marks.
Let, x_1, x_2, x_3 be marks obtained in 3 papers respectively. The total marks to be obtained is 150.

Therefore, Sum of marks obtained = 150

$$\Rightarrow x_1 + x_2 + x_3 = 150 \qquad (1)$$

$$60 \le x_1 \le 100;\ 60 \le x_2 \le 100;\ 0 \le x_3 \le 30.$$

The required number of ways = Number of integral solutions of Eq. (1)

= Coefficient of x^{150} in $\{(x^{60} + x^{61} + \cdots + x^{100})^2 (1 + x + x^2 + \cdots + x^{30})\}$
= Coefficient of x^{30} in $\{(1 + x + \cdots + x^{40})^2 (1 + x + \cdots + x^{30})\}$

= Coefficient of x^{30} in $\left(\dfrac{1-x^{41}}{1-x}\right)^2 \left(\dfrac{1-x^{31}}{1-x}\right)$

= Coefficient of x^{30} in $(1 - x)^{-3} = {}^{30+3-1}C_{3-1} = {}^{32}C_2$.

Thus, the student gets atleast 60% marks in first two papers to get 150 marks as total in ${}^{32}C_2$ ways. But the two papers, of atleast 60% marks, can be chosen out of 3 papers in ${}^3C_2$ ways.

Hence, the required number of ways = ${}^3C_2 \times {}^{32}C_2$.

Example 134 *Find the number of ways in which* 30 *marks can be allotted to* 8 *questions if each questions carries atleast* 2 *marks.*

Solution: Let $x_1, x_2, x_3, x_4, \ldots, x_8$ be marks allotted to 8 questions.
As total marks is 30, we can make following integral equation:

$$x_1 + x_2 + x_3 + \cdots + x_8 = 30.$$

It is given that every question should be of atleast 2 marks. It means

$$2 \le x_i \le 16\ \forall i = 1, 2, 3, \ldots, 8$$

The number of solutions of the integral equation is equal to number of ways to divide marks.

Number of solutions
= Coefficient of x^{30} in $(x^2 + x^3 + \cdots + x^{16})^8$
= Coefficient of x^{30} in $x^{16}(1 + x + \cdots + x^{14})^8$

= Coefficient of x^{14} in $\left(\dfrac{1-x^{15}}{1-x}\right)^8$

= Coefficient of x^{14} in $(1 - x)^{-8} = {}^{21}C_{14} = 116280$.

Alternate solution:

Let, the marks given in each question be;

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ [where x_i's ≥ 0 (i = 1, 2 … 8)]

and $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 30$

Let, $x_1 - 2 = y_1, x_2 - 2 = y_2, x_3 - 2 = y_3, x_4 - 2 = y_4, x_5 - 2 = y_5, x_6 - 2 = y_6, x_7 - 2 = y_7, x_8 - 2 = y_8$.

$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = 14$

where $0 \leq y_i$ $i = 1, 2, 3, \ldots, 8$

$\Rightarrow$Number of solutions $= {}^{14+8-1}C_{8-1} = {}^{21}C_7$.

Example 135 *In an examination the maximum marks for each of three papers is n and that for fourth paper is 2n. Find the number of ways in which a candidate can get 3n marks.*

Solution: Let x_1, x_2, x_3 and x_4 be the marks obtained in papers 1, 2, 3, 4 respectively. The total number of marks to be obtained by the candidate is $3n$.

Therefore, sum of marks obtained in various papers = $3n$.

$$\Rightarrow x_1 + x_2 + x_3 + x_4 = 3n \tag{1}$$

The total number of ways of getting $3n$ marks

= Number of solutions of the integral Eq. (1)

= Coefficient of x^{3n} in $(x^0 + x^1 + x^2 + \cdots + x^n)^3 \times (x^0 + x^1 + \cdots + x^{2n})$

= Coefficient of x^{3n} in $\left(\dfrac{1-x^{n+1}}{1-x}\right)^3 \left(\dfrac{1-x^{2n+1}}{1-x}\right)$

= Coefficient of x^{3n} in $(1 - x^{n+1})^3 (1 - x^{2n+1}) (1 - x)^{-4}$

= Coefficient of x^{3n} in $[(1 - 3x^{n+1} + 3x^{2n+2} - x^{3n+3}) (1 - x^{2n+1}) (1 - x)^{-4}]$

= Coefficient of x^{3n} in $[(1 - 3x^{n+1} - x^{2n+1} + 3x^{2n+2}) (1 - x)^{-4}]$

= Coefficient of x^{3n} in $(1 - x)^{-4}$ – 3 Coefficient of x^{2n-1} in $(1 - x)^{-4}$ – Coefficient of x^{n-1} in $(1 - x)^{-4}$ + 3 Coefficient of x^{n-2} in $(1 - x)^{-4}$

$= {}^{3n+4-1}C_{3n} - 3 \times {}^{2n-1+4-1}C_{2n-1} - {}^{n-1+4-1}C_{n-1} + 3 \times {}^{n-2+4-1}C_{n-2}$

$= {}^{3n+3}C_3 - 3 \times {}^{2n+2}C_3 - {}^{n+2}C_3 + 3 \times {}^{n+1}C_3$ [as ${}^nC_r = {}^nC_{n-r}$]

$$= \frac{(3n+3)(3n+2)(3n+1)}{6} - 3\frac{(2n+2)(2n+1)(2n)}{6} - \frac{(n+2)(n+1)(n)}{6} + 3\frac{(n+1)(n)(n-1)}{6}$$

$$= \frac{1}{2}(n + 1)(5n^2 + 10n + 6).$$

Example 136 *In a shooting competition a man can score* 5, 4, 3, 2 *or* 0 *points for each shot. Find the number of different ways in which he can score 30 in seven shots.*

Solution: Let $x_1, x_2, x_3, x_4, \ldots, x_7$ be the scores in 7 shots. As total score of 30 is

Sum of scores in 7 shots = 30

$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 30$[where $x_i \in \{0, 2, 3, 4, 5\}$ i = 1, 2, …, 7]

Number of solutions of above equation

Number of ways of making 30 in 7 shots to be taken,

Coefficient of x^{30} in $(x^0 + x^2 + x^3 + x^4 + x^5)^7$.

$\Rightarrow$ Coefficient of x^{30} in $\{(x^0 + x^2 + x^3) + x^4 (x + 1)\}^7$

$\Rightarrow$ Coefficient of x^{30} in $\{x^{28} (x + 1)^7 + {}^7C_1 x^{24}.(x + 1)^6.(1 + x^2 + x^3) + {}^7C_2\, x^{20} (x + 1)^5 (x^3 + x + 1)^2 + \cdots\}$ [using Binomial theorem]

= Number of ways to score 30

$\Rightarrow {}^7C_2 + {}^7C_1\,({}^6C_3 + {}^6C_2 + {}^6C_0) + {}^7C_2\,({}^5C_1 + 2)$

$\Rightarrow 21 + 252 + 147 = 420.$

Example 137 *Find the number of non-negative integral solutions of*

$$x_1 + x_2 + x_3 + 4x_4 = 20.$$

Solution: Number of non-negative integral solutions of the given equation

= Coefficient of x^{20} in $(1-x)^{-1}(1-x)^{-1}(1-x)^{-1}(1-x^4)^{-1}$

= Coefficient of x^{20} in $(1-x)^{-3}(1-x^4)^{-1}$

= Coefficient of x^{20} in $(1 + {}^3C_1x + {}^4C_2x^2 + {}^5C_3x^3 + {}^6C_4x^4 + \cdots)(1 + x^4 + x^8 + \cdots)$

$= 1 + {}^6C_4 + {}^{10}C_8 + {}^{14}C_{12} + {}^{18}C_{16} + {}^{22}C_{20} = 536.$

Build-up Your Understanding 6

1. Find the number of ways to select 10 balls from an unlimited number of red, white, blue and green balls.
2. Find the number of ordered triples of positive integers which are solutions of the equation $x + y + z = 100$.
3. Find the number of integral solutions of $x_1 + x_2 + x_3 = 0$, with $x_i \geq -5$.
4. Find the number of integral solutions for the equation $x + y + z + t = 20$, where x, y, z, t are all ≥ -1.
5. Find the number of integral solutions of $a + b + c + d + e = 22$, subject to $a \geq -3$, $b \geq 1$, $c, d, e \geq 0$.
6. If a, b, c are three natural numbers in AP and $a + b + c = 21$ then find the possible number of values of the ordered triplet (a, b, c).
7. If a, b, c, d are odd natural numbers such that $a + b + c + d = 20$ then then find the number of values of the ordered quadruplet (a, b, c, d).
8. Find the number of non-negative integral solution of the equation, $x + y + 3z = 33$.
9. Find the number of integral solutions of the equation $3x + y + z = 27$, where x, y, $z > 0$.
10. If a, b, c are positive integers such that $a + b + c \leq 8$ then find the number of possible values of the ordered triplet (a, b, c).
11. Find the number of non-negative integral solution of the inequation $x + y + z + w \leq 7$.
12. Find the number of non-negative even integral solutions of $x + y + z = 100$.
13. Find the number of non-negative integral solutions of $x + y + z + w \leq 23$.
14. Find the total number of positive integral solution of $15 < x_1 + x_2 + x_3 \leq 20$.
15. Find the number of non-negative integer solutions of $(a+b+c)(p+q+r+s) = 21$.
16. There are three piles of identical red, blue and green balls and each pile contains at least 10 balls. Find the number of ways of selecting 10 balls if twice as many red balls as green balls are to be selected.
17. Find the number of terms in a complete homogeneous expression of degree n in x, y and z.
18. In how many different ways can 3 persons A, B and C having 6 one rupee coins, 7 one rupee coins and 8 one rupee coins respectively donate 10 one rupee coins collectively.
    (i) If each one giving at least one coin
    (ii) If each one can give '0' or more coin.
    Also answer the above questions for 15 rupees donation.

19. In an examination, the maximum marks for each of the three papers are 50 each. Maximum marks for the fourth paper is 100. Find the number of ways in which a candidate can score 60% marks on the whole.
20. Between two junction stations A and B, there are 12 intermediate stations. Find the number of ways in which a train can be made to stop at 4 of these stations so that no two of these halting stations are consecutive.
21. The minimum marks required for clearing a certain screening paper is 210 out of 300. The screening paper consists of '3' sections each of Physics, Chemistry and Mathematics Each section has 100 as maximum marks. Assuming there is no negative marking and marks obtained in each section are integers, find the number of ways in which a student can qualify the examination (Assuming no subjectwise cut-off limit).
22. Find the number of ways in which the sum of upper faces of four distinct dices can be six.
23. How many integers > 100 and < 10^6 have the digital sum = 5?
24. In how many ways can 14 be scored by tossing a fair die thrice?
25. Find the number of positive integral solutions of $abc = 30$.
26. Find The number of positive integral solutions of the equation $x_1 x_2 x_3 x_4 x_5 = 1050$.
27. Let y be an element of the set $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and x_1, x_2, x_3 be positive integers such that $x_1 x_2 x_3 = y$, then find the number of positive integral solutions of $x_1 x_2 x_3 = y$.
28. Let $x_i \in \mathbb{Z}$ such that $|x_1 x_2 \ldots x_{10}| = 1080000$. Find number of solutions.
29. Let $x_i \in \mathbb{Z}$ such that $x_1 x_2 \ldots x_{10} = 180000$. Find Number of solutions.
30. Let $x_i \in \mathbb{Z}$, such that $|x_1| + |x_2| + \cdots + |x_{10}| = 100$. Find number of solutions.

7.11 Application of Recurrence Relations

Recurrence relation is a way of defining a series in terms of earlier member of the series with a few initial terms. It is complete description and much simpler than explicit formula. Here are some examples for use of recurrence relation.

Example 138 *Let there be n lines in a plane such that no two lines are parallel and no three are concurrent. Find the number of regions in which these lines divide the plane.*

Solution: Let a_n denotes required number of regions

Initial term $a_0 = 1, a_1 = 2, a_2 = 4$

Let number of region by $(n-1)$ lines be a_{n-1}.

Let us assume our plane be vertical and let us rotate it so that none of the $n - 1$ lines are horizontal.

Now draw nth line, horizontally, below all the point of intersections. All previous $n - 1$ lines meet the nth line at $n - 1$ different points. These points divides the nth line into n parts and each part falls in some old region and will divide the old region in two parts which will generate n new region.

n new regions are added to a_{n-1} regions

$$\Rightarrow a_n = a_{n-1} + n \quad \Rightarrow \quad a_n - a_{n-1} = n$$

$$\Rightarrow a_n - a_1 = \sum_{n=2}^{n} n$$

Hence, $$a_n = 1 + \sum n = 1 + \frac{n(n+1)}{2}. \qquad (\text{as } a_1 = 1)$$

Example 139 *Determine the number of regions that are created by n mutually overlapping circles in a plane. Assume that no three circles passing through same points and every two circles intersect in two distinct points.*

Solution: Let number of regions be h_n. Clearly $h_0 = 1; h_1 = 2, h_2 = 4, h_3 = 8$

It is tempting now to think $h_n = 2^n$ but by drawing diagram we see that $h_4 = 14$.

We obtain recurrence relation as follows:

Let $(n-1)$ mutually overlapping circle creating h_{n-1} regions.

Now draw nth circle. nth circle is intersected by each of $(n-1)$ circles in two points,

$\Rightarrow$ We are getting $2(n-1)$ distinct points, these points divides nth circle into $2(n-1)$ arcs. Each arc falls in some old region and will divide the old region in two parts and thus will generate $2(n-1)$ new regions.

$$\Rightarrow h_n = h_{n-1} + 2(n-1); n \geq 2$$

$$\Rightarrow h_n - h_{n-1} = 2(n-1)$$

$$\Rightarrow h_n = h_1 + 2\sum_{n=2}^{n}(n-1)$$

$$= h_n = 2 + 2\frac{n(n-1)}{2} = n^2 - n + 2. \qquad \text{(as } h_1 = 2\text{)}$$

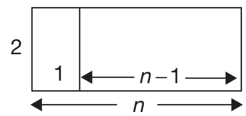
Example 140 *Determine number of ways to perfectly cover a $2 \times n$ board with dominoes (domino means a tile of size 2×1).*

Solution: Let number of ways be h_n. Then $h_0 = 1; h_1 = 1; h_2 = 2$

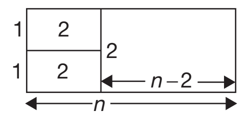
Let $n \geq 2$.

We divided the perfect covers of $2 \times n$ board into two parts A and B depending upon the domino placed at first place.

A: Those perfect covers in which there is a vertical domino at the first place as shown in figure.

(A) 

B: Those perfect covers in which there are two horizontal domino at the first place as shown in the figure.

(B) 

Now, perfect covers in A = perfect covers in $2 \times (n-1)$ board.

$\Rightarrow |A| = h_{n-1}$

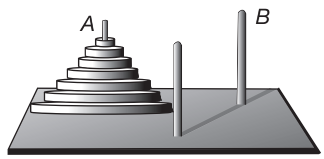
Similarly $|B| = h_{n-2}$

$\Rightarrow h_n = h_{n-1} + h_{n-2}$

This is our famous fibonacci sequence. Its general solution already discussed in the chapter of recurrence relation.

Example 141 *Tower of Brahma (or Tower of Hanoi) is a puzzle consisting of three pegs mounted on a board and n discs of different sizes. Initially all the n discs are stacked on the first peg so that any disc is always above a larger disc. The problem is to transfer all these discs to peg 2, with minimum number of moves, each move consisting of transferring one disc from any peg to another so that on the new peg the transferred disc will be on top of a larger disc (i.e., keeping a disc on a smaller one is not allowed).*

Find the total (minimum) number of moves required to do this.



Solution: Here again we shall give the explanation through four columns representing several number of the move: the positions of discs at each stage in peg 1, peg 2 and peg 3.

When there is just one disc, the problem is trivial, *i.e.*, in 1 move it is transferred directly to peg 2. We shall see the scheme of transfers for $n = 1, 2$ and 3, before finding the formula and proving it $n = 1$. Let name the discs as $d_1, d_2, \ldots, d_n$ with d_{i+1} to be smaller than d_i for all i, $1 \leq i \leq n - 1$.

Serial No. of the Moves	Peg 1	Peg 2	Peg 3
Initial stage	d_1	–	–
1		d_1	–

So in one move d_1 is transferred to peg 2, when $n = 1$, *i.e.*, total number of moves when $n = 1$ is 1.

$n = 2$, discs are d_1 and d_2, d_2 smaller than d_1.

Serial No. of the Moves	Peg 1	Peg 2	Peg 3
Initial stage	d_1, d_2	–	–
1	d_1		d_2
2	–	d_1	d_2
3	–	d_1, d_2	–

Thus, total no. of moves when $n = 2$ is 3.

$n = 3$, discs are d_1, d_2, d_3 with d_3 smaller than d_2, d_2 smaller than d_1.

Serial No. of the Moves	Peg 1	Peg 2	Peg 3
Initial stage	d_1, d_2, d_3	_	_
1	$d_1 d_2$	d_3	_
2	d_1	d_3	d_2
3	d_1	_	d_2, d_3
4	_	d_1	d_2, d_3
5	d_3	d_1	d_2
6	d_3	d_1, d_2	_
7	_	d_1, d_2, d_3	_

So, when there are 3 discs, *i.e.*, $n = 3$, the minimum number of moves is 7.

Note that here when the biggest disc alone is still in peg 1, all the discs are transferred to peg 3 and peg 2 is empty, so that the biggest one can now occupy peg 2. Then all the discs from peg 3 now can be transferred to peg 2 above the biggest one and it will again take as many times (to be transferred to peg 2), as it took to be transferred from peg 1 to peg 3.

Thus, to transfer two discs d_1 d_2 from peg 1 to peg 2:

d_2 goes to peg 3 in one move in the next move, d_1, goes to peg 2.

Now, disc d_2 takes the same 1 move to go to peg 2. Thus, the required number of moves is $1 + 2(1) = 3$.

Again, when there are 3 discs, as has been seen in the case of two discs, it takes 3 moves to transfer d_2 and d_3 to peg 3 (not peg 2 in this case) and it takes one move to transfer disc d_1 to peg 2 and it takes again another 3 moves to transfer discs d_2 and d_3 to peg 2.

So, the total number of moves $= 1 + 2 \times 3 = 7$. For 1 disc, there is one move; for 2 discs, there are $1 + (2 \times 1)$ moves or $2^2 - 1$; for 3 discs, there are $2\{1 + (2 \times 1)\} + 1$

$$= 2(2^2 - 1) + 1$$
$$= 2^3 - 2 + 1$$
$$= 2^3 - 1 \text{ moves.}$$

So, we can guess that when there are 4 discs, the number of moves is $2(2^3 - 1) + 1 = 2^4 - 2 + 1 = 2^4 - 1$.

Thus, to find the minimum number of moves, we can use the formula, $2^n - 1$, when there are n discs to be transferred from peg 1 to peg 2.

Now, proving this is very simple by using the principle of Mathematical induction.

We have already verified that this formula holds for the number of discs $n = 1, 2$ and 3.

So, let us assume that it holds for $n = k$, *i.e.*, when there are k discs, the minimum number of moves required to transfer the k discs from peg 1 to peg 2 is $2^k - 1$.

When there are $(k + 1)$ discs, we should verify if the number of moves is $2^{k+1} - 1$.

Serial No. of the Moves	Peg 1	Peg 2	Peg 3
After k discs are transferred $2^k - 1$	d_{k+1}	–	$d_1, d_2, \ldots, d_k$
2^{kth} move	–	d_{k+l}	$d_1, d_2, \ldots, d_k$

Now, by our assumption for $n = k$, it takes $2^k - 1$ moves to transfer $d_1, d_2, \ldots, d_k$ discs (k in all) to peg 2 from peg 3.

So, the total number of moves $= 2^k + 2^k - 1$

$$= 2 \cdot 2^k - 1 = 2^{k+1} - 1$$

Thus, whenever the formula to find the number of moves for $n = k$ (*i.e.*, no. of moves $= 2^k - 1$) is true, the formula is true for $n = k + 1$.

From the fact that the formula is true for $n = 1$, together with the last statement we find, that the formula is true for all $n \in \mathbb{N}$, *i.e.*, the minimum number of moves required to transfer n discs from peg 1 to peg 2, according to the given condition is $2^n - 1$.

Aliter: Let a_n be the minimum number of moves that will transfer n disks from one peg to other peg under given restriction. Then a_1 is obviously 1, and $a_2 = 3$.

Let as think when we can move the largest disk from the first peg? We first transfer the $n - 1$ smaller disk to peg 3 which requires a_{n-1} moves, then move the largest disk to peg 2 requiring one move and finally transfer the $n - 1$ smaller back to peg 2 on top of largest disk which require another a_{n-1} moves thus

$$a_n = a_{n-1} + 1 + a_{n-1}$$

$$\Rightarrow a_n = 2a_{n-1} + 1$$

$$\Rightarrow a_n + 1 = 2(a_{n-1} + 1)$$

$$\Rightarrow a_n + 1 = 2^{n-1}(a_1 + 1)$$

$$= 2^n \qquad (\text{as } a_1 = 1)$$

$$\Rightarrow a_n = 2^n - 1$$

7.12 Principle of Inclusion and Exclusion (PIE)

This principle is used in most counting situations.

The addition principle for counting is stated for disjoint sets as

$|A \cup B| = |A| + |B|$ or $n(A \cup B) = n(A) + n(B)$, where A and B are disjoint sets.

If A and B are not disjoint, then $|A \cup B| = |A| + |B| - (A \cap B)$.

We count the elements of A and B in turn and subtract the common elements of A and B, *i.e.*, the elements in $A \cap B$, as they are counted twice: firstly when we counted the elements of A and secondly, when we counted the elements of B.

Abraham de Moivre

26 May 1667–27 Nov 1754
Nationality: French

For three sets A, B and C, the counting principle states that

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

The general PIE is stated as follows:

For any sets $A_1 A_2, \ldots, A_n$, $n \geq 2$

$$|A_1 \cup A_2 \cup \cdots A_n|$$

$$= \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k}^{n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

In other words, consider properties $P_1, P_2, \ldots, P_n$. Let $n(A_k)$ or $|A_k|$ be the number of objects satisfying the property P_k, $k = 1, 2, \ldots, n$. A commonly asked question is 'how many elements satisfy atleast one of the properties '$P_1, P_2, \ldots, P_n$'?

This question is answered by the inclusion-exclusion principle which is stated below:

If $A_1, A_2, \ldots, A_m$ are m sets and $n(S)$ denotes the number of elements in the set S, then, $n\left(\bigcup_{k=1}^{m} A_k\right)$

$$= \sum_{k=1}^{m} n(A_k) - \sum_{1 \leq i < j \leq m} n(A_i \cap A_j) + \cdots + (-1)^{s-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_s \leq m} n\left(\bigcap_{k=1}^{s} A_{i_k}\right)$$

$$+ \cdots + (-1)^{m-1} n\left(\bigcap_{k=1}^{m} A_{i_k}\right)$$

Note that if $x \in \bigcup_{k=1}^{m} A_k$, then x belongs to at least one of $A_k, 1 \leq k \leq m$.

Note: For notational ease we may use $A_1 + A_2 + \cdots + A_k$ in place of $A_1 \cup A_2 \cup \ldots \cup A_k$ and $A_1 A_2 \ldots A_k$ in place of $A_1 \cap A_2 \cap \ldots \cap A_k$.

7.12.1 A Special Case of PIE

For any set $A_1, A_2, \ldots, A_n$, $n \geq 2$,

$$|A_1 + A_2 + \cdots + A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i A_j| + \sum_{i<j<k} |A_i A_j A_k| - \cdots + (-1)^{n-1} |A_1 A_2 \cdots A_n|$$

We consider here a special case of the principle of inclusion and exclusion.

In some applications we deal with properties, $a_1, a_2, \ldots, a_n$ and numerical values associated with properties, *i.e.,* $n(a_1), n(a_2), \ldots, n(a_n), n(a_1 a_2), \ldots, n(a_{n-1} a_n) \ldots$ and so on.

It is known that the numerical value assigned to a single property is a constant, and numerical values assigned to two properties $a_i a_j$, $i \neq j$ is also a constant and so on.

In other words

1. $n(a_1) = n(a_2) = \cdots = n(a_n)$
2. $n(a_1 a_2) = n(a_1 a_3) = \cdots = n(a_1 a_n) = n(a_2 a_3) = \cdots = n(a_{n-1} a_n)$
3. $n(a_1 a_2 a_3) = n(a_1 a_2 a_4) = \cdots = n(a_i a_j a_k)$, $i \neq j \neq k$
   and so on.

Again we denote by $N(1)$, the common value of the properties $a_1, a_2, \ldots, a_n$ taken one at a time, *i.e.,* $N(1) = n(a_1) = n(a_2) = \cdots = n(a_n)$.

$N(2)$ is the common value of the properties $a_1, a_2, \ldots, a_n$ when taken two at a time, etc. and $N(n)$ the number denoting the value $n(a_1a_2 \ldots a_n)$, *i.e.*, the number denoting the value of the properties when all of them are taken together and $N(0)$ is the value of $n(a_1'a_2' \ldots a_n')$ where a_i' is the complementary property of the property a_i and N is the value of collection of zero property or atleast one property.

Now, $$\sum_{i=1}^{n} n(a_i) = \binom{n}{1} N(1)$$

$$\sum_{i,j} n(a_i a_j) = \binom{n}{2} N(2)$$

$$\sum_{i,j,k,\ldots,r} n\left(\underbrace{a_i a_j a_k \ldots a_r}_{\text{taken } r \text{ at time}}\right) = \binom{n}{r} N(r)$$

$$n(a_1 a_2 \ldots a_n) = \binom{n}{n} N(n) = N(n)$$

Now, with this explanation, the principle of inclusion and exclusion takes the form

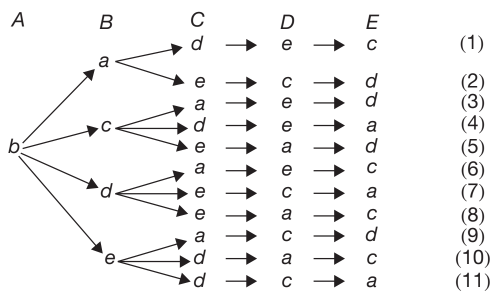
$$N(0) = N - \binom{n}{1}N(1) + \binom{n}{2}N(2) - \binom{n}{3}N(3) + \cdots + (-1)^{n+1}\binom{n}{n}N(n)$$

Example 142 *Five letters are written to five different persons and their addresses are written on five envelopes (one address on each envelope). In how many ways can the letters be placed in the envelopes so that no letter is placed in the correct envelope?*

Solution: Let us name the envelopes A, B, C, D, E and the corresponding letters a, b, c, d, e.

We shall now see, when the letter b is placed in envelope A, in how many ways the other 4 letters a, c, d, e can go to the wrong envelopes.

Envelopes



Thus for placing the letter b in envelope A, we have 11 different ways in which no letter goes to the correct envelope.

But we can also place c, d or e in envelope A, and in each case we get 11 different ways of placing letter in which no letter goes to the correct envelope.

Therefore, there are $11 \times 4 = 44$ different ways in which we can place the five letters, one in each of five envelopes so that no letter goes to the right envelope.

Aliter 1: Let us use special case of PIE

In to our problem of letters and envelopes, we take for each $i = 1, 2, 3, \ldots, 5$, k_i as the property that the letter a_t goes to the envelope A_i.

Here, $n = 5$,

$\therefore$ N = The total number of ways of 5 letters can be put into the envelopes = 5!

$$N(0) = N - \binom{5}{1}N(1) + \binom{5}{2}N(2) - \binom{5}{3}N(3) + \binom{5}{4}N(4) - \binom{5}{5}N(5)$$

$N(i)$ is the number of ways in which i letters go to i correct envelopes, so whatever happens to the other letters is $(5 - i)!$

Thus, $N(1) = 4! = 24$,

because $5 - 1 = 4$ letters can be placed in 4 envelopes in 4! ways and there is first one way of placing the letter in the correct envelope.

$$N(2) = 3! = 6,$$

since $5 - 2 = 3$ letters can be placed in 3 envelopes in $3! = 6$ different ways and again there is just one way of placing the two letters in their corresponding envelopes.

Similarly, $N(3) = (5 - 3)! = 2! = 2$

$N(4) = (5 - 4)! = 1$

$N(5) = (5 - 5)! = 0! = 1.$

$\therefore$ $N(0)$ = The number of ways that none of the letters go into the correct envelope is

$$5! - 5\times 4! + \frac{5\times 4}{1\cdot 2}\times 3! - \frac{5\times 4\times 3}{1\cdot 2\cdot 3}\times 2! + 5\times 1 - 1\times 1$$

$$= 120 - 120 + 60 - 20 + 5 - 1$$

$$= 44.$$

Aliter 2: See the formula given in derangement section 7.13

By using the given formula for $n = 5$, we get

$$D_5 = 5!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$$

$$= 5!\left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right]$$

$$= 60 - 20 + 5 - 1 = 44.$$

Example 143 *Find the number of positive integers from 1 to 1000, which are divisible by at least one of* 2, 3 *or* 5.

Solution: Let A_k be the set of positive integers from 1 to 1000, which are divisible by k.

Obviously we have to find $n(A_2 \cup A_3 \cup A_5)$.

$$n(A_2) = \left\lfloor \frac{1000}{2} \right\rfloor = 500, n(A_3) = \left\lfloor \frac{1000}{3} \right\rfloor = 333, n(A_5) = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

$$n(A_2 \cap A_3) = \left\lfloor \frac{1000}{6} \right\rfloor = 166, \text{similarly } n(A_3 \cap A_5) = 66, n(A_2 \cap A_5)$$

$$= 100, n(A_2 \cap A_3 \cap A_5) = 33.$$

Hence, $n(A_2 \cup A_3 \cup A_5) = 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734.$

Note that number of positive integers from 1 to 1000, which are not divisible by any of 2, 3 or 5 is

$1000 - n\ (A_2 \cup A_3 \cup A_5) = 266.$

Example 144 *Find the number of ways in which two Americans, two Britishers, one Chinese, one Dutch and one Egyptian can sit on a round table so that persons of the same nationality are separated.*

Solution: Total = 6!

$n(A)$ = when $A_1 A_2$ together = 5 ! 2 ! = 240

$n(B)$ = when $B_1 B_2$ together = 5 ! 2 ! = 240

$\Rightarrow n(A \cup B) = n(A) + n(B) - n(A \cap B) = 240 + 240 - 96 = 384$

Hence $n(\bar{A} \cap \bar{B})$ = Total $- n(A \cup B)$

$= 6! - 384$

$= 720 - 384$

$= 336.$

Example 145 *In how many ways can 5 cards be drawn from a complete deck (of 52 cards) so that all the suites are present? (Do not simplify.)*

Solution: Consider the notation: In a selection of 5 cards,

C: the set of selections in which clubs are absent

D: the set of selections in which diamonds are absent

S: the set of selections in which spades are absent

H: the set of selections in which hearts are absent

We have $|C| = |D| = |S| = |H| = {}^{39}C_5$,

$|C \cap D| = \ldots = {}^{26}C_5$,

$|C \cap D \cap S| = \ldots = {}^{13}C_5$,

and $|C \cap D \cap S \cap H| = 0$

Now $|C \cup D \cup S \cup H| = 4({}^{39}C_5) - 6({}^{26}C_5) + 4({}^{13}C_5) - 0$

Finally, the required number is

$$^{52}C_5 - 4\,{}^{39}C_5 + 6\,{}^{26}C_5 - 4\,{}^{13}C_5.$$

Example 146 *In how many ways can 6 distinguishable objects be distributed in four distinguishable boxes such that there is no empty box?*

Solution: The number of distributions such that:

(i) atleast one box is empty, is ${}^4C_1 \cdot 3^6$

(ii) atleast two boxes are empty, is ${}^4C_2 \cdot 2^6$

(iii) atleast three boxes are empty, is ${}^4C_3 \cdot 1^6$

The totality of distributions is 4^6.

Hence the required number is

$$4^6 - {}^4C_1\, 3^6 + {}^4C_2\, 2^6 - {}^4C_3\, 1^6 = 2260.$$

Note: If there should be exactly one empty box, then the number of distributions is

$$^4C_1(3^6 - {}^3C_1 .\, 2^6 + {}^3C_2 .\, 1^6) = 2160.$$

Example 147 *Find the number of ways to choose an ordered pair (a, b) of numbers from the set $\{1, 2, \ldots, 10\}$ such that $|a - b| \le 5$.*

Solution: Let $A_1 = [(a, b)/a, b \in \{1, 2, 3, \ldots, 10\}, /a - b/ = \{i\}, i = 0, 1, 2, 3, 4, 5.$

$A_0 = \{(i, i) | i = 1, 2, 3, \ldots, 10\}$ and $|A_0| = 10$

$A_1 = \{(i, i + 1) | i = 1, 2, 3, \ldots, 9\} \cup \{i, i - 1\} | i = 2, 3, \ldots, 10\}$ and $|A_1| = 9 + 9 = 18$

$$A_2 = \{(i, i+2) | i = 1, 2, 3, \ldots, 8\} \cup \{(i, i-2) | i = 3, 4, \ldots, 10\} \text{ and } |A_2| = 8 + 8 = 16$$
$$A_3 = \{(i, i+3) | i = 1, 2, \ldots, 7\} \cap \{(i, i-3) | i = 4, 5, \ldots, 10\} \text{ and } |A_4| = 6 + 6 = 12$$
$$A_4 = \{(i, i+4) | i = 1, 2, 3 \ldots, 6\} \cup \{(i, i-4) | i = 5, 6, \ldots, 10\} \text{ and } |A_4| = 6 + 6 = 12$$
$$A_5 = \{(i, i+5) | i = 1, 2, \ldots, 5\} \cup \{(i, i-5) | i = 6, 7, \ldots, 10\} \text{ and } |A_5| = 5 + 5 = 10$$

$\therefore$ The required set of pairs $(a, b) = \bigcup_{i=10}^{5} A_i$ and the number of such pairs, (which are disjoint)

$$\left|\bigcup_{i=10}^{5} A_i\right| = \sum_{i=10}^{5} |A_i| = 10 + 18 + 16 + 14 + 12 + 10 = 80.$$

Alternate: Total ways (without condition) = $10^2 = 100$

Let $b - a \geq 6$

$$1 \leq a < b \leq 10 \quad \Rightarrow \quad 1 \leq a < b - 5 \leq 5 \quad \Rightarrow \quad \binom{5}{2} = 10$$

Similarly for $a - b \geq 6$ we will get 10 ways.

Hence required answer = 100 – 10 – 10 = 80.

Example 148 *Identify the set S by the following information:*

(i) $S \cap \{3, 5, 8, 11\} = \{5, 8\}$
(ii) $S \cup (4, 5, 11, 13\} = \{4, 5, 7, 8, 11, 13\}$
(iii) $\{8, 13\} \subset S$
(iv) $S \subset \{5, 7, 8, 9, 11, 13\}$

Also, show that no three of the conditions suffice to identify S uniquely.

Solution: From (i),

$$5, 8 \in S \tag{1}$$

From (ii),

$$7, 8 \in S \tag{2}$$

From (iii),

$$8, 13 \in S \tag{3}$$

Therefore, from Eqs. (1), (2) and (3), we find that

$$5, 7, 8, 13 \in S \tag{4}$$
$$S \subset \{5, 7, 8, 9, 11, 13\} \qquad \text{(Given)}$$

If at all S contains any other element other than those given in (4), it may be 9 or 11 or both.

But $9 \notin S$. [$\because 9 \notin S \cup \{4, 5, 11, 13\} = \{4, 5, 7, 8, 11, 13\}$]

Again $11 \notin S$, for $11 \notin S \cap \{3, 5, 8, 11\} = \{5, 8\}$

$\therefore \quad S = \{5, 7, 8, 13\}$.

If condition (i) is not given, then S is not unique as S may be $\{7, 8, 13\}$ or $\{5, 7, 8, 13\}$ or $\{5, 7, 8, 11, 13\}$.

Similarly, deleting any other data leads to more than one solution to S (Verify.)

Example 149 *Suppose that in a poll made of 150 people, the following information was obtained: 70 of them read The Hindu, 80 read The Indian Express and 50 read Deccan Herald. 30 read both The Hindu and The Indian Express; 20 read both The Hindu and the Deccan Herald and 25 read both The Indian Express and Deccan Herald. Find at most how many of them read all the three.*

Solution: Let H, I and D be the set of those who read The Hindu, The Indian Express and the Deccan Herald, respectively.
So, the data given in mathematical symbols are as follows:

1. $|H \cup I \cup D| \leq 150$
2. $|H| = 70$
3. $|I| = 80$
4. $|D| = 50$
5. $|H \cap I| = 30$
6. $|H \cap D| = 20$
7. $|I \cap D| = 25$

We need to find the maximum possible value of $|H \cap I \cap D|$.

$$150 \geq |H \cup I \cup D| = |H| + |I| + |D| - |H \cap I| - |I \cap D| - |H \cap D| + |H \cap I \cap D|$$
$$\Rightarrow \quad 150 - 70 - 80 - 50 + 30 + 50 + 25 \geq |H \cap I \cap D|$$
$$\therefore \quad |H \cap I \cap D| \leq 25$$

$\therefore$ At most 25 of them read all the three. If every one of the 150 people interviewed read at least one of these three newspapers, then exactly 25 of them read all the three.

Example 150 *Lewis Carroll, the famous author of Alice in Wonderland, Through the Looking Glass, The hunting of the Shark and other wonderful works, was a mathematician whose real name was Charles Lutwidge Dodgson* (1832–1898). *Here is a problem from his book 'A Tangled Tale'.*

Let S be the set of pensioners, E the set of those who lost an eye, H those who lost an ear, A those who lost an arm and L those who lost a leg.

Given that $n(E) = 70\%$, $n(H) = 75\%$, $n(A) = 80\%$ *and* $n(L) = 85\%$. *Find what percentage at least must have lost all the four.*

Solution: Let $n(S)$ be 100.

$$\therefore \quad n(S) \geq n(E \cup H) = n(E) + n(H) - n(E \cap H)$$
$$\Rightarrow \quad 100 \geq 70 + 75 - n(E \cap H)$$
$$\Rightarrow \quad n(E \cap H) \geq 45.$$

Similarly $n(S) \geq n(L \cup A) = n(L) + n(A) - n(L \cap A)$

$$= 80 + 85 - n(L \cap A)$$
$$\Rightarrow \quad n(L \cap A) \geq 65.$$

Now, $n(S) = 100 \geq n[(E \cap H) \cup (L \cap A)]$

$$= n[(E \cap H) + n(L \cap A) - n(E \cap H \cap L \cap A)$$
$$\Rightarrow 100 \geq 45 + 65 - n(E \cap H \cap L \cap A)$$
$$\Rightarrow n(E \cap H \cap L \cap A) \geq 110 - 100 = 10.$$

That is at least 10% of the people must have lost all the four.

Example 151 *In the above problem, if those who lost all the four are more than* 10 *and less than* 70, *construct an example.*

Solution: Here we have to find

$$n(E \cap H \cap A \cap L) = 10 + k, \text{ where } 0 < k < 60.$$

We have $n[(E \cap H) \cup (A \cap L)] = n(E \cap H) + n(A \cap L) - n(E \cap H \cap A \cap L)$

But we know that $100 \geq n\,[(E \cap H) \cup (A \cap L)]$

$$\therefore \quad 100 \geq n(E \cap H) + n(A \cap L) - (10 + k)]$$
$$\Rightarrow \quad n(E \cap H) + n\,(A \cap L) \leq 110 + k.$$

$\therefore$ We can have $n(E \cap H)$ to be say $= (45 + k)$ and $n(A \cap L) = 65$.

But, $n(S) = 100 \geq (E \cup H) = n(E) + n(H) - n(E \cap H)$

$$\Rightarrow \quad 100 + n(E \cap H) \geq n(E \cup H) = n(E) + n(H) - n(E \cap H)$$
$$\Rightarrow \quad 145 + k \geq n(E) + n(H).$$

So, we can take $n(E) = 65 + k$, $n(H) = 80$.
Similarly, for $n\,(A \cap L)$

$100 \geq n(A) + n(L) - n(A \cap L)$

$\Rightarrow \quad 100 + n(A \cap L) \geq n(A) + n(L)$

$\Rightarrow \quad 165 \geq n(A) + n(L).$

We can take $n(A) = 75$, $n(L) = 90$

Now, we find

$$n(E) = 65 + k,\ n(H) = 80,\ n(A) = 75,\ n(L) = 90.$$

Let us check if we are correct in our choice of the cardinal number of each of these four.

$100 \geq n(E \cup H) = n(E) + n(H) - n(E \cap H)$

$\Rightarrow \quad n(E \cap H) \geq (65 + k) + 80 - 100 = 45 + k$

and again,

$$100 \geq n(A \cup L) = n(A) + n(L) - n(A \cap L)$$
$$= 75 + 90 - n(A \cap L)$$
$$\Rightarrow \quad n(A \cap L) \geq 65$$

again, $100 \geq n[(E \cap H) \cup (A \cap L)]$

$$= n(E \cap H) + n(A \cap L) - n(E \cap H \cap A \cap L)$$
$$\geq 45 + k + 65 - n(E \cap H \cap A \cap L)$$
$$\Rightarrow \quad n(E \cap H \cap A \cap L) \geq 10 + k \text{ as desired.}$$

In fact, this is just one solution. You can have yet a number of (only finite! Why don't you find them) other solutions. Once you get the cardinal number of the sets E, H, A and L, you can even combine E, A and H, L or E, L and H, A, as well. You shall get the same result.

For $n(S) = 100 \geq n(E \cup A) = n(E) + n(A) - n(E \cap A)$

$$\Rightarrow \quad n(E \cap A) \geq n(E) + n(A) - 100 = 65 + k + 75 - 100 = 40 + k$$

and Similarly $n(H \cap L) \geq n(H) + n(L) - 100 = 80 + 90 - 100 = 70$

$\therefore \quad n[(E \cap A) \cap (H \cap L)] \geq n(E \cap A) + n(H \cap L) - 100$

$\quad = 40 + k + 70 - 100 = 10 + k.$

You can verify this by taking the pairs of sets H, A and E, L.

Example 152 *a, b, c, d be integers ≥ 0, $d \leq a$, $d \leq b$, and $a + b = c + d$. Prove that there exist sets A and B satisfying $n(A) = a$, $n(B) = b$, $n(A \cup B) = c$, $n(A \cap B) = d$.*

Solution: $(A \cap B) \subset A$

$\Rightarrow \quad n(A \cap B) \leq n(A)$

or, $d \leq a$

Again, $(A \cap B) \leq B$

$$n(A \cap B) \leq n(B)$$

$$d \leq a$$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad n(A \cup B) + n(A \cap B) = n(A) + n(B)$$

$$\Rightarrow \quad c + d = a + b.$$

Example 153 *How many positive integers of n digits exist such that each digit is* 1, 2 *or* 3? *How many of these contain all three of the digits* 1, 2 *and* 3 *at least once?*

Solution: There are three digits 1, 2, 3 and an n-digit number is to be formed, repetitions allowed.

Thus, number of possibilities is $\underbrace{3 \times 3 \times 3 \times \cdots \times 3}_{n\,\text{times}} = 3^n$

For the second part of the question:

In (1), we include the possibility that all the n digits consist of (a) 1 only, (b) 2 only, (c) 3 only and again in (2), we include the possibility that the n digits consist of only (i) 1 and 2 (ii) 2 and 3 (iii) 1 and 3.

The number of n-digit numbers all of whose digits are 1 or 2 or 3 is 3^n.

(i) The number of n-digit numbers all of whose digits are 1 and 2, each of 1 and 2 occurring at least once is $2^n - 2$.

(ii) The number of n-digit numbers all of whose digits are 2 and 3, each of 2 and 3 occurring at least once is again $2^n - 2$.

(iii) The number of n-digit numbers all of whose digits are 1 and 3, each of 1 and 3 occurring at least once is $2^n - 2$.

Thus, the total numbers made up of the digits 1, 2 and 3 is

$$3^n - 3(2^n - 2) - 3 = 3^n - 3 \cdot 2^n + 3.$$

Example 154 *A, B and C are the set of all the positive divisors of* 10^{60}, 20^{50} *and* 30^{40}, *respectively. Find* $n(A \cup B \cup C)$.

Solution: Let $n(A)$ = number of positive divisors of

$$10^{60} = 2^{60} \times 5^{60} \text{ is } 61^2$$

$n(B)$ = number of positive divisors of

$$20^{50} = 2^{100} \times 5^{50} \text{ is } 101 \times 51 \text{ and}$$

$n(C)$ = number of positive divisors of

$$30^{40} = 2^{40} \times 3^{40} \times 5^{40} = 41^3$$

The set of common factors of A and B will be of the form $2^m \cdot 5^n$ where $0 \leq m \leq 60$ and $0 \leq n \leq 50$.

So, $\quad n(A \cap B) = 61 \times 51.$

Similarly, since the common factors of B and C and A and C are also of the form $2^m \times 5^n$,

and in the former case $0 \leq m \leq 40$,

$$0 \leq n \leq 40,$$

and in the latter case $0 \leq m \leq 40$,

$$0 \leq n \leq 40,$$

$\therefore \quad n(B \cap C) = 41^2 \quad$ also $\quad n(A \cap C) = 41^2$

and, $n(A \cap B \cap C)$ is also 41^2.

$\therefore$ $n(A \cap B \cap C)$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$
$$= 61^2 + 101 \times 51 + 41^3 - 61 \times 51 - 41^2 - 41^2 + 41^2$$
$$= 61(61 - 51) + 41^2(41 - 1) + 101 \times 51$$
$$= 610 + 1681 \times 40 + 5151 = 73001.$$

Example 155 *Find the number of integer solutions to the equation $x_1 + x_2 + x_3 = 28$ where $3 \le x_1 \le 9$, $0 \le x_2 \le 8$ and $7 \le x_3 \le 17$.*

Solution: Consider three numbered boxes whose contents are denoted as x_1, x_2, x_3, respectively. The problem now reduces to distributing 28 balls in the three boxes such that the first box has at least 3 and not more than 9 balls, the second box has at most 8 balls and the third box has at least 7 and at most 17 balls. At first, put 3 balls in the first box, and 7 balls in the third box. This takes care of the minimum needs of the boxes. So, now the problem reduces to finding the number of distribution of 18 balls in 3 boxes such that the first has at most $(9 - 3) = 6$, the second at most 8 and the third at most $(17 - 7) = 10$. The number of ways of distributing 18 balls in 3 boxes with no condition is $\binom{18+3-1}{3-1} = \binom{20}{2} = 190$.

[See article 7.14: The number of ways of distributing n identical objects in r distinct boxes is $\binom{n+r-1}{r-1}$ where 'r' stands for the numbers of boxes and n for balls.]

Let d_1 be the distribution where the first box gets at least 7; d_2, the distributions where the second box gets at least 9 and d_3, the distributions where the third gets at least 11.

$$|d_1| = \binom{18-7+3-1}{3-1} = \binom{13}{2} = \frac{13 \times 12}{1.2} = 78$$

$$|d_2| = \binom{18-9+3-1}{3-1} = \binom{11}{2} = \frac{11 \times 10}{1.2} = 55$$

$$|d_3| = \binom{18-11+3-1}{3-1} = \binom{9}{2} = \frac{9 \times 8}{1.2} = 36$$

$$\therefore \quad |d_1 \cap d_2| = \binom{18-7-9+3-1}{3-1} = \binom{4}{2} = 6$$

$$|d_2 \cap d_3| = \binom{18-9-11+3-1}{3-1} = \binom{0}{2} = 0,$$

$$|d_3 \cap d_1| = \binom{18-11-7+3-1}{3-1} = \binom{2}{2} = 1.$$

Also, $|d_1 \cap d_2 \cap d_3| = 0$,

$$\Rightarrow \quad |d_1 \cup d_2 \cup d_3| = 78 + 55 + 36 - 6 - 0 - 1 + 0 = 162.$$

So, the required number of solutions = 190 − 162 = 28.

Note: The number of ways the first box gets at most 6, the second at most 8 and the third at most 10 = Total number of ways of getting 18 balls distributed in 3 boxes − (the number of ways of getting at least 7 in the first box, or at least 9 in the second box or at least 11 in the third box).

Example 156 *I have six friends and during a certain vacation, I met them during several dinners. I found that I dined with all the six exactly on 1 day, with every five of them on 2 days, with every four of them on 3 days, with every three of them on 4 days and with every two of them on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?*

Solution: For $i = 1, 2, 3, \ldots, 6$, let A_i be the set of days on which ith friend is present at dinner.

Then given $n(A_i)$ or $|A_i| = 7$ and $|A_i'| = 7$.

So, $|A_i \cap A_j| = 5$, $|A_i \cap A_j \cap A_k| = 4$, $|A_i \cap A_j \cap A_k \cap A_l| = 3$, $|A_i \cap A_j \cap A_k \cap A_l \cap A_m| = 2$,

and, $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| = 1$.

where i, j, k, l, m vary between 1 to 6 and are distinct.

$$|A_1 \cup A_2 \cup A_3 \ldots \cup A_6|$$

$$= \sum_{i=1}^{6} |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \sum |A_i \cap A_j \cap A_k \cap A_l|$$

$$+ \sum |A_i \cap A_j \cap A_k \cap A_l \cap A_m| - |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6|$$

$$= \binom{6}{1}\times 7 - \binom{6}{2}\times 5 + \binom{6}{3}\times 4 - \binom{6}{4}\times 3 + \binom{6}{5}\times 2 - \binom{6}{6}\times 1$$

$= 42 - 75 + 80 - 45 + 12 - 1 = 13$.

The total number of dinners $|A_i| + |A_i'| = 7 + 7 = 14$.

The number of dinners in which at least one friend was present $= |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = 13$.

The number of dinners I dine alone $= 14 - 13 = 1$.

Aliter: Let the proposer of the problem be called X, and the friends be denoted as A, B, C, D, E, F. Since X dines with all the 6 friend exactly on one day. We have the combination $XABCDEF$ (1) for one day.

Thus, every five of A, B, C, D, E, F had already dined with X for a day. According to the problem, every five of them should dine on another day. It should happen in ${}^nC_5 = 6$ days. The combination is $XABCDEF$ (2), $XABCDF$ (3), $XABCEF$ (4), $XABDEF$ (5), $XACDEF$ (6), $XBCDEF$ (7).

In (1) and (2) together, X has already dined with every four friends three times, for example, with $ABCD$, he dined on the first day the numbers above the combinations can be taken as the rank of the days X dined with his friends. 2nd and 3rd days, X has dined with every three friends of them on four days, for example, with ABC, 1st, 2nd, 3rd and 4th days, X has dined with every two friends, of them for five days for example, with AB, 1st, 2nd, 3rd, 4th and 5th days,

With just one of them he has dined so far 6 days (with A, 1st, 2nd, 3rd, 4th, 5th and 6th days).

So, he has to dine with every one of them for one more day he should dine with XA, XB, XC, XD, XE and XF for 6 more days. Thus, the total number of days he dined so far with at least one of his friends is 1 + 6 + 6 = 13 days. In this counting, we see that he has dined with every one of them for 7 days. That shows that he has not dined with every one of them for 6 days.

But it is given that every friend was absent for 7 days. Since each one of them has been absent for 6 days already, all of them have to be absent for one more day.

Thus, he dined alone for 1 day and the total number of dinners he had is 13 + 1 = 14.

Example 157 *A student on vacation for d days observed that* (a) *it rained seven times morning or afternoon;* (b) *when it rained in the afternoon, it was clear in the morning;* (c) *there were five clear afternoon and* (d) *there were six clear mornings. Find d.*

Solution: Let the set of days it rained in the morning be M_r and the set of days it rained in the afternoon be A_r.

Then, clearly the set of days when there were clear morning is M'_r and the set of days when there were clear afternoon is A'_r.

By condition (b), we get $M_r \cap A_r = \phi$,

By (d), *we* get $M'_r = 6$,

By (c), we get $A'_r = 5$,

and by (a), we get $M_r \cup A_r = 7$.

M_r and A_r are disjoint sets and $n(M_r) = d - 6$, $n(A_r) = d - 5$.

$\therefore$ Applying the principle of inclusion and exclusion, we get'

$$n(M_r \cup A_r) = n(M_r) + n(A_r) - n(M_r \cap A_r)$$
$$\Rightarrow \quad 7 = (d - 6) + (\mathrm{d} - 5) - 0$$
$$\Rightarrow \quad 2d = 18$$
$$\Rightarrow \quad d = 9.$$

Aliter: Observe the tabular columns for rainy mornings, rainy afternoons, clear mornings and clear afternoons.

	Rainy afternoon	Clear afternoon
Rainy morning	x	y
Clear morning	z	w

Now, by the hypothesis, we have

$$x + y + z + w = d \tag{1}$$
$$x + y + z = 7 \tag{2}$$
$$y + w = 5 \tag{3}$$
$$z + w = 6 \tag{4}$$

By condition (b), $x = 0$.

From Eqs. (3) and (4),

$$y + z + 2w = 11 \tag{5}$$

From Eq. (2),

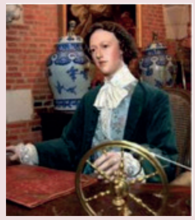
$$y + z = 7 \tag{6}$$

Solving Eqs. (5) and (6), we get

$$2w = 4 \quad \text{or}$$
$$w = 2$$
$$\therefore \quad d = x + y + z + w = 0 + y + z + w$$
$$= 0 + 7 + 2 = 9.$$

7.13 Derangement

Pierre Raymond de Montmort



27 Oct 1678–7 Oct 1719
Nationality: French

A derangement of 1, 2, …, n is a permutation of the numbers such that no number occupies its natural position. Thus (2, 3,1) and (3, 1, 4, 2) are derangements. On the other hand, (2, 4, 3, 5, 1) is not a derangement as 3 is at the 3rd position.

The total number of derangements of 1, 2, …, n will be denoted by D_n.

It is easy to realise that $D_1 = 0$, $D_2 = 1$ and $D_3 = 2$, etc.

$$D_n = n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!}\right]$$

Proof: Let A_i be the collection of all ways such that i be at ith position. Now we need to get D_n which is $N(A'_1 A'_2 A'_3 \ldots A'_n)$. Using special inclusion and exclusion formula we get

$$N(A'_1A'_2\ldots A'_n) = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^r\binom{n}{r}(n-r)! + \cdots$$

$$= n! - \frac{n!}{(n-1)!\times 1!}\times(n-1)! + \frac{n!}{(n-2)!\times 2!}\times(n-2)! - \cdots + (-1)^r\frac{n!}{(n-r)!\times r!}(n-r)! + \cdots + (-1)^n$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^r\frac{n!}{r!} + \cdots + (-1)^n$$

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{(-1)^r}{r!}+\cdots+\frac{(-1)^n}{n!}\right].$$

For an alternate proof see the Example 158.

Note that $\lim\limits_{n\to\infty} D_n = 1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\cdots+\dfrac{(-1)^n}{n!} = e^{-1}$.

For example, let S_1, S_2, S_3 are three slots where objects A, B, C should be placed. Number of ways to place A, B, C in S_1, S_2, S_3 such that A goes to S_1, B goes to S_2 and C goes to S_3, *i.e.*, all object are placed in there correct places = 1. Number of way to place only one object in a wrong slot is not possible because if A is placed in say S_2, then B, whose correct slot is S_2, would take either S_1 or S_3. It means B is also placed in the wrong slot. So it is not possible to place only one object in wrong slot. To place objects A, B, C in S_1, S_2, S_3 such that all objects are placed in wrong slots we use derangement formulae, *i.e.*,

Number of way to place A, B, C all in wrong slots

$$= \underline{|3}\left[1-\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}\right] = 2 \text{ ways.}$$

Example 158 *On a rainy day n people go to a party. Each of them leaves his raincoat at the counter of the gate. Find the number of ways in which the raincoats are handed over to the guests after the function is over so that no one receives his/her own raincoat.*

Solution: Let us name the guests as $g_1, g_2, \ldots, g_n$ and their raincoats as $r_1, r_2, \ldots, r_n$, respectively.

Let us denote number of ways for the event that no one gets his/her raincoat by D_n.

We shall find a recurrence relation for D_n, as follows:

For g_1 there are $(n-1)$ possible ways of getting the wrong raincoats.

If g_1 is given the raincoat r_2, Case (1) r_1 may be given to g_2 or Case (2) r_1 may not be given to g_2.

In case (1) if g_2 receives r_1 then the remaining $(n-2)$ guests may not get their raincoats in D_{n-2} different ways.

In case (2) if g_2 does not receive the raincoat r_1 then the number of ways in which g_2 does not receive r_1, g_3 does not receive $r_3, \ldots, g_n$ does not receive r_n is D_{n-1} as there are $(n-1)$ guests and also $(n-1)$ raincoats.

Thus, the total number of ways in which the remaining $(n-1)$ guests do not receive their raincoats is $D_{n-1} + D_{n-2}$ as the two cases mutually exclusive.

For each one way of giving the wrong raincoat to g_1 there are $D_{n-1} + D_{n-2}$ ways that the remaining $(n-1)$ guests get the wrong raincoats.

But there are $(n-1)$ different ways in which g_1 can get a wrong raincoat.

So, $$D_n = (n-1)[D_{n-1} + D_{n-2}]$$

or $$D_n = nD_{n-1} - D_{n-1} + (n-1)\, D_{n-2}$$

or
$$\begin{aligned} D_n - nD_{n-1} &= -[D_{n-1} - (n-1)D_{n-2}] \qquad (1)\\ &= (-1)^2[D_{n-2} - (n-2)\, D_{n-3}] \qquad (2)\\ &= (-1)^3[D_{n-3} - (n-3)D_{n-4}]\\ &\vdots \quad \vdots \quad \vdots\\ &= (-1)^{n-2}[D_2 - 2D_1] \end{aligned}$$

[Here replacing n by $(n-1)$ in Eq. (1), we get $D_{n-1} - (n-1)D_{n-2} = -\{D_{n-2} - (n-2)\, D_{n-3}\}$ and hence from Eq. (1), we get Eq. (2) and so on.]

$\therefore$ We have,

$$D_n - nD_{n-1} = (-1)^{n-2}[D_2 - 2D_1].$$

Now, $D_1 = 0$, $D_2 = 1$, since D_1 stands for just one guest that does not get his/her raincoat, which is clearly zero.

Also $D_2 = 1$, since there are just two guests, there is only one way of getting their raincoats exchanged so that neither of the two get their raincoat.

$$\therefore\ D_n - nD_{n-1} = (-1)^{n-2}(1-0) = (-1)^{n-2} = (-1)^n$$

$$\therefore\ \frac{D_n}{n!} - \frac{nD_{n-1}}{n!} = \frac{(-1)^n}{n!}$$

$$\Rightarrow\ \frac{D_n}{n!} - \frac{D_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!}.$$

Substituting $n-1, n-2, \ldots$ for n successively, we get

$$\frac{D_{n-1}}{(n-1)!} - \frac{D_{n-2}}{(n-2)!} = \frac{(-1)^{n-1}}{(n-1)!}$$

$$\frac{D_{n-2}}{(n-2)!} - \frac{D_{n-3}}{(n-3)!} = \frac{(-1)^{n-2}}{(n-2)!}$$

$$\vdots \quad \vdots$$

$$\frac{D_2}{2!} - \frac{D_1}{1!} = \frac{(-1)^2}{2!}.$$

Adding both the sides, we get,

$$\frac{D_n}{n!}-\frac{D_1}{1!}=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{(-1)^n}{n!}$$

$$\Rightarrow \quad D_n=n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!}\right](\because\ D_1=0)$$

Note that $1-\frac{1}{1!}=0,$ and thus zero is added to the right hand side to get the formula in the proper format.

Aliter: Use derangement formulae (which was obtained by using the special inclusion and exclusion principle).

Example 159 *Find D_4.*

Solution: The totality of permutations of 1, 2, 3, 4 is 4!
The number of permutations, which leave fixed

(i) atleast one of 1, 2, 3, 4, is $^4C_1\ 3!$
(ii) atleast two of 1, 2, 3, 4 is $^4C_2\ 2!$
(iii) atleast three of 1, 2, 3, 4 is $^4C_3\ 1!$ and, finally,
(iv) all of 1, 2, 3, 4, is 1

By the inclusion-exclusion principle,

$$D_4=4!-{}^4C_1\ 3!+{}^4C_2\ 2!-{}^4C_3\ 1!+1=9.$$

Example 160 *Find the number of permutations of 1, 2, 3, 4, 5 in which exactly one number occupies its natural position.*

Solution: Choose the number which should occupy its natural position ($^5C_1$)
The number of arrangements of the others is D_4.
Hence the required number $={}^5C_1\cdot D_4=45$.

Example 161 *There are 5 boxes of 5 different colours. Also there are 5 balls of colours same as those of the boxes. In how many ways we can place 5 balls in 5 boxes such that*

(i) *all balls are placed in the boxes of colours not same as those of the ball.*
(ii) *at least 2 balls are placed in boxes of the same colour.*

Solution:

(i) All the balls should be placed in the wrong boxes.
That is, boxes not of the colour same as balls.
Using derangement formulae, number of ways in which this can be done.

$$=\lfloor 5\left[1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}-\frac{1}{\lfloor 5}\right]$$

$$=120\left[1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\right]$$

$$=60-20+5-1=44.$$

(ii) Atleast 2 balls are placed in the correct boxes, *i.e.*, boxes of the colour same as ball = Total number of ways to place balls in boxes – Number of ways to place balls such that all balls are placed in wrong boxes – Number of ways to place balls in boxes such that 1 ball is placed in the correct box (*i.e.*, box of the same colour as balls).

$= \lfloor 5 - 44 -$ Number of ways to select a ball that will be in correct box $\times$ Number of ways in which remaining 4 balls can be placed in 4 boxes such that all balls go in wrong boxes (boxes of colour different from balls).

$$= \lfloor 5 - 44 - {}^5C_1 \times \lfloor 4 \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4}\right]$$

$= 120 - 44 - 5 \times 9$ [using answer of (i) part and derangement formulae]

$= 120 - 44 - 45$

$= 31.$

Example 162 *In how many ways 6 letters can be placed in 6 envelopes such that*

(i) *No letter is placed in its corresponding envelope.*
(ii) *at least 4 letters are placed in correct envelopes.*
(iii) *at most 3 letters are placed in wrong envelopes.*

Solution:

(i) Using derangement formulae:

Number of ways to place 6 letters in 6 envelopes such that all are placed in wrong envelopes.

$$= 6!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{6!}\right]$$

$$= 360 - 120 + 30 - 6 + 1 = 265.$$

(ii) Number of ways to place letters such that at least 4 letters are placed in correct envelopes

$=$ 4 letters are placed in correct envelopes and 2 are in wrong $+$5 letters are placed in correct envelopes and 1 in wrong $+$ All 6 letters are placed in correct envelopes

$= {}^6C_4 \times 1 + 0$ (not possible to place 1 in wrong envelope) $+1 = \frac{6 \times 5}{2} + 1 = 16.$

(iii) Number of ways to place 6 letters in 6 envelopes such that at most 3 letters are placed in wrong envelopes

$=$ 0 letter is wrong envelope and 6 in correct $+$ 1 letter in wrong envelop and 5 in correct $+$ 2 letters in wrong envelopes and 4 are in correct $+$ 3 letters in wrong envelopes and 3 in correct

$= 1 + 0$ (not possible to place 1 in wrong envelope) $+ {}^6C_4 \times 1 + {}^6C_3 \lfloor 3 \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3}\right]$

$$= 1 + \frac{6 \times 5}{2} + \frac{6 \times 5 \times 4}{6}\left(\frac{\lfloor 3}{\lfloor 2} - \frac{-\lfloor 3}{\lfloor 3}\right)$$

$= 1 + 15 + 20 \times 2 = 56.$

Build-up Your Understanding 7

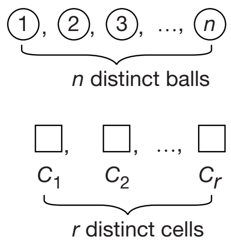
1. Find the numbers from 1 to 100 which are neither divisible by 2 nor by 3 nor by 7.
2. Find the number of numbers, from amongst 1, 2, 3, …, 500, which are divisible by none of 2, 3, 5.
3. Find the number of 3 element subsets of the set $\{1, 2, \ldots, 10\}$, in which the least element is 3 or the greatest element is 7.
4. Find the number of n digit numbers, which contain the digits 2 and 7, but not the digits 0, 1, 8, 9.
5. How many integers from 1 through 999 do not have any repeated digits?
6. Find the number of natural numbers less than or equal to 10^8 which are neither perfect squares, nor perfect cubes, nor perfect fifth powers.

7. In a certain state, license plates consist of from zero to three letters followed by from zero to four digits, with the provision, however, that a blank plate is not allowed.
(i) How many different license plates can the state produce?
(ii) Suppose 85 letter combinations are not allowed because of their potential for giving offense. How many different license plates can the state produce?
8. If the number of ways of selecting K coupons one by one out of an unlimited number of coupons bearing the letters A, T, M so that they cannot be used to spell the word MAT is 93, then find K.
9. How many positive integers divide 10^{40} or 20^{30}?
10. Find the number of permutations of letters a, b, c, d, e, f, g taken all together if neither 'beg' nor 'cad' pattern appear.
11. Find the number of permutations of the letters of the word HINDUSTAN such that neither the pattern 'HIN' nor 'DUS' nor 'TAN' appears.
12. Find the number of permutations of the 8 letters AABBCCDD, taken all at a time, such that no two adjacent letters are alike.
13. Find the number of non-negative integer solutions of $x_1 + x_2 + x_3 = 15$, subject to $x_1 \leq 5$, $x_2 \leq 6$, and $x_3 \leq 7$.
14. According to the Gregorian calendar, a leap year is defined as a year n such that (i) n divides 4 but not 100; or (ii) n divides 400.
Find the number of leap years from the year 1000 to the year 3000, inclusive.
15. Find the number of onto functions from a set containing 6 elements to a set containing 3 elements.
16. How many 6-digit numbers contain exactly three different digits?
17. Let D_n be the nth derangement number. Prove that
(i) $D_n = (n-1)(D_{n-1} + D_{n-2}), n > 2;$
(ii) $\lim\limits_{n\to\infty} \dfrac{D_n}{n!} = \dfrac{1}{e}$
18. Show that n letters in n corresponding envelopes can be put such that none of the letters goes to the correct envelop is $n!\left(1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\cdots+\dfrac{(-1)^n}{n!}\right)$ ways.
19. Five pairs of hand gloves of different colours are to be distributed to each of five people. Each person must get a left glove and a right glove. Find the number of distributions so that,exactly one person gets a proper pair.
20. Prove (combinatorially) that $\sum_{r=1}^{n} r!r = (n+1)! - 1.$
21. In maths paper there is a question on 'Match the column' in which column A contains 6 entries and each entry of column A corresponds to exactly one of the 6 entries given in column B written randomly. 2 marks are awarded for each correct matching and 1 mark is deducted from each incorrect matching. A student having no subjective knowledge decides to match all the 6 entries randomly. Find the number of ways in which he can answer, to get atleast 25% marks in this question.
22. Ten parabolas are drawn in a plane. Any two parabola intersect in four real, and distinct, points. No three parabola are concurrent. Find the total number of disjoint regions of the plane.
23. In how many ways can a 12 step staircase be climbed taking 1 step or 2 steps at a time?
24. A coin is tossed 10 times. Find the number of outcomes in which 2 heads are not successive.
25. Find the number of ways to pave a 1×7 rectangle by 1×1, 1×2, 1×3 tiles, if tiles of the same size are indistinguishable.

7.14 Classical Occupancy Problems

The problems of the number of distributions of balls into cells are called occupancy problems. We distinguish several cases as described below:

7.14.1 Distinguishable Balls and Distinguishable Cells



1. Number of ways to divide n non-identical balls in r different cells such that each cell gets 0 or more number of balls (empty cells are allowed) $= r^n$.
2. If no cell is empty, then the number is determined by the inclusion/exclusion principle or by recurrence relation or by generating function method. Using any one of them we can get number of ways to divide n non-identical balls in r different cells such that each cell gets at least one object (empty cells are not allowed)

$$= r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n - {}^rC_3 (r-3)^n + \dots (-1)^{r-1}\, {}^rC_{r-1} 1^n.$$

Example 163 *Find the number of distributions of 5 distinguishable balls in 3 distinguishable cells, if*

(i) *an empty cell is allowed;*
(ii) *no cell is empty.*

Solution:

(i) $3^5 = 243$.
(ii) Method 1:

The five balls can be distributed in 3 non-identical boxes in the following 2 ways:

Boxes	Box1	Box 2	Box 3
Number of balls	3	1	1
Number of balls	2	2	1

Case 1: 3 in one Box, 1 in another and 1 in third Box (3, 1, 1) (1)
Number of ways to divide balls corresponding to (1)

$$= \frac{5!}{3!\,1!\,1!}\frac{1}{2!} = 10$$

But corresponding to each division there are 3! ways of distributing the balls into 3 boxes.

So number of ways of distributing balls corresponding to (1)
= (Number of ways to divide balls) × 3! = 10 × 3! = 60

Case 2: 2 in one Box, 2 in another and 1 in third Box (2, 2, 1) (2)
Number of ways to divide balls corresponding to (2)

$$= \frac{5!}{2!\,2!\,1!}\frac{1}{2!} = 15$$

But corresponding to each division there are 3! ways of distributing balls into 3 boxes.

So number of ways of distributing balls corresponding to (2)
= (Number of ways to divide balls) × 3!
= 15 × 3! = 90
Hence, required number of ways = 60 + 90 = 150.

Method 2:
Let us name the Boxes as A, B and C. Then there are following possibilities of placing the balls.

Box A	Box A	Box A	Number of ways
1	2	2	${}^5C_1 \times {}^4C_2 \times {}^2C_2 = 30$
1	1	3	${}^5C_1 \times {}^4C_1 \times {}^3C_3 = 20$
1	3	1	${}^5C_1 \times {}^4C_3 \times {}^1C_1 = 20$
2	1	2	${}^5C_2 \times {}^3C_1 \times {}^2C_2 = 30$
2	2	1	${}^5C_2 \times {}^3C_2 \times {}^1C_1 = 30$
3	1	1	${}^5C_3 \times {}^2C_1 \times {}^1C_1 = 20$

Therefore required number of ways of placing the balls

$$= 30 + 20 + 20 + 30 + 30 + 20 = 150$$

Method 3:

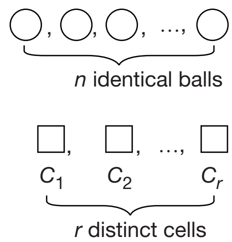
Number of ways of distributing 5 balls in 3 boxes so that no Box is empty

$$r^n - {}^rC_1\,(r-1)^n + {}^rC_2\,(r-2)^n - {}^rC_2\,(r-3)^n + \cdots$$

Put $n = 5$ and $r = 3$ to get:

Number of ways $= 3^5 - {}^3C_1\, 2^5 + {}^3C_1\, 1^5 = 243 - 3 \times 32 + 3 = 246 - 96 = 150$ ways.

7.14.2 Identical Balls and Distinguishable Cells



If an empty cell is allowed, then the number of distributions is $\binom{n+r-1}{r-1}$ (use binary sequences).

In other words the number of ways to divide n identical objects into r groups (different) such that each gets 0 or more objects (empty groups are allowed) $= {}^{n+r-1}C_{r-1}$.

Proof:

Let $x_1, x_2, x_3, \ldots, x_r$ be the number of objects given to groups 1, 2, 3, …, r respectively. As total objects to be divided is n, we can take

Sum of the objects given to all groups $= n$

$$\Rightarrow x_1 + x_2 + x_3 + x_4 + \cdots + x_r = n.$$

This equation is known as integral equation as all variables are integer.

As each group can get 0 or more, following are constraints on integer variables.

$$0 \le x_1 \le n\,;\, 0 \le x_2 \le n, \ldots\, 0 \le x_r \le n, \text{ i.e., } 0 \le x_i \le n\ i = 1, 2, 3, \ldots, r.$$

We can observe that number of integral solutions of the above equation is equal to number of ways to divide n identical objects among r groups such that each gets 0 or more.

$$= {}^{n+r-1}C_n = {}^{n+r-1}C_{r-1}.$$

If no cell is allowed to remain empty, then the number is ${}^{n-1}C_{r-1}$.

In other words the number of ways to divide n identical objects into r groups (different) such that each group receives at least one object (empty groups are not allowed).

$$= {}^{n-1}C_{r-1}.$$

Example 164 *How many terms are there in the expansion of* $(a + b + c + d)^{24}$?

Solution: A typical term is $a^{k_1} \cdot b^{k_2} \cdot c^{k_3} \cdot d^{k_4}$, where k_1, k_2, k_3, k_4 are non-negative integers whose sum $= 24$.

The number of terms is the same as the number of distributions of 24 identical balls in four distinguishable cells, empty cell allowed. This is ${}^{24+4-1}C_{24} = {}^{27}C_{24}$.

Example 165 *Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.*

Solution: Let x_1, x_2 and x_3 be the number of balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.
The number of ways of distributing 5 balls into Boxes 1, 2 and 3 is the number of integral solutions of the equation $x_1 + x_2 + x_3 = 5$ subjected to the following conditions on x_1, x_2, x_3. (1)

Conditions on x_1, x_2 and x_3:
According to the condition that the boxes should contain at least one ball, we can find the range of x_1, x_2 and x_3, *i.e.*,

Min $(x_i) = 1$ and Max $(x_i) = 3$ for $i = 1, 2, 3$ [using: Max $(x_1) = 5 -$ Min $(x_2) -$ Min(x_3)]
or $1 \le x_i \le 3$ for $i = 1, 2, 3$
So, number of ways of distributing balls
= Number of integral solutions of (1)
= Coefficient of x^5 in the expansion of $(x + x^2 + x^3)^3$
= Coefficient of x^5 in $x^3 (1 - x^3) (1 - x)^{-3}$
= Coefficient of x^2 in $(1 - x^3) (1 - x)^{-3}$
= Coefficient of x^2 in $(1 - x)^{-3}$ [as x^3 cannot generate x^2 terms]
$= {}^{3+2-1}C_2 = {}^4C_2 = 6.$

Alternate solution:
The number of ways of dividing n identical objects into r groups so that no group remains empty

$$= {}^{n-1}C_{r-1}$$
$$= {}^{5-1}C_{3-1} = {}^4C_2 = 6.$$

Example 166 *Find the number of ways of distributing* 10 *identical balls in* 3 *boxes so that no box contains more than four balls and less than* 2 *balls.*

Solution: Let x_1, x_2 and x_3 be the number of balls placed in Boxes 1, 2 and 3 respectively.
Number of ways of distributing 10 balls in 3 boxes
= Number of integral solutions of the equation $x_1 + x_2 + x_3 = 10$ (1)

Conditions on x_1, x_2 and x_3:
As the boxes should contain atmost 4 ball and at least 2 balls, we can make

Max $(x_i) = 4$ and Min $(x_i) = 2$ for $i = 1, 2, 3$
or $2 \le x_i \le 4$ for $i = 1, 2, 3$
So the number of ways of distributing balls in boxes
= Number of integral solutions of equation (i)
= Coefficient of x^{10} in the expansion of $(x^2 + x^3 + x^4)^3$
= Coefficient of x^{10} in $x^6 (1 - x^3)^3 (1 - x)^{-3}$
= Coefficient of x^4 in $(1 - x^3)^3 (1 - x)^{-3}$
= Coefficient of x^4 in $(1 - {}^3C_1 x^3 + {}^3C_2 x^6 + \cdots) (1 - x)^{-3}$
= Coefficient of x^4 in $(1 - x)^{-3}$ – Coefficient of x in ${}^3C_1 (1 - x)^{-3}$
$= {}^{4+3-1}C_4 - 3 \times {}^{3+1-1}C_1 = {}^6C_4 - 3 \times {}^3C_1 = 15 - 9 = 6.$

Example 167 *Find the number of ways in which* 14 *identical toys can be distributed among three boys so that each one gets atleast one toy and no two boys get equal number of toys.*

Solution: Let the boys get a, $a + b$ and $a + b + c$ toys respectively.

$$a + (a + b) + (a + b + c) = 14,\ a \geq 1,\ b \geq 1,\ c \geq 1$$

$$\Rightarrow 3a + 2b + c = 14,\ a \geq 1,\ b \geq 1,\ c \geq 1$$

$\therefore$ The number of solutions

$= \text{Coefficient of } t^{14} \text{ in } \{(t^3 + t^6 + t^9 + \ldots)(t^2 + t^4 + \cdots)(t + t^2 + \cdots)\}$

$= \text{Coefficient of } t^8 \text{ in } \{(1 + t^3 + t^6 + \cdots)(1 + t^2 + t^4 + \ldots)(1 + t + t^2 + \cdots)\}$

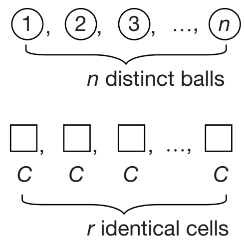
$= \text{Coefficient of } t^8 \text{ in } \{(1 + t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8)(1 + t + t^2 + \cdots + t^8)\}$

$= 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 = 10.$

Since, three distinct numbers can be assigned to three boys in 3! ways.

So, total number of ways $= 10 \times 3! = 60$.

7.14.3 Distinguishable Balls and Identical Cells



Label the balls by the natural numbers 1, 2, …, n. A partition of $\{1, 2, \ldots, n\}$ in r part is a set of r non-empty subsets, $A_1, A_2, \ldots, A_r$ of $\{1, 2, \ldots n\}$ such that $A_1 \cup A_2 \cup \ldots \cup A_r = \{1, 2, \ldots, n\}$ and any two of $A_1, \ldots A_r$ are disjoint.

For example, $\{\{1\}, \{2, 3\}, \{4\}\}$ is a 3 partition of $\{1, 2, 3, 4\}$.

Denote the number of r partitions of $\{1, 2, \ldots, n\}$ by $S(n, r)$.

$S(n, r)$ is called a Stirling number of the second kind**.**

It is easy to see that:

$S(n, 1) = 1$, $S(n, n) = 1$, $S(n, r) = 0$, if $r > n$.

To determine $S(n, r)$ for $1 < r < n$.

There are two possibilities:

1. The number n is by itself is a partition.
   $\Rightarrow$ The numbers 1, 2, …, $n - 1$ must form a $r - 1$ partition.
   The number of such partitions $= S\,(n - 1, r - 1)$.
2. The number n is along with atleast one of 1, 2, …, $n - 1$ in a partition.
   $\Rightarrow$ The numbers 1, 2, …, $n - 1$ must form a r partition and n must be inserted in any one of the r subsets. So n can be put in r ways.

The number of such partitions $= r\,S(n - 1, r)$

Hence $S(n, r) = S(n - 1, r - 1) + r\,S(n - 1, r), 1 < r < n$

Use this to show that $S(n, 2) = 2^{n-1} - 1$

In general, we can easily get

$$S(n, r) = \frac{1}{r!}\left[r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \cdots + (-1)^{r-1}\binom{r}{r-1}1^n\right]$$

Note: If n distinguishable balls are to be distributed in r identical cells, an empty cell allowed, then the number of distributions is $\sum_{k=1}^{r} S(n, k)$.

Example 168 *Find the number of distributions of* 5 *distinguishable balls in* 3 *identical cells, an empty cell allowed.*

Solution: The sought after number is $S(5, 1) + S(5, 2) + S(5, 3)$.
Now $S(5, 1) = 1$, $S(5, 2) = 2^{5-1} - 1 = 15$, and

$$\begin{aligned} S(5, 3) &= S(4, 2) + 3S(4, 3) \\ &= (2^3 - 1) + 3(S(3, 2) + 3\,S(3, 3)) \\ &= 7 + 3((2^2 - 1) + 3) \\ &= 25 \end{aligned}$$

Hence, the answer is $1 + 15 + 25 = 41$.

Leonhard Euler

15 Apr 1707–18 Sep 1783
Nationality: Swiss

7.14.4 Identical Balls and Identical Cells

Consider the problem of distributing n identical balls in k identical cells, no cell remaining empty.

The number of distributions = The number of ways of writing n as the sum $\underbrace{x_1 + x_2 + \cdots + x_k}_{\text{positive integers}}$, the order of terms being ignored = number of **Partition** of n in k parts.

This is equivalent to number of integral solution of $x_1 + x_2 + x_3 + \cdots + x_k = n$ with $1 \le x_1 \le x_2 \le x_3 \le \cdots \le x_k$ which is equal to $[x^n]$ in $\dfrac{x^k}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^k)}$

Aleternatively denote this number by $P_k(n)$.

Clearly, $P_1(n) = P_n(n) = 1$, $P_2(n) = \left\lfloor \dfrac{n}{2} \right\rfloor$, $P_k(n) = 0$, $k > n$

For example, $\left.\begin{aligned} 5 &= 2+2+1 \\ &= 3+1+1 \end{aligned}\right\} \Rightarrow P_3(5) = 2$

To determine $P_k(n)$, $1 < k < n$

Let us divide all partitions in two types:

(A) Atleaset one partition of size 1

(B) No partition of size 1

Number of partitions of type A is $p_{k-1}(n-1)$ (As make one partition of size 1 and remaining $n - 1$ in $k - 1$ parts). Number of partitions of type B is $p_k(n - k)$ (As first remove k objects and divide $n - k$ objects in k parts). Now add one object in each part so that each part will be of size atleast 2.

Hence, $P_k(n) = P_{k-1}(n-1) + P_k(n-k)$, $1 < k \le \left\lfloor \dfrac{n}{2} \right\rfloor$

Using the above reccurence we can easily prove $P_3(n) = \left\langle \dfrac{n^2}{12} \right\rangle$. Read it "nearest integer" (see the Example 169).

Note: If n identical balls are to be distributed in r identical cells, an empty cell allowed, then the number is $\sum_{k=1}^{r} P_k(n)$.

Example 169 *What is the number of necklaces that can be made from 6n identical blue beads and 3 identical red beads?*

Solution: The sought after number is $P_3(6n) + P_2(6n) + P_1(6n)$.

We have

$$P_k(n) - P_k(n-k) = P_{k-1}(n-1)$$

$$\Rightarrow \quad P_3(6n) - P_3(6n-3) = P_2(6n-1) = \left\lfloor \frac{6n-1}{2} \right\rfloor = 3n - 1 \qquad (1)$$

and $P_3(6n-3) - P_3(6n-6) = P_2(6n-4) = \left\lfloor \dfrac{6n-4}{2} \right\rfloor = 3n-2$ (2)

Adding (1) and (2), we get, $P_3(6n) - P_3(6(n-1)) = 3(2n-1)$ (3)

Let $P_3(6n) = a_n$, then the Eq. (3) becomes $a_n - a_{n-1} = 3(2n-1)$ (4)

Now plugging $n = 2, 3, \ldots, n$ in Eq. (4) and adding all, we get $a_n - a_1 = 3(n^2 - 1)$

As, $a_1 = P_3(6) = 3$

$\Rightarrow \quad a_n = 3n^2$

$\Rightarrow \quad P_3(6n) = 3n^2$

Also $P_2(6n) = \left\lfloor \dfrac{6n}{2} \right\rfloor = 3n$

and $P_1(6n)$

$\therefore$ The required number is $3n^2 + 3n + 1$.

Build-up Your Understanding 8

1. Find the number of ways in which n distinct objects can be put into two different boxes so that no box remains empty.
2. Find the number of ways in which n distinct objects can be kept into two identical boxes so that no box remains empty.
3. 10 identical balls are to be distributed in 5 different boxes kept in a row and labeled A, B, C, D and E. Find the number of ways in which the balls can be distributed in the boxes if no two adjacent boxes remain empty.
4. Find the number of distributions of 6 distinguishable objects in three distinguishable boxes such that each box contains an object.
5. Find the number of ways in which 12 identical coins can be distributed in 6 different purses, if not more than 3 and not less than 1 coin goes in each purse.
6. Find the number of ways in which 30 coins of one rupee each be given to six persons so that none of them receive less than 4 rupees.
7. Find the number of ways of wearing 8 distinguishable rings on 5 fingers of right hand.
8. 15 identical balls have to be put in 5 different boxes. Each box can contain any number of balls. Find total number of ways of putting the balls into box so that each box contains atleast 2 balls.
9. In how many ways can 3 blue, 4 red and 2 green balls be distributed in 4 distinct boxes? (Balls of the same colour are identical)
10. How many different ways can 15 Candy bars be distributed to Tanya, Manya, Shashwat and Adwik, if Tanya cannot have more than 5 candy bars and Manya must have at least two. Assume all Candy bars to be alike.
11. In how many ways, 16 identical coins can be distributed to 4 beggars when
    (i) any beggar may get any number of coins?
    (ii) every beggar gets atleast one coin?
    (iii) every beggar gets atleast two coins?
    (iv) every beggar gets atleast three coins?
12. Prove that the number of n digit quaternary sequences (whose digits are 0, 1, 2, and 3), in which each of the digits 2 and 3 appear atleast once, is $4^n - 2 \cdot 3^n + 2^n$.
13. Shivank has 15 ping-pong balls each uniquely numbered from 1 to 15. He also has a red box, a blue box, and a green box.
    (i) How many ways can Shivank place the 15 distinct balls into the three boxes so that no box is empty?
    (ii) Suppose now that Shivank has placed 5 ping-pong balls in each box. How many ways can he choose 5 balls from the three boxes so that he chooses at least one from each box?

14. In how many ways we can place 9 different balls in 3 different boxes such that in every box at least 2 balls are placed?

15. In how many ways can we put 12 different balls in three different boxes such that first box contains exactly 5 balls.

16. Five balls are to be placed in three boxes. Each can hold all the five balls. In how many different ways can we place the balls so that no box remains empty, if
 (i) balls and boxes are all different?
 (ii) balls are identical but boxes are different?
 (iii) balls are different but boxes are identical?
 (iv) balls as well as boxes are identical?

17. A man has 3 daughters. He wants to bequeath his fortune of 101 identical gold coins to them such that no daughter gets more share than the combined share of the other two. Find the number of ways of accomplishing this task.

18. There are six gates in an auditorium. suppose 20 delegates arrive. How many records could be there?

19. A man has to move 9 steps. He can move in 4 directions: left, light, forward, backward.
 (i) In how many ways he can take 9 steps in 4 direction?
 (ii) In how many ways he can move 9 steps if he has to take atleast one step in every direction.
 (iii) In how many ways he can move 9 steps such that he finish his journey one step away (either left or right or forward or backward) from the starting position.

Johann Peter Gustav Lejeune Dirichlet

13 Feb 1805–5 May 1859
Nationality: German

7.15 Dirichlet's (Or Pigeon Hole) Principle (PHP)

Let $k, n \in \mathbb{N}$. If at least $kn + 1$ objects are distributed among k boxes, then atleast one of the box, must contain atleast $(n + 1)$ objects. In particular, if atleast $(n + 1)$ objects are put into n boxes, then atleast one of the box must contain atleast two objects. For arbitrary n objects and m boxes this generalizes to atleast one box will contain atleast $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ objects.

Example 170 *Divide the numbers* 1, 2, 3, 4, 5 *into two arbitrarily chosen sets. Prove that one of the sets contains two numbers and their difference.*

Solution: Let us try to divide 1, 2, 3, 4, 5 into two sets in such a way that neither set contains the difference of two of its numbers.

2 cannot be in the same set as 1, 4, because if 2 and 1 are in the same sets $2 - 1 = 1$ belongs to the set; again if 2 and 4 are in the same set then $4 - 2 = 2$ belongs to the set and hence, if we name the sets as A and B, and if $2 \in A$, then 1, 4 both belong to B.

$$\begin{array}{cc} A & B \\ \{2, \square, \square\} & \{1, 4, \square\} \end{array}$$

We cannot put 3 in set B as $4 - 3 = 1$ belongs to B, so 3 belongs to A.

$$A = \{2, 3, \square\} \; B = \{1, 4\, \square\}$$

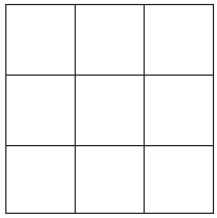
Now, 5 is the only number left out. Either 5 should be in set A or in B, but then if $5 \in A \Rightarrow 5 - 3 = 2 \in A$.

So, 5 cannot be in A.

However, if 5 is put in set B, then $5 - 4 = 1 \in B$. So, 5 cannot be in set B.

Thus, we cannot put 5 in either set and hence, the result.

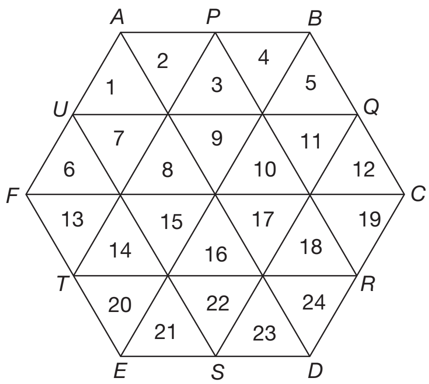
Example 171 *Show that for any set of* 10 *points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance is at most* $\sqrt{2}$.



Solution: Divide the square into 9 unit squares as given in the figure. Out of the 10 points distributed in the big square, at least one of the small squares must have at least two points by the Pigeon hole principle. These two points being in a unit square, are at the most $\sqrt{2}$ units distance apart as $\sqrt{2}$ is the length of the diagonal of the unit square.

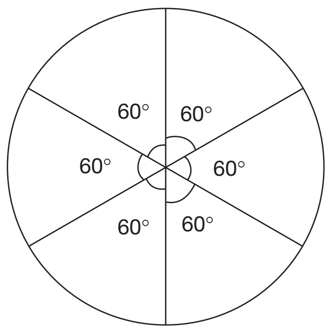
Example 172 *Show that given a regular hexagon of side* 2 *cm and* 25 *points inside it, there are at least two points among them which are at most 1 cm distance apart.*

Solution: If *ABCDE* is the regular hexagon of side 2 cm and *P*, *Q*, *R*, *S*, *T* and *U* are respectively the midpoints of *AB*, *BC*, *CD*, *DE*, *EF* and *FA*, respectively, then by joining the opposite vertices, and joining *PR*, *RT*, *TP*, *UQ*, *QS* and *SU*, we get in all 24 equilateral triangles of side 1 cm.



We have 25 points. So, of these 25 points inside the hexagon *ABCDEF*, at least 2 points lie inside any one triangle whose sides are 1 cm long. So, at least two points among them, will be at most 1 cm apart.

Example 173 *If* 7 *points are chosen on the circumference or in the interior of a unit circle, such that their mutual distance apart is greater than or equal to* 1, *then one of them must be the centre.*



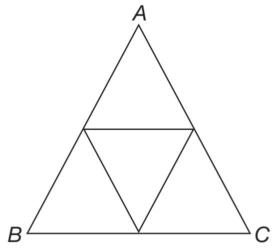
Solution: Divide the circle into six equal parts by drawing radii with two adjacent radii making an angle of 60°. Then, two of the seven points cannot lie in the interior of any one of the six sectors, since the distance between any two points is greater than or equal to 1.

If at all, in any sector, with boundaries included, two of the points may lie on the circular arc as end points (of the arc of any one of these sectors) or one on the arc and one at the centre of the circle.

Even if two lie on the ends of each circular arc, we have only 6 points satisfying the condition, thus forcing the seventh point to lie at the centre.

Example 174 $4^n + 1$ *points lie within an equilateral triangle of side* 1 *cm. Show that it is possible to choose out of them, at least two, such that the distance between them is at most* $\frac{1}{2^n}$ *cm.*

Solution: *ABC* is an equilateral triangle of side 1 cm. If the sides are divided into two equal parts, we get 4 equilateral triangles with side $\frac{1}{2}$ cm.



Again, if each of these four triangles is subjected to the above method, we get 4×4 triangles of side $\frac{1}{2} \times \frac{1}{2} = \frac{1}{2^2}$ cm.

Thus, after n steps we get, 4^n triangles of side $\frac{1}{2^n}$ cm.

Now, if we take $4^n + 1$ points inside the original equilateral ΔABC, then at least two of the points lie on the same triangle out of 4^n triangles by Pigeon hole principle. Hence, the distance between them is less than or at the most equal to the length of the side of the triangle, in which they lie, *i.e.*, they are $\frac{1}{2^n}$ cm apart or they are less than $\frac{1}{2^n}$ cm apart.

Example 175 *Let A be any set of* 19 *distinct integers chosen from the Arithmetic Progression* 1, 4, 7, …, 100. *Prove that there must be two distinct integers in A, whose sum is* 104.

Solution: There are $\frac{(100-1)}{3} + 1 = 34$ elements in the progression.
1, 4, 7, …, 100. Consider the following pairs:

$$(4, 100), (7, 97), (10, 94), \ldots, (49, 55).$$

There are in all $\frac{49-4}{3} + 1 = 16$ pairs $\left(\text{or } \frac{100-55}{3} + 1\right)$.

Now, we shall show that we can choose eighteen distinct numbers from the AP, such that no two of them add up to 104. In the above 16 pairings of the AP the numbers 1 and 52 are left out.

Now, taking one of the numbers from each of the pairs, we can have 16 numbers and including 1 and 52 with these 16 numbers, we now have 18 numbers.

But, no pair of numbers from these 18 numbers can sum up to 104, since just one number is selected from each pair and the other number of the pair (not selected) is 104, the number chosen.

Also $1 + 52 \neq 104$. Thus, we can choose 18 numbers, so that no two of them sum up to 104.

For getting 19 numbers (all these should be distinct), we should choose one of the 16 not chosen numbers, but then this number chosen is the 104 complement of one of the 16 numbers chosen already (among the 18 number). Thus, if a set of 19 distinct elements are chosen, then we must have at least one pair whose sum is 104.

Example 176 *Let* $X \subset \{1, 2, 3, \ldots, 99\}$ *and* $n(X) = 10$. *Show that it is possible to choose two disjoint non-empty proper subsets Y, Z of X such that* $\sum_{y \in Y} y = \sum_{z \in Z} z$.

Solution: Since $n(X) = 10$, the number of non-empty, proper subsets of X is $2^{10} - 2 = 1022$.
The sum of the elements of the proper subsets of X can possibly range from 1 to $\sum_{i=1}^{9} (90 + i)$. That is 1 to $(91 + 92 + \cdots + 99)$, *i.e.*, 1 to 855.

$i = l$

That is, the 1022 subsets can have sums from 1 to 855.

By Pigeon hole principle, at least two distinct subsets B and C will have the same sum.

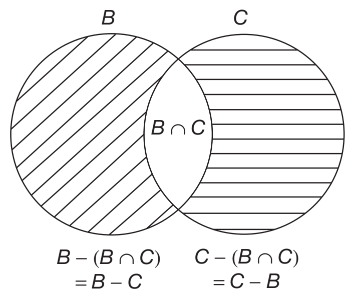
($\because$ There are 855 different sums, and so if we have more than 855 subsets, then at least two of them have the same sum.)

If B and C are not disjoint, then let

$$X = B - (B \cap C)$$

and, $$Y = C - (B \cap C).$$

Clearly, X and Y are disjoint and non-empty and have the same sum of their elements.



Define $s(A)$ = sum of the elements of A. We have B and C not necessarily disjoint such that $s(B) = s(C)$.

Now, $$s(X) = s(B) - s(B \cap C)$$

$$s(Y) = s(C) - s(B \cap C)$$

but, $$s(B) = s(C).$$

Hence, $s(X) = s(Y)$.

Also $X \neq \phi$. For if X is empty, then $B \subset C$ which implies $s(B) < s(C)$ (a contradiction). Thus, X and Y are non-empty and $s(X) = s(Y)$.

Example 177 *If repetition of digits is not allowed in any number* (*in base* 10)*, show that among three four-digit numbers, two have a common digit occurring in them.*

Also show that in base 7 *system any two four-digit numbers without repetition of digits will have a common number occurring in their digits.*

Solution: In base 10, we have ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Thus, for 3 four-digit numbers without repetition of digits, we have to use in all 12 digits, but in base 10 we have just 10 digits. Thus, at least any two of the three four-digit numbers have a common number occurring in their digits by Pigeon hole principle. Again for base 7 system, we have seven digits 0, 1, 2, 3, 4, 5, 6. For two four-digit numbers without repetition we have to use eight digits and again by Pigeon hole Principle, they have atleast one common number in their digits.

Example 178 *In base* $2k$*,* $k \geq 1$ *number system, any* 3 *non-zero, k-digit numbers are written without repetition of digits. Show that two of them have a common digit among them.*

In base $2k + 1$*,* $k \geq 1$ *among any* $3k + 1$ *digit non-zero numbers, there is a common number occurring in any two digits.*

Solution:

Case 1: In case $k' = 1$, we have the digits 0, 1 and the k-digit non-zero number(s) is 1 only. Thus, all the three numbers in this case are trivially the same 1.

For $k > 1$: Three k-digit (non-zero) numbers will have altogether $3k$ digits and the total number of digits in base $2k$ system is $2k$. Since repetition of digits is not allowed and $3k > 2k$ implies that among the digits of at least two of the numbers, there is at least one digit common among them (by Pigeon hole principle).

Case 2: In the case of $k = 1$, $2k + 1 = 3$, the three digits in base $2k + 1 = 3$ systems are 0, 1 and 2.

$k + 1 = 1 + 1 = 2$ and the digits non-zero numbers here are 10, 20, 12, 21.

So, we can pick up 10, 20 and 12, or 10, 20, 21, …. In each of the cases there is a common digit among two of them. (In fact, any two numbers will have a common digit 1.)

In general case, $3(k+1)$ digit numbers will have $3k+3$ digits in all. But it is a base $(2k+1)$ system.

The numbers are written without repetition of digits, since $3k+3 > 2k+1$ (In fact, any two $k+1$ digit numbers could also have the same property as $2k+2 > 2k+1$, again by the Pigeon hole principle at least two of the numbers, will have at least one common number in their digits.

Example 179 *Let A denote the subset of the set $S = \{a, a+d, \ldots, a+2nd\}$ having the property that no two distinct elements of A add up to $2(a+nd)$. Prove that A cannot have more than $(n+1)$ elements. If in the set S, 2nd is changed to $a+(2n+1)d$, what is the maximum number of elements in A if in this case no two elements of A add up to $2a+(2n+1)d$?*

Solution: Pair of the elements of S as $[a, a+2nd]$, $[a+d, a+(2n-1)d]$, …, $[a+(n-1)d, a+(n+1)d]$ and one term $a+nd$ is left out.
Now, sum of the terms in each of the pairs is $2(a+nd)$. Thus, each term of the pair is $2(a+nd)$ complement of the other term.

Now, there are n pairs. If we choose one term from each pair, we get n term. To this collection of terms include $(a+nd)$ also.

Now, we have $(n+1)$ numbers. Thus, set A can be taken as the set of the above $(n+1)$ numbers. Here no two elements of the set A add up to $2(a+nd)$ as no element has its $2(a+nd)$ complement in A except $a+nd$, but then, we should take two distinct elements.

If we add any more terms to A so that A contains more than $(n+1)$ elements, then some of the elements will now have then $2(a+nd)$ complement in A, so that sum of these two elements will be $2(a+nd)$, and hence, the result.

In the second case, we have

$$S = \{a, a+d, \cdots, a+\{2n+1)\,d\}$$

There are $2(n+1)$ elements. So, pairing them as before gives $(n+1)$ pairs, *i.e.*, $[a, a+(2n+1)\,d]$, $[a+d, a+2nd]$, …, $[a+nd, a+(n+1)d]$.

Now, we can pick exactly one term from each of these $(n+1)$ pairs.

We get a set A of $(n+1)$ elements where no two of which add up to $[2a+2(n+1)d]$.

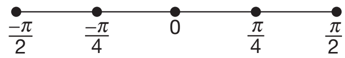
Note: Here we need not use distinct numbers, even if the same number is added to itself, the sum will not be $[2a+2(n+1)d]$. Here again, even choosing one more term from the numbers left out and adding it to A; A will have a pair which adds up to $[2a+2(n+1)d]$. Thus, the maximum number of elements in A satisfying the given condition is $(n+1)$.

Example 180 *Given any five distinct real numbers, prove that there are two of them, say x and y, such that* $0 < \dfrac{(x-y)}{(1+xy)} \le 1$.

Solution: Here we are using the property of tangent functions of trigonometry.

Given a real number a, we can find a unique real number A, lying between $\dfrac{-\pi}{2}$ and $\dfrac{\pi}{2}$, *i.e.*, lying in the real interval $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ such that $\tan A = a$, as the tangent function in the open interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is continuous and strictly increasing and covers R completely. Therefore, corresponding to the five given real numbers $a_i (i = 1, 2, 3, 4, 5)$, we can find five distinct real numbers A_i $(i = 1, 2, 3, 4, 5)$ lying between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ such that $\tan A_i = a_i$.

Divide the open interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ into four equal intervals, each of length $\frac{\pi}{4}$. Now, by Pigeon hole principle at least two of the A_i's must lie in one of the four intervals. Suppose A_k and A_l with $A_k > A_l$ lie in the same interval, then



$$0 < A_k - A_l \le \frac{\pi}{4}.$$

$\Rightarrow \tan 0 < \tan (A_k - A_l) < \tan \frac{\pi}{4}$

[It is because tan function increases in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$]

i.e.,
$$0 < \frac{\tan A_k - \tan A_l}{1 + \tan A_k \tan A_l} < 1$$

$$0 < \frac{a_k - a_l}{1 + a_k a_l} < 1.$$

Hence, there are two real numbers $x = a_k$, $y = a_1$ such that $0 < \frac{x - y}{1 + xy} \le 1.$

Build-up Your Understanding 9

1. Prove that, among any 52 integers, two can always be found, such that the difference of their squares, is divisible by 100.
2. Show that, for any set of 10 points, chosen within a square, whose side is 3 units, there are two points, in the set, whose distance is at most $\sqrt{2}$.
3. There are 7 persons in a group, show that, some two of them, have the same number of acquaintances among them.
4. 51 points are scattered inside a square, with a side of one metre. Prove that some set of three of these points can be covered by a square, with side 20 cm.
5. Let $1 < a_1 < a_2 < a_3 < \ldots < a_{51} < 142$. Prove that, among the 50 consecutive differences $(a_i - a_{i-1})$ where $i = 1, 2, 3, \ldots, 51$, some value, must occur at least twelve times.
6. You are given 10 segments, such that, every segment is larger than 1 cm but shorter than 55 cm. Prove that, you can select three sides of a triangle, among these segments.
7. There are 9 cells in a 3×3 square. When these cells are filled by numbers 1, 2, 3 only, prove that, of the eight sums obtained, at least, two sums are equal.
8. Let there be given 9 lattice points in a 3-D Euclidean space. Show that, there is a lattice point, on the interior of one of the line segments joining two of these nine points.
9. Consider seven distinct positive integers, not exceeding 1706. Prove that, there are three of them, say a, b, c such that, $a < b + c < 4a$.

10. One million pine trees grow in a forest. It is known that, no pine tree, has more than 60000 pine needles in it. Show that, two pine trees in the forest must have the same number of pine needles.
11. In a circle of radius 16, there are placed 650 points; Prove that there exists a ring (annulus) of inner radius 2 and outer radius 3, which contains not less than 10 of the given points.
12. On a rectangular table of dimensions 120″ by 150″, we set 14001 marbles of size 1″ by 1″. Prove that, no matter how these are arranged, one can place a cylindrical glass with diameter of 5″ over atleast 8 marbles.
13. Let A be the set of 19 distinct integers, chosen from the AP 1, 4, 7, 10, …, 100. Prove that, there should be two distinct integers in A, such that, their sum is 104.
14. If a line is coloured in 11 colours, show that, there exist two points, whose distance apart, is an integer, which have the same colour.
15. Show that, given 12 integers, there exists two of them whose difference is divisible by 11.
16. Given eleven triangles, show that, some three of them belong to the same type (such as equilateral, isosceles, etc.)
17. A is a subset of the AP 2, 7, 12, …, 152. Prove that, there are two distinct elements of A whose sum is 159. What can you conclude if A has only 14 elements?
18. Given three points, in the interior of a right angled triangle, show that, two of them are at a distance not greater than the maximum of the lengths of the sides containing the right angle.
19. There are 90 cards numbered 10 to 99. A card is drawn and the sum of the digits of the number in the card is noted; show that if 35 cards are drawn, then, there are some three cards, whose sum of the digits are identical.
20. If in a class of 15 students, the total of the marks in a subject is 600, then show that, there is a group of 3 students, the total of whose marks is at least 120.
21. Let $ABCD$ be a square of side 20. Let T_i $(i = 1, 2, \ldots, 2000)$ be points in the interior of the square, such that, no three points from the set $S = \{A, B, C, D\} \subset T_i \ \forall\ i = 1, 2, 3, \ldots, 2000$ are collinear, Prove that, at least one triangle, with the vertices in S has area less than $\frac{1}{10}$.
22. 5 points are plotted inside a circle. Prove that, there exist two points, which form an acute angle with the centre of the circle.
23. Let A denote a subset of $\{1, 11, 21, 31, \ldots, 551\}$ having the property that, no two elements of A, add up to 552. Prove that A cannot have more than 28 elements.
24. Prove that, there exist two powers 3, which differ by a multiple of 2005.
25. All the points in the plane are coloured, using three colours. Prove that, there exists a triangle with vertices, having the same colour, such that, either it is isosceles or its angles are in geometric progression.

Solved Problems

Problem 1 *In how many ways can a pack of* 52 *cards be*

(i) *distributed equally among four players in order?*
(ii) *divided into* 4 *groups of* 13 *cards each?*
(iii) *divided into four sets of* 20, 15, 10, 7 *cards?*
(iv) *divided into four sets, three of them having* 15 *cards each and the fourth having* 7 *cards?*

Solution:

(i) From 52 cards of the pack, 13 cards can be given to the first player in ${}^{52}C_{13}$ ways.
From the remaining 39 cards, 13 cards can be given to the second player in ${}^{39}C_{13}$ ways.
From the remaining 26 cards, 13 cards can be given to the third player in ${}^{26}C_{13}$ ways.
The remaining 13 cards can be given to the fourth player in ${}^{13}C_{13} = 1$ way.
By fundamental theorem, the number of ways of dividing 52 cards equally among four players $= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13} = \dfrac{52!}{13!39!} \times \dfrac{39!}{13!26!} \times \dfrac{26!}{13!13!} \times 1 = \dfrac{52!}{(13!)^4}$.

(ii) By standard result, the number of ways of forming 4 groups, each of 13 cards $= \dfrac{52!}{4!(13!)^4}$.

(iii) Here the sets have unequal number of cards, hence the required number of ways

$$= {}^{52}C_{20} \times {}^{32}C_{15} \times {}^{17}C_{10} \times {}^{7}C_{7} = \frac{52!}{20!32!} \times \frac{32!}{15!17!} \times \frac{17!}{10!7!} \times 1 = \frac{52!}{20!15!10!7!}.$$

(iv) By standard result, the required number of ways $= \dfrac{52!}{15!15!15!7!3!} = \dfrac{52!}{(15!)^3.3!7!}$.

Problem 2 *Find the number of ways of filling three boxes (named A, B and C) by 12 or less number of identical balls, if no box is empty, box B has at least 3 balls and box C has at most 5 balls.*

Solution: Suppose box A has x_1 balls, box B has x_2 balls and box C has x_3 balls. Then,

$$x_1 + x_2 + x_3 \le 12,\ x_1 \ge 1,\ x_2 \ge 3,\ 1 \le x_3 \le 5$$

Let $x_4 = 12 - (x_1 + x_2 + x_3)$. Then

$$x_1 + x_2 + x_3 + x_4 = 12\ (1 \le x_1 \le 8,\ 3 \le x_2 \le 10,\ 1 \le x_3 \le 5 \text{ and } 0 \le x_4 \le 7)$$

The required number = Coefficient of x^{12} in

$$(x^1 + x^2 + \cdots + x^8)(x^3 + x^4 + \cdots + x^{10})(x^1 + x^2 + \cdots + x^5)(x^0 + x^1 + \cdots + x^7)$$

$=$ Coefficient of x^{12} in $(x + x^2 + x^3 + \cdots)(x^3 + x^4 + x^5 + \cdots)(x + x^2 + \cdots + x^5)(1 + x + x^2 + \cdots)$

$=$ Coefficient of x^7 in $(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + \cdots)$

$=$ Coefficient of x^7 in $(1 - x)^{-4}(1 - x^5)$
$=$ Coefficient of x^7 in $(1 - x^5)(1 + {}^4C_1 x + {}^5C_2 x^2 + {}^6C_3 x^3 + \cdots)$
$= {}^{10}C_7 - {}^5C_2 = 110$.

Problem 3 *A person writes letters to six friends and address the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that*

(i) *at least two of them are in the wrong envelopes?*
(ii) *all the letters are in the wrong envelopes?*

Solution:

(i) The number of all the possible ways of putting 6 letters into 6 envelopes is 6!.
There is only one way of putting all the letters correctly into the corresponding envelopes.
Hence if there is a mistake, at least 2 letters will be in the wrong envelope.
Hence the required answer is $6! - 1 = 719$.

(ii) Using the result of derangements, the required number of ways

$$= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$= 720\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$$

$$= 360 - 120 + 30 - 6 + 1 = 265.$$

Problem 4 *Find the number of integers which lie between* 1 *and* 10^6 *and which have the sum of the digits equal to* 12.

Solution: Consider the product $(x^0 + x^1 + x^2 + \cdots + x^9)\,(x^0 + x^1 + x^2 + \cdots + x^9)\ldots 6$ factors. The number of ways in which the sum of the digits will be equal to 12 is equal to the coefficient of x^{12} in the above product. So, required number of ways = Coefficient of x^{12} in $(x^0 + x^1 + x^2 + \cdots + x^9)^6$.

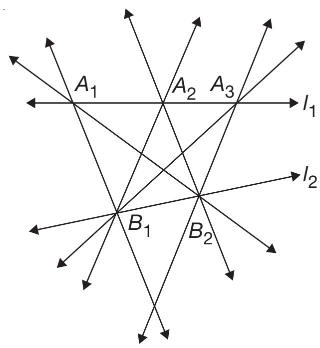
$$= \text{Coefficient of } x^{12} \text{ in } (1 - x^{10})^6\,(1 - x)^{-6}$$

$$= \text{Coefficient of } x^{12} \text{ in } (1 - x)^{-6}\,(1 - {}^6C_1\,x^{10} + \cdots)$$

$$= \text{Coefficient of } x^{12} \text{ in } (1 - x)^{-6} - {}^6C_1 \cdot \text{Coefficient of } x^2 \text{ in } (1 - x)^{-6}$$

$$= {}^{12+6-1}C_{6-1} - {}^6C_1 \times {}^{2+6-1}C_{6-1} = {}^{17}C_5 - 6 \times {}^7C_5 = 6062.$$

Problem 5 *Straight lines are drawn by joining m points on a straight line to n points on another line. Then excluding the given points, prove that the lines drawn will intersect at* $\frac{1}{2}mn(m-1)(n-1)$ *points. (No two lines drawn are parallel and no three lines are concurrent.)*



Solution: Let $A_1, A_2, \ldots, A_m$ be the points on the first line (say l_1) and let $B_1, B_2, \ldots, B_n$ be the points on the second line (say l_2). Now any point on l_1 can be chosen in m ways and any point on l_2 can be chosen in n ways. Hence number of ways of choosing a point l_1 and a point on l_2 is mn.
Hence number of lines obtained on joining a point on l_1 and a point on l_2 is mn. Now any point of intersection of these lines, which can be done in ${}^{mn}C_2$ ways. Hence number of point is ${}^{mn}C_2$. But some of these points are the given points and counted many times. For example, the point A_1 has been counted ${}^nC_2$ times. Hence required number of points is

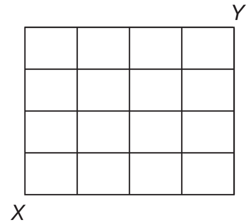
$${}^{mn}C_2 - m \cdot {}^nC_2 - n \cdot {}^mC_2 = \frac{1}{2}mn(m-1)(n-1)$$

Aliter: If we select two points from first line and two from the second line then we will have 2 required points from every such selection

Hence number of such points $= 2 \times \binom{m}{2} \times \binom{n}{2} = \frac{1}{2}mn(m-1)(n-1).$

Problem 6 *In the figure you have the road plan of a city. A man standing at X wants to reach the cinema hall at Y by the shortest path. What is the number of different paths that he can take?*

Solution: A path from X to Y is shown by dark line segments which corresponds $y\,x\,x\,x\,y\,y\,x\,y$. It is easy to see that any path of required type corresponds to an arrangement of x, x, x, x, y, y, y and y and vice versa. Hence required number of ways = number of arrangements of 4x's and 4y's, which is $\frac{\lfloor 8}{\lfloor 4 \lfloor 4}$.



Problem 7 *Show that the number of combinations of n letters out of 3n letters of which n are a's, n are b's and the rest are unequal is* $(n+2)\cdot 2^{n-1}$.

Solution: From n we have 0, 1, 2, 3 . . ., n From n we may have 0, 1, 2, 3 …, n, while for each of the rest n letters we may have 2 combinations 0 or 1. Thus the required number of combinations is thus

$= \text{Coefficient of } x^n \text{ in}$

$$(1+x+x^2+\cdots+x^n)\,(1+x+x^2+\cdots+x^n)\,(1+x)\,(1+x)+\cdots(1+x)$$

$= \text{Coefficient of } x^n \text{ in } \dfrac{(1-x^{n+1})^2}{(1-x)^2}\cdot(1+x)^n$

$= \text{Coefficient of } x^n \text{ in } (1-x^{n+1})^2\,(1+x)^n\,(1-x)^{-2}$

Since $(1-x^{n+1})^2$ will not contain x^n, we have required number of combinations

$= \text{Coefficient of } x^n \text{ in } (1+x)^n\cdot(1-x)^{-2}$

$= \text{Coefficient of } x^n \text{ in } [2-(1-x)]^n\,(1-x)^{-2}$

$= \text{Coefficient of } x^n \text{ in } 2^n\,(1-x)^{-2} - {}^nC_1\,2^{n-1}\,(1-x)^{-1} + {}^nC_1.\,2^{n-2}\cdot(1-x)^0$

$- {}^nC_3\cdot 2^{n-3}\,(1-x) + \cdots + (-1)^n\cdot {}^nC_n\,(1-x)^{n-2}$

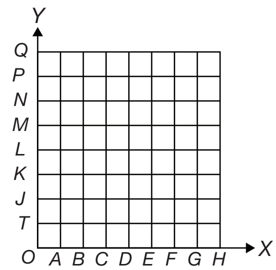
$= \text{Coefficient of } x^n \text{ in } 2^n\,(1-x)^{-2} - n\cdot 2^{n-1}\cdot(1-x)^{-1}$

$= 2^n\dfrac{(n+1)!}{n!} - n\cdot 2^{n-1} = 2^n\cdot(n+1) - n\cdot 2^{n-1} = 2^{n-1}\cdot(n+2).$

Problem 8 *Show that the number of rectangles of any size on a chess board is* $\sum_{k=1}^{8} k^3$.

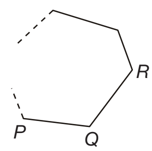
Solution: A rectangle can be fixed on the chess board if and only if we fix two points on x-axis and two points on y-axis. For example, in order to fix the rectangle $RSTU$, we fix B and G on x-axis and K and M on y-axis and vice-versa.

Hence total number of rectangles on the chess board is the number of ways of choosing two points on x-axis (which can be done in ${}^9C_2$ ways) and two points on y-axis (which can also be done is ${}^9C_2$ ways). Hence require number is $({}^9C_2)^2 = \sum_{k=1}^{8} k^3$.



Problem 9 *Find the number of triangles whose angular points are at the angular points of a given polygon of n sides. but none of whose sides are the sides of the polygon.*

Solution: A n-sided polygon has n angular points. Number of triangles formed from these n angular points = ${}^nC_3$. But it also includes the triangles with sides on the polygon.



Let us consider a side PQ. If each angular point of the remaining $(n-2)$ points is joined with PQ, we get a triangle with one side PQ.

$\therefore$ Number of triangles with PQ as one side = $n-2$. In similar ways n sides like QR can be considered. Hence number of triangle = $n\,(n-2)$. But some triangles have been counted twice. For example, PQ side with R gives ΔPQR. and QR side with P gives same ΔPQR.

Number of such triangles = n

[As for each side, one triangle is repeated. Hence for n sides, n triangle's have been counted more.]

Hence, the number of triangles of which one side is the side of the triangle $= n(n-2) - n = n(n-3)$

Hence number of required triangles

$$= {}^nC_3 - n(n-3) = \frac{n(n-1)(n-2)}{6} - n(n-3) = \frac{n}{6}(n^2 - 9n + 20) = \frac{n}{6}(n-4)(n-5).$$

Problem 10 *Find the number of all whole numbers formed on the screen of a calculator which can be recognized as numbers with (unique) correct digits when they are read inverted. The greatest number formed on its screen is* 999999.

Solution: The digits 0, 1, 2, 5, 6, 8 and 9 can be recognized as digits when they are seen inverted hence number can contain these digits only.
Note that number can be of 1 digit to 6 digit number. But in more than one digit numbers, 0 cannot come in first place and also in unit place (Imagine inverted case).

Number of digits	Total numbers
1	7
2	$6 \times 6 = 36$
3	$6 \times 7 \times 6 = 252$
4	$6 \times 7^2 \times 6 = 1764$
5	$6 \times 7^3 \times 6 = 12348$
6	$6 \times 7^4 \times 6 = 86436$
	Total = 100843

Problem 11 *Find the number of positive integral solutions of* $x + y + z + w = 20$ *under the following conditions:*

(i) *Zero values of x, y, z, w are included.*
(ii) *Zero values are excluded.*
(iii) *No variable may exceed 10; zero values excluded*
(iv) *Each variable is an odd number.*
(v) $0 < x < y < z < w$.

Solution:

(i) $x + y + z + w = 20; x \geq 0, y \geq 0, z \geq 0, w \geq 0$
Coefficient of a^{20} in $(a^0 + a^1 + a^2 + \cdots)^4$
$= (1-a)^{-4} = {}^{20+4-1}C_{20}$
$= {}^{23}C_3 = 1771$

Note: You can directly use the result ${}^{n+r-1}C_{r-1}$ or ${}^{n+r-1}C_n$

(ii) Number of ways = Coefficient of a^{20} in $(a + a^2 + a^3 + \cdots)^4$
= Coefficient of a^{20} in $a^4(1-a)^{-4}$
= Coefficient of a^{16} in $(1-a)^{-4} = {}^{19}C_{16}$
= 969.

Note: that you can directly use ${}^{n-1}C_{r-1}$

(iii) If no variable exceeds 10, then sum of rest should be less than or equal to 10 [as $20 - 10 = 10$]

Let $x \leq 10$, then $y + z + w \geq 10$

and $\quad \max (y + z + w) = 20 - \min (x)$

$\max (y + z + w) = 20 - 1 = 19$

$\therefore\ 10 \leq y + z + w \leq 19$ [where $y \geq 1, z \geq 1, w \geq 1$]

$\Rightarrow 10 \leq (y_1 + 1) + (z_1 + 1) + (w_1 + 1) \leq 19$

$\Rightarrow 7 \leq y_1 + z_1 + w_1 \leq 16;\ 0 \leq y_1 \leq 9,\ 0 \leq z_1 \leq 9,\ 0 \leq w_1 \leq 9$

Number of solutions = (Number of solutions of $y_1 + z_1 + w_1 < 16$) – (Number of solutions of $y_1 + z_1 + w_1 \leq 6$)

Now,

Number of solutions of $y_1 + z_1 + w_1 \leq 16$ can be obtained by adding a dummy variable x_1 ($x_1 \geq 0$) such that $x_1 + y_1 + z_1 + w_1 = 16$.

Number of solutions = Coefficient of x^{16} in $(1 - x^{10})^3\ (1 - x)^{-4} = {}^{19}C_4 - 3\,{}^{9}C_3$

Again,

Number of solutions of $y_1 + z_1 + w_1 \leq 6$ can be obtained by adding a dummy variable l_1 ($l_1 \geq 0$) such that $l_1 + y_1 + z_1 + w_1 = 6$

Number of solutions = Coefficient of x^6 in $(1 - x^{10})^3\ (1 - x)^{-4} = {}^{9}C_3$

Hence, Total number of solutions = ${}^{19}C_3 - 4\,{}^{9}C_3 = 633$.

(iv) Each variable is an odd number.

$\therefore\ x = 2x_1 + 1 \quad y = 2y_1 + 1$

$z = 2z_1 + 1 \quad w = 2w_1 + 1$ [where $x_1, y_1, z_1, w_1 \geq 0$]

$x + y + z + w = 20$

$\Rightarrow (2x_1 + 1) + (2y_1 + 1) + (2z_1 + 1) + (2w_1 + 1) = 20$

$2x_1 + 2y_1 + 2z_1 + 2w_1 = 16$

$\Rightarrow x_1 + y_1 + z_1 + w_1 = 8$ [where $x_1, y_1, z_1, w_1 \geq 0$]

Number of solutions $= {}^{8+4-1}C_{4-1}$

$= {}^{11}C_3 = 165$

(v) Assume $0 < x < y < z < w$

Let $x = x_1$

$y = x + x_2 = (x_1) + x_2$

$z = y + x_3 = (x_1 + x_2) + x_3$

$w = z + x_4 = (x_1 + x_2 + x_3) + x_4$ [where $x_1, x_2, x_3, \geq 1$]

$x + y + z + w = 20$

$\Rightarrow x_1 + (x_1 + x_2) + (x_1 + x_2 + x_3) + (x_1 + x_2 + x_3 + x_4) = 20$

$4x_1 + 3x_2 + 2x_3 + x_4 = 20$ (1) [where $x_1, x_2, x_3, x_4 \geq 1$]

Let us again change the variables

$x_1 = y_1 + 1;\ x_2 = y_2 + 1;\ x_3 = y_3 + 1;\ x_4 = y_4 + 1$ [where $y_1, y_2, y_3, y_4 \geq 0$]

Substituting above values in Eq. (1), we get

$4(y_1 + 1) + 3\,(y_2 + 1) + 2\,(y_3 + 1) + (y_4 + 1) = 20.$

$\Rightarrow \quad 4y_1 + 3y_2 + 2y_3 + y_4 = 10$ [where $y_1, y_2, y_3, y_4 \geq 0$]

Y_1	$3y_2 + 2y_3 + y_4$	Number of solutions
0	10	14 (Use Table-1)
1	6	7 (Use Table-2)
2	2	2 (Use Table-3)

Total Number of solutions = 23

Table 1

$3y_2 + 2y_3 + y_4 = 10$		
y_2	$2y_3 + y_4$	Number of solutions
0	10	6
1	7	4
2	4	3
3	1	1
		14

Table 2

$3y_2 + 2y_3 + y_4 = 6$		
y_2	$2y_3 + y_4$	Number of solutions
0	6	4
1	3	2
2	0	1
		7

Table 3

$3y_2 + 2y_3 + y_4 = 6$		
y_2	$2y_3 + y_4$	Number of solutions
0	2	2
		2

Problem 12 *There are* 12 *seats in the first row of a theater of which* 4 *are to be occupied. Find the number of ways of arranging 4 persons so that:*

(i) *no two persons sit side by side.*
(ii) *there should be atleast 2 empty seats between any two persons.*
(iii) *each person has exactly one neighbour.*

Solution:

(i) We have to select 4 seats for 4 persons so that no two persons are together. It means that there should be atleast one empty seat vacant between any two persons.
To place 4 persons we have to put 4 seats between the remaining 8 empty seats so that all persons should be separated.
Between 8 empty seats 9 gaps are available for 4 seats to put.
We can select 4 gaps in ${}^9C_4$ ways.
Now we can arrange 4 persons on these 4 seats in 4! ways. So total number of ways to give seats to 4 persons so that no two of them are together
$= {}^9C_4 \times 4! = {}^9P_4 = 3024$.

(ii) Let x_0 denotes the empty seats to the left of the first person, x_i (i = 1, 2, 3) be the number of empty seats between ith and $(i + 1)$th person and x_4 be the number of empty seats to the right of 4th person.
Total number seats are 12. So we can make this equation :
$$x_0 + x_1 + x_2 + x_3 + x_4 = 8 \qquad (1)$$
Number of ways to give seats to 4 persons so that there should be two empty seats between any two persons is same as the number of integral solutions of the Eq. (1) subjected to the following conditions.
Conditions on x_1, x_2, x_3, x_4:
According to the given condition, these should be two empty seats between any two persons. That is,
Min $(x_i) = 2$ for $i = 1, 2, 3$ and Min $(x_0) = 0$
Max $(x_0) = 8 -$ Min $(x_1 + x_2 + x_3 + x_4) = 8 - (2 + 2 + 2 - 0) = 2$
Max $(x_4) = 8 -$ Min $(x_0 + x_1 + x_2 + x_3) = 8 - (2 + 2 + 2 - 0) = 2$
Similarly,
Max $(x_i) = 4$ for $i = 1, 2, 3$
Number of integral solutions of the equation (i) subjected to the above condition
= Coefficient of x^8 in the expansion of $(1 + x + x^2)^2 (x^2 + x^3 + x^4)^3$
= Coefficient of x^8 in $x^6 (1 + x + x^2)^5$
= Coefficient of x^2 in $(1 - x^3)^5 (1 - x)^{-5}$
= Coefficient of x^2 in $(1 - x)^{-5}$
$= {}^{5+2-1}C_2 = {}^6C_2 = 15$.

Number of ways to select 4 seats so that there should be atleast two empty seats between any two persons = 15. But 4 persons can be arranged in 4 seats in 4! ways. So total number of ways to arrange 4 persons in 12 seats according to the given condition = $15 \times 4! = 360$.

(iii) As every person should have exactly one neighbour, divide 4 persons into groups consisting two persons in each group.

Let G_1 and G_2 be the groups in which 4 persons are divided.

According to the given condition G_1 and G_2 should be separated from each other.

Number of ways to select seats so that G_1 and G_2 are separated = ${}^{8+1}C_2$

But 4 persons can be arranged in 4 seats in 4! ways.

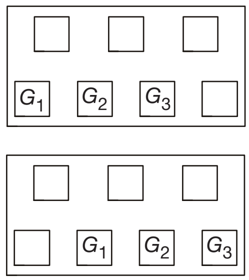
So total number of ways to arrange 4 persons so that every person has exactly one neighbour = ${}^9C_2 \times 4! = 864$

Problem 13 *In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats?*

Solution:

(i) Out of 14 seats (7 in each Van), we have to select 12 seats for 3 girls and 9 boys.
12 seats from 14 available seats can be selected in ${}^{14}C_{12}$ ways.
Now on these 12 seats we can arrange 3 girls and 9 boys in 12! ways.
So total number of ways ${}^{14}C_{12} \times 12! = 91 \times 12!$

(ii) One van out of two available can be selected in ${}^2C_1$ ways.



Out of two possible arrangements (see figure) of adjacent seats, select one in ${}^2C_1$ ways.
Out of remaining 11 seats, select 9 seats for 9 boys in ${}^{11}C_9$ ways.

Arrange 3 girls on 3 seats in 3! ways and 9 boys on 9 seats 9! ways.

So possible arrangement of sitting (for 3 girls and 9 boys in 2 vans) is:

${}^2C_1 \times {}^2C_1 \times {}^{11}C_9 \times 3! \times 9! = 12!$ ways.

Problem 14 *How many seven-letters words can be formed by using the letter of the word SUCCESS so that:*

(i) *the two C are together but not two S are together?*
(ii) *no two C and no two S are together?*

Solution:

(i) Considering CC as single object, U, CC, E can be arranged in 3! ways.
X U X C C X E X
Now the three S are to be placed in the 4 available places (X) so that C C are not separated but S are separated.
Number of ways to place S S S = (No of ways to select 3 places) $\times 1 = {}^4C_3 \times 1 = 4$
$\Rightarrow$ Number of words = $3! \times 4 = 24$.

(ii) Let us first find the words in which no two S are together. To achieve this, we have to do following operations.

(a) Arrange the remaining letter U C C E in $\dfrac{4!}{2!}$ ways.

(b) Place the three S S S in any arrangement from (a)

X U X C X C X E X

There are five available places for three S S S.

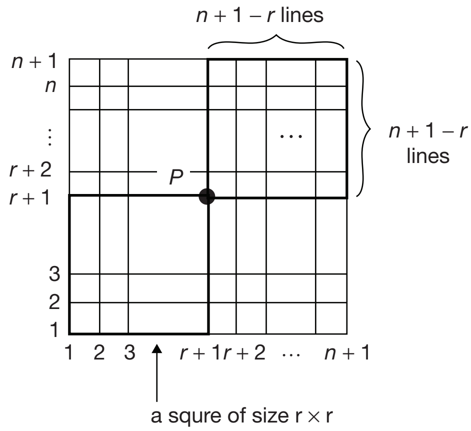
Number of placements = ${}^5C_3$

Hence total number of words with no two S together $= \dfrac{4!}{2!}\,{}^5C_3 = 120.$

Number of words having C C separated and S S S separated = (Number of words having S S S separated) – (Number of words having S S S separated but C C together) = 120 – 24 = 96 [using result of part (i)].

Problem 15 *A square of n units by n units is divided into n^2 squares each of area 1 sq. units. Find the number of ways in which 4 points (out of $(n + 1)^2$ vertices of unit squares) can be chosen so that they form the vertices of a square.*

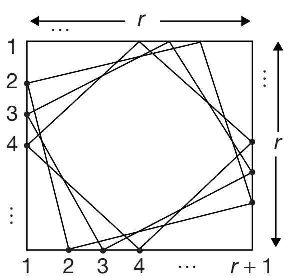
Solution:



We can easily see that number of squares of size $r \times r$ with its sides along the horizontal and vertical lines is equal to number of positions of P on the lattice points formed by $(n + 1 - r)$ horizontal and $(n + 1 - r)$ vertical lines which is $(n + 1 - r) \times (n + 1 - r)$.

$\Rightarrow$ Number of squares of size $r \times r = (n + 1 - r)^2$

In addition to these squares there are squares whose sides are not parallel to horizontal/vertical lines. Each of these squares is inscribed in some previously counted squares. So we will first count how many are inscribed in our $r \times r$ size square. Then we will sum over 'r'.



From the adjacent figure we can see that these are r inscribed squares, including the $r \times r$ square itself.

Now total number of squares

$$= \sum_{r=1}^{n} r(n+1-r)^2$$

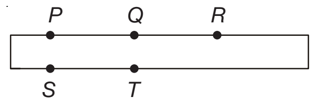
$$= \sum_{r=1}^{n} (n+1-r)\cdot r^2 = (n+1)\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r^3$$

$$= \frac{(n+1)n(n+1)(2n+1)}{6} - \left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

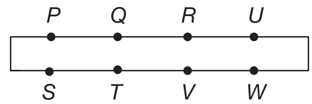
Problem 16 *A boat's crew consists of 8 men, 3 of whom can only row on one side and 2 only on the other. Find the number of ways in which the crew can be arranged.*

Solution: Let the man P, Q, R, S, T, U, V, W and suppose P, Q, R can row only on one side and S, T on the other as represented in the figure.



Then, since 4 men must row on each side, of the remaining 3, one must be placed on the side of P, Q, R and the other two on the side S, T; and this can evidently be done in 3 ways, for we can place any one of the three side of P, Q, R.

Now 3 ways of distributing the crew let us first consider one, say that in which *U* is on the side of P, Q, R as shown in the figure.



Now, P, Q, R, U can be arranged in 4! ways and S, T, V, W can be arranged in 4! ways.

Hence total number of ways arranging the men = 4! × 4! = 576

Hence the number of ways of arranging the crew = 3 × 576 = 1728.

Problem 17 *How many integers between* 1 *and* 1000000 *have the sum of the digits equal to* 18.

Solution: Integers between 1 and 1000000 will be, 1,2, 3, 4, 5 or 6-digits numbers, and given sum of digits = 18

Thus we need to obtain the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18 \qquad (1)$$

Where $0 \le x_i \le 9$, i = 1, 2, 3, 4, 5, 6

Therefore, the number of solutions of the Eq. (1), will be

$= \text{Coefficient of } x^{18} \text{ in } (x^0 + x^1 + x^2 + x^3 + \cdots + x^9)$

$= \text{Coefficient of } x^{18} \text{ in } \left(\dfrac{1-x^{10}}{1-x}\right)^6$

$= \text{Coefficient of } x^{18} \text{ in } (1 - x^{10})^6 (1 - x)^{-6}$

$= \text{Coefficient of } x^{18} \text{ in } (1 - 6x^{10}) (1 - x)^{-6}$

$= {}^{6+18-1}C_{18} - 6 \cdot {}^{6+8-1}C_8$

$= {}^{23}C_{18} - 6 \cdot {}^{13}C_8 = {}^{23}C_5 - 6 \cdot {}^{13}C_5$

$= 33649 - 7722 = 25927.$

Problem 18 *How many three digit numbers are of the form xyz with x < y; z < y and* $x \neq 0$.

Solution: Since, $x \ge 1$, then $y \ge 2$ ($\therefore x < y$)

If $y = n$ then n take the values from 1 to $n - 1$ and z can take the value from 0 to $n - 1$ (*i.e.*, n values) thus for each value of y $(2 < y < 9)$, x and z take $n(n - 1)$ values.

Hence, the 3-digit numbers are of the from *xyz*

$$= \sum_{n=2}^{9} n(n-1) = \sum_{n=1}^{9} n(n-1)\left\{\sum 1 \times (1-1) = 0\right\}$$

$$= \sum_{n=1}^{9} n^2 - \sum_{n=1}^{9} n == \frac{9(9+1)\,(18+1)}{6} - \frac{9(9+1)}{2}.$$

$$= 285 - 45 = 240.$$

Problem 19 *Find the number of polynomials of the form* $x^3 + ax^2 + bx + c$ *which are divisible by* $x^2 + 1$ *and where a, b, c belong to* $(1, 2, \ldots, n)$.

Solution: Let $f(x) = x^3 + ax^2 + bx + c$ be the polynomial divisible by $x^2 + 1$ or $(x + i)(x - i)$.

$$f(i) = 0 \quad \Rightarrow \quad i^3 + ai^2 + bi + c = 0$$
$$(b - 1)\, i + (c - a) = 0$$
$$b - 1 = 0 \text{ and } c - a = 0$$
$$b = 1,\ c = a$$

Hence, number of polynomials = Number of values which a or c can take.

As a or c can takes n values, therefore number of polynomials = n.

Problem 20 *John has x children by his first wife. Mary has (x + 1) children by her first husband. They marry and have children of their own. The whole family has 24 children. Assuming that two children of the same parents do not fight. Prove that the maximum number of fights thats can take place in* 191.

Solution: Let number of children of John and Mary are y and No. of children of John and his first wife is x. Hence, number of children of Mary from his first husband are $(x + 1)$.

$$x + x + 1 + y = 24 \tag{1}$$

Total number of fights between two children subject to the condition that any children of same parents do not flight.

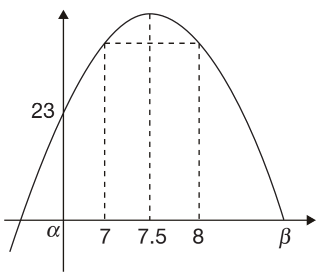
$$N(x) = {}^{24}C_2 - \left[{}^{x}C_2 + {}^{x+1}C_2 + {}^{y}C_2\right]$$

$$N(x) = 276 - \left[\frac{x(x-1) + (x+1)x}{2} + {}^{y}C_2\right]$$

$$= 276 - \left[x^2 + \frac{y(y-1)}{2}\right]$$

$$= 276 - \left[x^2 - \frac{(23-2x)(22-2x)}{2}\right] \quad \text{[using Eq. (1)]}$$

$$N(x) = 276 - (3x^2 - 45x + 253) = -3x^2 + 45x + 23$$



Maximum value of $N(x)$ can occur at $x = -\dfrac{(45)}{2(-3)} = 7.5$

But $x \in I$ hence $x = 7$ or 8 [as Graph is symmetrical about $x = 7.5$]

Maximum value $= 23 - 3(7)^2 + 45(7)$
$= 191$.

Problem 21 *There are 2n guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another, and that there are two specified guests who must not be placed next to one another, find the number of ways in which the company can be placed.*

Solution: Let the M and M' represent seats of the master and mistress respectively, and let $a_1, a_2 \ldots, a_{2n}$ represent the $2n$ seats.

Let the guests who must not be placed next to one another be called P and Q.

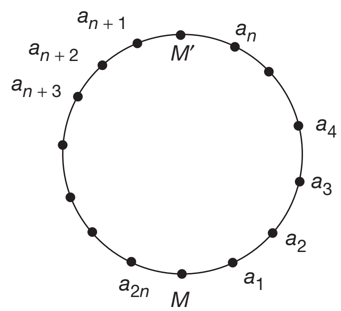
Now put P at a_1, and Q at any position, other than a_2, say at a_3; then remaining $2n - 2$ guests can be positioned in $(2n - 2)!$ ways. Hence there will be altogether $(2n - 2)(2n - 2)!$ arrangements of the guests when P is at a_1.

The same number of arrangements when P is at a_n or a_{n+1} or a_{2n}.

Hence, for these position (a_1, a_n, a_{n+1}, a_{2n}) of P, there are altogether in 4 $(2n-2)$ $(2n-2)!$ ways. (1)

If P is at a_2 there are altogether $(2n-3)$ positions for Q.

Hence, there will be altogether $(2n-3)$ $(2n-2)!$ arrangements of the guests when P is at a_2.



The same number of arrangements can be made when P is at any other position except the four position a_1, a_n, a_{n+1}, a_{2n}.

Hence, for these $(2n-4)$ positions of P there will be altogether in $(2n-4)$ $(2n-3)$ $(2n-2)!$ arrangements of the guests. (1)

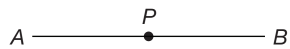
Hence, from Eqs. (1) and (2), the total number of ways of arranging the guests

$$= 4(2n-2)\,(2n-2)! + (2n-4)\,(2n-3)\,(2n-2)!$$
$$= (4n^2 - 6n + 4)\,(2n-2)!$$

Problem 22 *There are n straight lines in a plane, no two of which are parallel and no three passes through the same point. Their point of intersection are joined. Show that the number of fresh lines thus introduced is:*

$$\frac{1}{8}n(n-1)(n-2)(n-3)$$

Solution: Let AB be any one of the n straight lines and suppose it is intersected by some other straight line CD at P.



Then it is clear that AB contains $(n-1)$ points of intersection because it is intersected by the remaining $(n-1)$ straight lines in $(n-1)$ different points. Hence, the aggregate number of points contained in the n straight lines $= n(n-1)$. But in making up this aggregate each point has evidently been counted twice. For instance, the point P has been counted one among the points situated on AB and again among those on CD.

Hence, the actual number of points $= \dfrac{n(n-1)}{2}$

Now we have to find the number of new lines formed by joining these points. The number of new lines passing through P is evidently equal to the number of points lying outside the lines AB and CD for we get a new lines joining P with each of these points only.

Now, since, each of the lines AB and CD contained $(n-2)$ points besides the point P, the number of points situated on AB and CD.

$$= 2(n-2) + 1 = (2n-3)$$

Thus, the number of points outside AB and CD are $\dfrac{n(n-1)}{2}(2n-3)$ = The number of new lines passing through P and similarly through each other points.

So, the aggregate number of new lines passing through the points.

$$= \frac{n(n-1)}{2}\left\{\frac{n(n-1)}{2} - (2n-3)\right\}$$

But in the making up this aggregate every new line is counted twice; for instance if Q be one of the points outside AB and CD, the line PQ is counted once among the lines passing through P and again among these passing through Q.

Hence, actual number of fresh lines introduced

$$= \frac{1}{2}\left[\frac{n(n-1)}{2}\left\{\frac{n(n-1)}{2} - (2n-3)\right\}\right]$$

$$= \frac{1}{8}n(n-1)(n-2)(n-3).$$

Problem 23 *Let set $S = \{a_1, a_2, a_3, \ldots, a_{12}\}$ where all twelve elements are distinct, we want to form sets each of which contains one or more of the elements of set S (including the possibility of using all the elements of S). The only restriction is that the subscript of each element in a specific set must be an integral multiple of the smallest subscript in the set. For example, $\{a_2, a_6, a_8\}$ is one acceptable set, as is $\{a_6\}$. How many such sets can be formed? Can you generalize the result?*

Solution: Every (positive) integer is a multiple of 1.

So, we will first see a set consisting of a_1 and other elements:

There are 11 elements other than a_1. So the set with a_1 and another element, with one other element, 2 other elements, and all the 11 other elements, … and all the 11 other elements, *i.e.*, we have to choose a_1 and 0, 1, 2,…, 11 other elements out of a_2, a_3, …, a_{12}. This could be done in $\binom{11}{0}+\binom{11}{1}+\cdots+\binom{11}{11} = 2^{11}$ ways.

If a set contains a_2, as the element with the least subscript, then besides a_2, the set can have a_4, a_6, a_8, a_{10}, a_{12} elements, none or one or more of them. This could be done in $\binom{5}{0}+\binom{5}{1}+\cdots+\binom{5}{5} = 2^5$ ways.

Similarly, for having a_3 as the element with the least subscript 3, we have a_6, a_9, a_{12} to be the elements such that the subscripts (6, 9, 12) are divisible by 3.

So, the number of subsets with a_3 as one element is ${}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 2^3$.

For a_4, one of the elements, the number of subsets (other elements being a_8 and a_{12}) is 2^2.

For a_5 it is 2^1 (there is just an element a_{10} such that 10 is a multiple of 5).

For a_6, it is again 2^1 (as 6/12)

For a_7, a_8, a_9, a_{10}, a_{11} and a_{12}, there is just one subset, namely, the set with these elements. This is total up to 6.

So, the total number of acceptable set according to the condition is

$$2^{11} + 2^5 + 2^3 + 2^2 + 2^1 + 2^1 + 6$$
$$= 2048 + 32 + 8 + 4 + 2 + 2 + 6 = 2102$$

If there are n elements in the set a_1, a_2, a_3, …, a_n then there are n multiples of 1.

$\left\lfloor \frac{n}{2} \right\rfloor$ multiples of 2

$\left\lfloor \frac{n}{3} \right\rfloor$ multiples of 3

… … … … … … …

… … … … … … …

$\left\lfloor \frac{n}{n} \right\rfloor$ multiples of n

So that the total number of such sets is given by

$$2^{n-1}+2^{\left\lfloor \frac{n}{2}\right\rfloor -1}+2^{\left\lfloor \frac{n}{3}\right\rfloor -1}+\cdots+2^{\left\lfloor \frac{n}{n}\right\rfloor -1}.$$

Problem 24 *Find the number of* 6*-digit natural numbers where each digit appears at least twice.*

Solution: We consider numbers like 222222 or 233200 but not 212222, since the digit 1 occurs only once.

The set of all such 6-digits can be divided into the following classes.

S_1 = the set of all 6-digit numbers where a single digit is repeated six times.

$n(S_1) = 9$, since '0' cannot be a significant number when all its digits are zero.

Let S_2 be the set of all 6-digit numbers, made up of three distinct digits.

Here we should have two cases: $S_2(a)$ one with the exclusion of zero as a digit and other $S_2(b)$ with the inclusion of zero as a digit.

***S*₂(*a*):** The number of ways, three digits could be chosen from 1, 2, …, 9 is ${}^9C_3$. Each of these three digits occurs twice. So, the number of 6-digit numbers in this case is

$$= {}^9C_3 \times \frac{6!}{2!\times 2!\times 2!} = \frac{9\times 8\times 7}{1\times 2\times 3}\times\frac{720}{8} = 9\times 8\times 7\times 15 = 7560.$$

***S*₂(*b*):** The three digits used include one zero, implying, we have to choose the other two digits from the 9 non-zero digits.

This could be done in ${}^9C_2 = \dfrac{9\times 8}{1.2} = 36.$ Since zero cannot be the leading digit, so let us fix one of the fixed non-zero numbers in the extreme left. Then the other five digits are made up of two zeroes, two fixed non-zero numbers and another non-zero number, one of which is put in the extreme left.

In this case the number of 6-digit numbers that could be formed is $\dfrac{5!}{2!\times 2!\times 1!}\times 2$ (since from either of the pairs of fixed non-zero numbers, one can occupy the extreme left digit) = 60.

So, the total number in this case = 36 × 60 = 2160.

$\therefore \quad n(S_2) = n(S_2a) + n(S_2b) = 7560 + 2160 = 9720.$

Now, let S_3 be the set of 6-digit numbers whose digits are made up of two distinct digits each of which occurs thrice. Here again, there are two cases: $S_3(a)$ excluding the digit zero and $S_3(b)$ including the digit zero.

***S*₃(*a*)** is the set of 6-digit numbers, each of whose digits are made up of two non-zero digits each occurring thrice.

$$\therefore\ n[S_3(a)] = {}^9C_2 \times \frac{6!}{3!\times 3!} = 36\times 20 = 720.$$

S₃(*b*) consists of 6-digit numbers whose digits are made up of three zeroes and one of non-zero digit, occurring thrice. If you fix one of the nine non-zero digit, use that digit in the extreme left. This digit should be used thrice. So in the remaining 5 digits, this fixed non-zero digit is used twice and the digit zero occurs thrice.

So, the number of 6-digit numbers formed in this case is

$$9\times\frac{5!}{3!\times 2!} = 90.$$

$\therefore \quad n(S_3) = nS_3(a) + nS_3(b) = 720 + 90 = 810.$

Now, let us take S_4, the case where the 6-digit number consists of exactly two digits, one of which occurs twice and the other four times.

Here again, there are two cases: $S_4(a)$ excluding zero and $S_4(b)$ including zero.

$\boldsymbol{S_4(a)}$**:** If a and b are the two non-zero numbers, then when a is used twice and b is four times, we get $\dfrac{6!}{2! \times 4!}$ and when a is used four times and b is used twice, we again get $\dfrac{6!}{4! \times 2!}$.

So, when two of the nine non-zero digits are used to form the 6-digit number in this case, the total numbers formed is

$$^9C_2 \times 2 \times \frac{6!}{4! \times 2!} = 36 \times 5 \times 6 = 1080.$$

Thus, $\qquad n[S_4(a)] = 1080.$

$\boldsymbol{S_4(b)}$**:** In this case we may use four zeroes and a non-zero number twice or two zeroes and a non-zero number four times.

In the former case, assuming the one of the fixed non-zero digit occupying the extreme left, we get the other five digits consisting of four zeroes and one non-zero number.

This results in $9 \times \dfrac{5!}{4! \times 1!}$ 6-digit numbers.

When we use the fixed non-zero digit four times and use zero twice, then we get $9 \times \dfrac{5!}{3! \times 2!} = 90$ six-digit numbers, as the fixed number occupies the extreme left and for the remaining three times it occupies 3 of the remaining digits, other digits being occupied by the two zeroes.

So, $\qquad n(S_4) = n[S_4(a)] + n[S_4(b)]$

$$= 1080 + 45 + 90 = 1215.$$

Hence, the total number of 6-digit numbers satisfying the given condition

$$= n(S_1) + n(S_2) + n(S_3) + n(S_4)$$
$$= 9 + 720 + 810 + 1215$$
$$= 2754.$$

Problem 25 *Let $X = \{1, 2, 3, \ldots, n\}$, where $n \in N$. Show that the number of r combinations of X which contain no consecutive integers is given by*

$$\binom{n-r+1}{r} \text{ where } 0 \le r \le \frac{n+1}{2}.$$

Solution: Each such r combination can be represented by a binary sequence b_1, b_2, b_3, …, b_n where $b_i = 1$, if i is a member of the r combination and 0, otherwise with no consecutive b_i's = 1 (the above r combinations contain no consecutive integers). The number of 1's in the sequence is r.

Now, this amounts to counting such binary sequences.

Now, look at the arrangement of the following boxes and the balls in them.

1	2	3	4	5	6	7
00	000	00	0000	0	0	000

00 | 000 | 00 | 0000 | 0 | 0 | 000

Here, the balls stand for the binary digits zero, and the boundaries on the left and right of each box can be taken as the binary digit one. In this display of boxes and balls as interpreted gives previously how we want the binary numbers. Here, there are 7 boxes, and 6 left/right boundary for the boxes (stating from 2 to 6). So, this is an illustration of 6 combinations of non-consecutive numbers.

The reason for zeroes in the front and at the end is that we can have leading zeroes and trailing zeroes in the binary sequence $b_1, b_2, \ldots, b_n$.

Now, clearly finding the r combination amounts to distribution of $(n - r)$ balls into $(r + 1)$ distinct boxes [$(n - r)$ balls $= (n - r)$ zeroes as these are r ones, in the n number sequence] such that the 2nd, 3rd, …, rth boxes are non-empty. (The first and the last boxes may or may not be empty—in the illustration 1st and the 7th may have zeroes or may not have balls as we have already had six combinations!). Put $(r - 1)$ balls one in each of 2nd, 3rd, …, rth boxes, (so that no two l's occur consecutively). Now we have $(n - r) - (r - 1)$ balls to be distributed in $(r + 1)$ distinct boxes.

This could be done in $\binom{[(n-r)-(r-1)+(r-1)+1]}{[(n-r)-(r-1)]}$ ways,

i.e., $\binom{n-r+1}{n-2r+1}$ ways which is equal to

$$\binom{n-r+1}{(n-r+1)-(n-2r+1)} = \binom{n-r+1}{r} \text{ ways.}$$

Here $(n - 2r + 1)$ is the number of that of identical objects (zeroes of the binary representation) and (the distinct boxes is $(r + 1 - 1) = r$. Thus, we apply the formula for distributing r identical objects in n distinct boxes as given by $\binom{n-r+1}{r}$.

[Distribution formula]

Problem 26 *Let $S = \{1, 2, 3, \ldots, (n + 1)\}$, where $n \geq 2$ and let $T = \{(x, y, z) / x, y, z \in S, x < z, y < z\}$. By counting the members of T in two different ways, prove that* $\sum_{k=1}^{n} k^2 = \binom{n+2}{2} + 2\binom{n+1}{3}$.

Solution: T can be written as $T = T_1 \cup T_2$, $T_1 = \{(x, y, z) \mid x, y, z \in S, x = y < z)$ and $T_2 = \{(x, y, z) \mid x, y, z \in S, x \neq y, x, y < z\}$.

The number of elements in T_1 is the same as choosing two elements from the set S, where $n(S) = (n + 1)$, *i.e.*, $n(T_1) = \binom{n+1}{2}$, (as every subset of two elements, the larger element will be z and the smaller will be x and y.)

In T_2 we have $2\binom{n+1}{3}$ elements, after choosing three elements from the set S, two of the smaller elements will be x and y and they may be either taken as (x, y, z) or as (y, x, z) or in other words, every three element subset of S, say $\{a, b, c\}$, the greatest is z, and the other two can be placed in two different ways in the first two positions,

$$\therefore\ n(T)(\text{or}\,|T|) = \binom{n+1}{2} + 2\binom{n+1}{3}$$

T, can also be considered as $\bigcup_{i=2}^{n+1} S_i$, where $S_i = \{(x, y, i) / x, y < i, x, y \in S\}$. All these sets are pair-wise disjoint as for different i, we get different ordered triplets (x, y, i).

Now in S_i the first two components of (x, y, i) namely (x, y), can be any element from me set $1, 2, 3, \ldots, i - 1$.

x and y can be any member from $1, 2, 3, \ldots, (i - 1)$, equal or distinct.

$\therefore$ The number of ways of selecting (x, y), $x, y \in \{1, 2, 3, \ldots, (i - 1)\}$ is $(i - 1)^2$.

Thus, $n(S_i)$ for each i is $(i-1)^2$, $i > 2$. For example, $n(S_2) = 1$, $n(S_3) = 2^2 = 4$ and so on.

Now,
$$n(T) = n\left(\bigcup_{i=2}^{n+1} S_1\right)$$
$$= \sum_{i=2}^{n+1} n(S_i)$$

(because all S_i's are pair-wise disjoint)

$$= \sum_{i=2}^{n+1} (i-1)^2 = \sum_{i=1}^{n} i^2$$

and hence,
$$\binom{n+1}{2} + 2\binom{n+1}{3} = \sum_{k=1}^{n} k^2.$$

Problem 27 *Show that the number of ways in which three numbers in AP can be selected from 1, 2, 3,…, n is* $\frac{1}{4}(n-1)^2$ *or* $\frac{1}{4}n(n-2)$ *accordingly as n is odd or n is even.*

Solution: Let three numbers be a, b, c with common diffrence 'd', Now $c - a = 2d$ $\Rightarrow c \equiv a (\text{mod } 2) \Rightarrow c, a$ both ever or odd.

Let $n = 2m$ then there are m even numbers and m odd numbers. For c, a both even $\binom{m}{2}$ choices and for both odd $\binom{m}{2}$ choices. Hence for $n = 2m$, $2\binom{m}{2}$ AP's. For n even, $2 \cdot \frac{m(m-1)}{2} = \frac{n}{4}(n-2)$ AP's.

Similarly for $n = 2m + 1$, $\binom{m}{2} + \binom{m+1}{2} = m^2 = \left(\frac{n-1}{2}\right)^2$ AP's.

Problem 28 *A train going from station X to station Y, has* 11 *stations in between, as halts.* 9 *persons enter the train during the journey with* 9 *different tickets of the same class. How many different sorts of tickets they may have had?*

Solution: 9 people enter the train during the journey, that is, they enter possibly from halt 1 to halt 11. But they can have tickets from halt i to halt j, $1 \le i \le j < 12$ (where 12th station is Y).

$\therefore$ The total number of different tickets

$$= {}^{12}C_2 = \frac{12 \times 11}{2} = 66.$$

So, the total number of different sort of available tickets is

$$^{12}C_2 = \frac{12\times 11}{1.2} = 66.$$

From these 66, we have to choose 9 tickets.

This can be done in $^{66}C_9$ ways.

Aliter: Halt 1 issues 11 different tickets.

Halt 2 issues 10 different tickets.

$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots$$

Halt 11 issues 1 ticket.

As the travellers might have got into the train from Halt 1 to 11. So, the total number of different types of available tickets is

$$1+2+3+\cdots+10+11 = \frac{11\times 12}{1.2} = 66.$$

So, there are 66 possible types of tickets to be issued to 9 persons. This could be done in $^{66}C_9$ ways.

Problem 29 *There are two bags, each containing m numbered balls. A person has to select an equal number of balls from both the bags. Find the number of ways in which he can select at least one ball from each bag.*

Solution: He may choose one ball or two balls or m balls from each bag.

Choosing one ball from one of the bags can be done in $^mC_1$ ways. Then, choosing one ball from the other bag also can be done in $^mC_1$ ways.

Thus, there are $^mC_1 \times {}^mC_1$ ways of choosing one ball from each bag. Similarly, if r balls, $1\times r\times m$ are chosen from each of the two bags, the number of ways of doing this is

$$(^mC_r)\cdot(^mC_r) = (^mC_r)^2$$

Thus, the total number of ways of choosing at least one ball from both the bags is

$$\sum_{r=1}^{m}(^mC_r)^2 = \sum_{r=0}^{m}(^mC_r)^2 + (^mC_0)^2 \ = {}^{2n}C_n - 1$$

$$= \frac{(2n)!}{n!\cdot n!} - 1 \text{ as } {}^mC_0 = 1\left[\sum_{r=0}^{m}(^mC_r)^2 = {}^{2m}C_m\right].$$

Problem 30 *If n points (no three of which are collinear) in a plane be joined in all possible ways by straight lines and if no two of the straight lines coincide or are parallel and no three lines pass through the same point (with the exception of the n original points), then prove that the number of points of intersection, exclusive of these n points is*

$$\frac{1}{8}n(n-1)(n-2)(n-3).$$

Solution: Every pair of distinct points determines a straight line. Given n points, no three of which are collinear, we get $^nC_2$ lines, *i.e.*, the number of lines determined by n distinct points, no three of which are collinear is $^nC_2 = \dfrac{n(n-1)}{2}$.

In turn these lines, taken two at a time, intersect. However, through joining each one of points to the other $(n-1)$ points, we see that there are $(n-1)$ lines passing through each one of these original points. Thus, each of these original points will be counted ${}^{n-1}C_2$ times and all the original points will be counted as $n \times {}^{n-1}C_2$ points.

The total number of points of intersection of the lines $\frac{n(n-1)}{2}$ including these n original points counted $n \cdot {}^{n-1}C_2$ times is, thus, ${}^{\frac{n(n-1)}{2}}C_2$.

So, the points of intersection other than the original points is thus

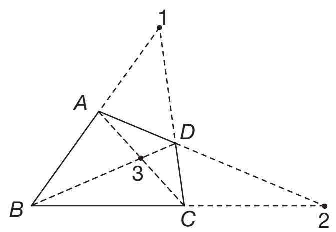
$$ {}^{\frac{n(n-1)}{2}}C_2 - n \times {}^{n-1}C_2 $$

$$ = \frac{\frac{n(n-1)}{2}\left[\frac{n(n-1)}{2}-1\right]}{2} - \frac{n(n-1)(n-2)}{1.2} $$

$$ = \frac{n(n-1)[n(n-1)-2]}{8} - \frac{n(n-1)(n-2)}{2} $$

$$ = \frac{n(n-1)}{8}[n^2 - n - 2 - 4(n-2)] $$

$$ = \frac{n(n-1)}{8}[n^2 - 5n + 6] = \frac{n(n-1)(n-2)(n-3)}{8}. $$



Aliter: Selection of any four points out of n points corresponds to a complete quadrilateral for a complete quadrilateral we get three new points of intersection as shown in the figure.

Hence $3 \cdot \binom{n}{4}$ points $= 3\frac{n(n-1)(n-2)(n-3)}{4 \times 3 \times 2 \times 1} = \frac{n(n-1)(n-2)(n-3)}{8}$.

Problem 31 *You have n objects, each of weight w. When they are weighed in pairs, the sum of the weights of all the possible pairs is* 120. *When they are weighed in triplets, the sum of the weights of all possible triplets is* 480. *Find n.*

Solution: The number of all possible pairs of objects that could be obtained from n objects is

$$ {}^nC_2 = \frac{n(n-1)}{2} $$

Total weight of $\frac{n(n-1)}{2}$ pairs $= \frac{n(n-1)}{2} \times 2 \times w$

$$ = n(n-1)w \text{ units} = 120 \tag{1} $$

The number of all possible triplets of objects that could be obtained from n objects $= {}^nC_3 = \frac{n(n-1)(n-2)}{6}$.

The total weight of all these triplets $= \frac{n(n-1)(n-2)}{6} \times 3w$

$$ = \frac{n(n-1)(n-2) \times w}{2} = 480 \tag{2} $$

Dividing Eq. (2) by (1), we get

$$\frac{n-2}{2} = \frac{480}{120} = 4$$

$\Rightarrow$ $n - 2 = 8$ or $n = 10$.

Problem 32 *Find the number of permutations* $(p_1, p_2, p_3, p_4, p_5, p_6)$ *of* $(1, 2, 3, 4, 5, 6)$ *such that for any k,* $1 \le k \le 5$ $(p_1, p_2, p_3, \ldots, p_k)$ *does not form a permutation of* $1, 2, 3, \ldots, k$, *i.e.,* $p_1 \ne 1$, (p_1, p_2) *is not a permutation of* $(1, 2)$ (p_1, p_2, p_3) *is not a permutation of* $(1, 2, 3)$, *etc.* **[INMO, 1992]**

Solution For each positive integer k, $1 \le k \le 5$, let N_k denote the number of permutations $(p_1, p_2, \ldots, p_6)$ such that $p_1 \ne 1$, (p_1, p_2) is not a permutation of (l, 2), … $(p_1, p_2, \ldots, p_k)$ is not a permutation of $(1, 2, \ldots, k)$. We are required to find N_5.

We shall start with N_1.

The total number of permutations of (1, 2, 3, 4, 5, 6) is 6! and the permutations of (2, 3, 4, 5, 6) is 5!. Thus, the number of permutations in which $p_1 = 1$ is 5!.

So, the permutation in which $p_1 \ne 1$ is $6! - 5! = 720 - 120 = 600$. Now, we have to remove all the permutations with (1, 2) and (2, 1) as the first two elements to get N_2. Of these, we have already taken into account (1, 2) in considering N_1 and subtracted all the permutations starting with 1. So, we should consider the permutation beginning with (2, 1). When $p_1 = 2$, $p_2 = 1$ $(p_3, p_4, p_5$ and $p_6)$ can be permuted in 4! ways.

So, $N_2 = N_1 - 4! = 600 - 24 = 576$.

Having removed the permutations beginning with (1, 2), we should now remove those beginning with (1, 2, 3). But, corresponding to the first two places (1, 2) and (2, 1), we have removed all the permutations. So, we should now remove the permutations with first three places (3, 2, 1), (3, 1, 2), (2, 3, 1).

Note that the first 3 positions being 1, 2, 3 is included in the permutations beginning with 1.

For each of these three arrangements, there are 3! ways of arranging 4th, 5th and 6th places and hence,

$N_3 = N_2 - 3 \times 3! = 576 - 18 = 558$. To get N_4, we should remove all the permutations beginning with the permutations of (1, 2, 3, 4), of which the arrangement of (1, 2, 3) in the first three places have already been removed. We have to account for the rest. So, when 4 is in the first place, 3! arrangements of 1, 2, 3 in the 2nd, 3rd and 4th places are possible. Also, when 4 is in the second place, since we have removed the permutation when 1 occupies the first place, there are two choices for the first place with 2 or 3 and for each of these there are 2 arrangements, *i.e.*, (2, 4, 1, 3), (2, 4, 3, 1), (3, 4, 2, 1), (3, 4, 1, 2). When 4 is in the third place, then there are first 3 arrangements (2, 3, 4, 1), (3, 2, 4, 1) and (3, 1, 4, 2).

So, the total permutations of (1, 2, 3, 4) to be removed from N_3 to get N_4 is $(6 + 4 + 3) \times 2 = 26$, because there are 2 ways of arranging the 5th and 6th places for each one of the above permutations of (1, 2, 3, 4).

$$\therefore N_4 = N_3 - 26$$
$$= 558 - 26 = 532.$$

To get N_5, we should remove from N_4 all the permutations of (1, 2, 3, 4, 5) which have not been removed until now. When 5 occupies the first position, there are 4! = 24 ways of getting 2nd, 3rd, 4th and 5th places which have not been removed so far. When $p_2 = 5$, p_1 cannot be 1, so p_1 can be chosen from the other 3, *viz.*, 2, 3 and 4, in 3 ways and 3rd, 4th and 5th places can be filled for each of the first place choice in $3 \times 2 \times 1 = 6$ ways.

So, when $p_2 = 5$, there 18 different arrangements to be removed.

When $p_3 = 5$, the first two places cannot be (1, 2) so that they can be filled in (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3) and for the fourth and fifth places there are exactly two choices for each of the above arrangements for first and second place.

So, when $p_3 = 5$, the number of arrangements to be removed is $8 \times 2 = 16$. When $p_4 = 5$, $p_1\, p_2\, p_3$ can be removed (241, 412, 421, 234, 243, 342, 324, 423, 432, 314, 341, 413, 431) and since there is only one choice left, we have now to remove 13 arrangements when

$$p_4 = 5.$$

When $p_5 = 5$, we have already removed the permutations of (1, 2, 3, 4) of the first four places to find S_4.

So, now
$$S_5 = S_4 - (24 + 18 + 16 + 13)$$
$$= 534 - 71 = 463.$$

So, 463 is the desired number of permutations.

Problem 33 *Consider the collection of all three element subsets drawn from the set* $\{1, 2, 3, 4, \ldots, 299, 300\}$. *Determine the number of subsets for which, the sum of the elements is a multiple of 3.*

Solution: The given set $S = \{1, 2, 3, 4, \ldots, 299, 300\}$ can be realised as the union of the three disjoint sets S_1, S_2 and S_3 with

$$S_1 = \{x \in S : x = 3n + 1, 0 \le n \le 99\},$$
$$S_2 = \{x \in S : x = 3n + 2, 0 \le n \le 99\},$$
$$S_3 = \{x \in S : x = 3n, 1 \le n \le 100\}.$$

Now, to get the set of all three element subsets of S, with the sum of the elements of the subset a multiple of 3, we should choose all three elements from the same set S_1, S_2 or S_3 or we should choose one element from each of the set S_1, S_2 and S_3.

We see that, $\quad n(S_1) = n(S_2) = n(S_3) = 100.$

Choosing all the three elements from either S_1, S_2 or S_3 will give $3 \times {}^{100}C_3$ triplets and its sum is also divisible by 3.

Choosing the three elements, one each from S_1, S_2 and S_3 will give ${}^{100}C_1 \times {}^{100}C_1 \times {}^{100}C_1$ triplets and its sum is also divisible by 3.

So, the total number of 3 element subsets with the required property is

$$3 \times {}^{100}C_3 + {}^{100}C_1 \times {}^{100}C_1 \times {}^{100}C_1$$
$$= \frac{3 \times 100 \times 99 \times 98}{1 \times 2 \times 3} + 100^3$$
$$= 100 \times 99 \times 49 + 1000000$$
$$= 485100 + 1000000$$
$$= 14{,}85{,}100.$$

Problem 34 *A normal die bearing the numbers* 1, 2, 3, 4, 5, 6 *on its faces is thrown repeatedly until the running total first exceeds* 12. *What is the most likely total that will be obtained?*

Solution: Consider the throws before the last one. After this penultimate throw, the running total 's' should be such that $7 \le s \le 12$; since, if we take the least value of s, *i.e.*,

$s = 7$, then we would just cross 12, if the final throw gives 6, and the maximum value of s is 12; in the final throw by getting any number 1 to 6, the running total exceeds 12. Thus, the possible values of the running total in the penultimate throw is 7, 8, 9, 10, 11 and 12.

Let us tabulate the possible running totals after the final throw.

Possible Running totals after the penultimate throw	**Possible running totals after the final throw**					
7	13	14	15	16	17	18
8	13	14	15	16	17	
9	13	14	15	16		
10	13	14	15			
11	13	14				
12	13					

Thus, the number that occurs most number of times in the possible running total after the final throw is 13.
[Since, the die is a fair die and so getting any one of 1 to 6 is equally likely and hence, the possible running totals 7, 8, 9, 10, 11 and 12 in the penultimate throw is also equally likely.]

Problem 35 *Create two fair dice which when rolled together have an equal probability of getting any sum from* 1 *to* 12.

Solution: The only sums that we want are from 1 to 12, using two dice with faces marked, say $a_1, a_2, \ldots, a_6$ and $b_1, b_2, b_3, \ldots, b_6$. We have totally $6 \times 6 = 36$ outcomes.

So, each number from 1 to 12 should occur $\frac{36}{12} = 3$ times.

If one die has numbers 1, 2, 3, 4, 5, 6 on its faces, then for 1 to 6 occur thrice, there should be three zeroes on the three faces of the second die. For each of 7, 8, …, 12 to occur thrice, three should be 3 sixes on the other three faces, so that (1, 6), (2, 6), (3, 6), …, (6, 6) can occur thrice.

Thus, the probability of getting 1 from the first die is $\frac{1}{6}$ and the probability of getting zero from the second die is $\frac{3}{6} = \frac{1}{2}$. So, probability of getting the pair (1, 0) is $\frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$ and similarly for each of numbers from 1 to 12 [$1 = 1 + 0, 2 = 2 + 0, \ldots, 6 = 6 + 0, 7 = 1 + 6, 8 = 2 + 6, \ldots, 12 = 6 + 6$].

Problem 36 *If the numbers x, y are chosen at random from* $1, 2, \ldots, n$ *with replacement,* $n \geq 3$, *show that* $P(x^3 + y^3$ *is a multiple of* $3)$ *is less than* $P(x^3 + y^3$ *is a multiple of* $7)$.

Solution: Let $S = \{1, 2, 3, \ldots, n\}$.

We shall first take $n = 2$, $n = 3$ and $n = 4$ and find, in how many ways we get $(x^3 + y^3)$ and how many of them are divisible by (*a*) 3; (*b*) 7.

For $n = 2$,

$$(x, y) = (1, 1), (1, 2), (2, 1), (2, 2),$$

$$(x^3, y^3) = (1, 1), (1, 8), (8, 1), (8, 8)$$

and $(x^3 + y^3)$ is divisible by 3 for $x^3 = 1, y^3 = 8$ and $x^3 = 8, y^3 = 1$.

Thus, $P[(x^3 + y^3)$ is a multiple of 3] in this case is $\frac{2}{4} = \frac{1}{2}$ and $P[(x^3 + y^3)$ is a multiple of 7] is an impossible event. Therefore, the statement does not hold for $n = 2$.

For $n = 3$, $\{(x, y) \mid (x, y) \in S) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

and, $\{(x^3, y^3) \mid (x, y) \in S) = \{(1, 1), (1, 8), (1, 27), (8, 1), (8, 8), (8, 27), (27, 1), (27, 8), (27, 27)\}$.

Of these set of ordered pairs, we get $(x^3 + y^3)$ divisible by 3 as $(1 + 8)$, $(8 + 1)$, $(27 + 27) = 3$.

So, here $P[(x^3 + y^3)$ is a multiple of 3] $= \frac{3}{9} = \frac{1}{3}$ and the set of ordered pairs we get set $(x^3 + y^3)$ is divisible by 7 is $(1 + 27)$, $(8 + 27)$, $(27 + 1)$, $(27 + 8) = 4$.

$\therefore$ In this case, $P[(x^3 + y^3)$ is a multiple of 7] $= \frac{4}{9}$,

and clearly, $P[(x^3 + y^3)$ is a multiple of 7] $> P[(x^3 + y^3)$ is a multiple of 3].

Now, we shall pass on to the general case where $n > 3$.

For any number, the possible remainders when n is divided by 3 is 0, 1 or 2.

So, the possible ordered pairs (x, y) (mod 3) is $\{(0, 0), (0, 1), (1, 0), (0, 2), (2, 6), (1, 1), (1, 2), (2, 1), (2,2)\}$.

Here $P\{(x^3 + y^3)$ is a multiple of 3$\}$ $= \frac{1}{3}$ as has already been seen.

$$\begin{aligned} T &= \{(x^3 + y^3) \mid (x, y) \in N \text{ (mod 3)}\} \\ &= \{(0^3 + 0^3), (0^3 + 1^3), (1^3 + 0^3), (0^3 + 2^3), (2^3 + 0^3), \\ &\quad (1^3 + 1^3), (1^3 + 2^3), (2^3 + 1^3), (2^3 + 2^3)\}. \end{aligned}$$

The subset of T which contains elements $x^3 + y^3$ is a multiple of 3 is $\{(0^3 + 1^3)$, $(1^3 + 2^3)$, $(2^3 + 1^3)\}$ and hence, the probability is $\frac{1}{3}$.

Again, when S is listed so that the elements are written in mod 7, we get

$S_7 = (0, 1, 2, 3, 4, 5, 6)$.

Now, the set of the cubes of the elements of S_7 is

$$S_c = \{0, 1, 8, 27, 64, 125, 216\}.$$

The pairs (x^3, y^3) such that $(x^3 + y^3)$ is a multiple of 7 are $\{(0, 0), (1, 27), (27, 1), (1, 125), (125, 1) (1, 216), (216, 1), (8, 27), (27, 8), (8, 125), (125, 8), (8, 216), (216, 8), (64, 27), (27, 64), (64, 125), (125, 64), (64, 216), (216, 64)\}$.

Thus, this set of ordered pairs (x^3, y^3) contains 19 elements such that $(x^3 + y^3)$ is a multiple of 7.

So, $P[(x^3 + y^3)$ is a multiple of 7] in this case is $\frac{19}{7 \times 7} = \frac{19}{49}$.

$$[\because\ n(S_c \times S_c) = n(S_c) \times n(S_c) = 7 \times 7 = 49]$$

$P[(x^3 + y^3)$ is a multiple of 3] $= \frac{1}{3}$

and hence, $P[(x^3 + y^3)$ is a multiple of 3] $< P[(x^3 + y^3)$ is a multiple of 7] $= \frac{19}{49}$.

$$\left[\frac{1}{3} < \frac{19}{49} \quad \text{as} \quad \frac{1 \times 49}{3 \times 49} < \frac{3 \times 19}{3 \times 49}\right]$$

Notes:

1. Here we have assumed that n is both a multiple of 3 as well as 7. Actually, we need to prove it for the general case where n need not be either a multiple of 3 or 7. But this can also be enumerated and verified.
2. S_c can be considered as the set of possible remainders as $\{0, 1, 1, -1, 1, -1, -1\}$ in the case of mod 7 and to get $(x^3 + y^3)$ to be divisible by 7, we can choose $(1, -1), (0, 0)$.

Probability of choosing 1 is $\frac{3}{7}$ and probability of choosing -1 is also $\frac{3}{7}$.

$\therefore$ Probability of choosing $(1, -1)$ or $(-1, 1)$ is

$$2\times\frac{3}{7}\times\frac{3}{7}=\frac{18}{49}.$$

Probability of choosing $(0, 0)$ is $\frac{1}{7}\times\frac{1}{7}=\frac{1}{49}$

$\therefore$ Probability of $(x^3 + y^3)$ is a multiple of 7 is $\frac{18}{49}+\frac{1}{49}=\frac{19}{49}$.

In the case of mod 3, also we have the set of possible remainders of x^3 or y^3 on dividing by 3 to be $\{0, 1, -1\}$.

For $(x^3 + y^3)$ to be a multiple of 3, we should choose $x^3 = 0 = y^3$ and $x^3 = 1$ and $y^3 = -1$ or $x^3 = -1$ and $y^3 = 1$.

0 can be chosen in $\frac{1}{3}$ ways.

So, probability of choosing a zero and again a zero is $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$.

Probability of choosing $(1, -1)$ or $(-1, 1)$ is

$$\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}+\frac{1}{9}=\frac{2}{9}.$$

$\therefore$ $P[(x^3 + y^3)$ is divisible by $3] = \frac{1}{9}+\frac{2}{9}=\frac{3}{9}=\frac{1}{3}$ and hence, the result.

Problem 37 *Show that the number of triplets (a, b, c) with $(a + b + c) < 95$ is less than the number of those with $(a + b + c) > 95$. where $a, b, c \in S = \{1, 2, 3, \ldots, 63\}$.*

Solution: Let $S = \{1, 2, 3, \ldots, 63\}$

Let A be the set of all triplets of S such that $(a + b + c) < 95$, *i.e.*,

$$A = \{(a, b, c):(a + b + c) < 95;\ a, b, c \in S\}.$$

Similarly, let B be the set of all triplets of S such that $(a + b + c) > 95$, where $\{a, b, c\} \in S$,

i.e., $\quad B = \{(a, b, c): (a + b + c) > 95;\ a, b, c \in S\}$

and $\quad C = \{(a, b, c): (a + b + c) > 97;\ a, b, c \in S\}$.

Clearly, C is a proper subset of B because $a, b, c \in S$, if $(a + b + c) = 96$ then $(a, b, c) \in B$ and $(a, b, c) \notin C$.

However, every element of $C \in B$,

as, $\quad a + b + c > 97 \quad\Rightarrow\quad a + b + c > 95$

hence, $\quad (a, b, c) \in C \quad\Rightarrow\quad (a, b, c) \in B$.

Now, it is enough if we show that $n(A) = n(C)$ as $n(C) < n(B)$ and $n(A) = n(C)$

$$\Rightarrow \quad n(A) < n(B).$$

If $(a, b, c) \in A$, then $1 \leq a + b + c < 95$ and also $1 \leq a, b, c \leq 63$.
Therefore, $1 \leq (64 - a), (64 - b), (64 - c) \leq 63$ and as $(a + b + c) < 95$,
$(64 - a) + (64 - b) + (64 - c) = 192 - (a + b + c) > 192 - 95 = 97$.
Thus to each element of A, there is a unique element in C.
Conversely, if $(a, b, c) \in C$, then $((64 - a), (64 - b), (64 - c)) \in A$ for
$(64 - a) + (64 - b) + (64 - c) = 192 - (a + b + c)$,
and since $(a, b, c) \in C, (a + b + c) > 97$

$\therefore$ $192 - (a + b + c) < 192 - 97 = 95$

and thus $((64 - a), (64 - b), (64 - c)) \in A$, which shows that for every element of C there corresponds a unique element in A.

Thus, there is a $1 - 1$ correspondence between the sets A and C.

$\therefore$ $n(A) = n(C) < n(B)$.

Problem 38 *Prove that it is impossible to load a pair of dice* (*each die has numbers* 1 *to* 6 *on their* 6 *faces*) *so that every sum* 2, 3, …, 12 *is equally likely. As customary, assume that the dice are distinguishable* (*For example, a* 2 *on the first die with a* 4 *on the second is different from a* 4 *on the first die and a* 2 *on the second, even though the same total* 6 *is obtained*).

Solution: Let p_i denote the probability of i coming up on the first die and q_i, the probability of i on the second die where $i = 1, 2, \ldots, 6$. The probability of getting the sum 2 is p_1q_1.
The probability of getting the sum 12 is p_6q_6.

If the probability of getting all the 11 sums are same, then probability of each would be $\frac{1}{11}$.

The probability of getting a 7 is also $\frac{1}{11}$ and is equal to

$$\begin{aligned}
\frac{1}{11} &= p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1 \\
&\geq p_1q_6 + p_6q_1 \\
&= p_1q_6\left(\frac{q_1}{q_1}\right) + p_6q_1\left(\frac{q_6}{q_6}\right) \\
&= p_1q_1\left(\frac{q_6}{q_1}\right) + p_6q_6\left(\frac{q_1}{q_6}\right) \\
&= \frac{1}{11}\left(\frac{q_6}{q_1}\right) + \frac{1}{11}\left(\frac{q_1}{q_6}\right) = \frac{1}{11}\left(\frac{q_6}{q_1} + \frac{q_1}{q_6}\right)
\end{aligned}$$

$$\Rightarrow \quad 1 \geq \frac{q_6}{q_1} + \frac{q_1}{q_6}.$$

But $\frac{q_6}{q_1}$ and $\frac{q_1}{q_6}$ are reciprocals of one another and hence their sum should be ≥ 2.

i.e., $\frac{q_6}{q_1} + \frac{q_1}{q_6}$ cannot be less than 1.

It is a contradiction and hence, the result.

Aliter: The probability mass function of the first die can be written as a probability generating function (*pgf*) as

$$p_1x + p_2x^2 + p_3x^3 + p_4x^4 + p_5x^5 + p_6x^6.$$

For the second die, the *pgf* is

$$q_1x + q_2x^2 + q_3x^3 + q_4x^4 + q_5x^5 + q_6x^6.$$

Now, the *pgf* of the sum is given by $\frac{1}{11}(x^2 + x^3 + \cdots + x^{12})$

$$\left(\sum_{i=1}^{6} p_i x^i\right)\left(\sum_{i=1}^{6} q_i x^i\right) \equiv \frac{1}{11}\left(\sum_{i=2}^{12} x^i\right).$$

Cancelling x^2 on both sides, we get

$$\left(\sum_{i=1}^{6} p_i x^{i-1}\right)\left(\sum_{i=1}^{6} q_i x^{i-1}\right) \equiv \frac{1}{11}\left(\sum_{i=0}^{10} x^i\right)$$

The RHS is the product $\frac{1}{11}$ $(x - \omega)$ $(x - \omega^2)$ … $(x - \omega^{10})$, where ω is the 11th roots of unity. All the roots of the RHS are complex and they occur in conjugate pairs. On the LHS we have two real polynomial factors each of degree 5. This is impossible. We cannot have a real 5th degree polynomial factor for $1 + x + x^2 + \cdots + x^{10}$.

Hence, such dice do not exist.

Problem 39 *There are 6 red balls and 8 green balls in a bag. Five balls are drawn out at random and placed in a red box. The remaining 9 balls are put in a green box. What is the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number?*

Solution: Let g denote the number of green balls in the red box.

So, the red box contains $(5 - g)$ red balls.

There are 8 green balls in all. So, the number of green balls in the green box

$$= (8 - g)$$

There are 6 red balls in all.

So, the number of red balls in the green box

$$= 6 - (5 - g) = (1 + g)$$

Green Box

5 balls

g green balls

$(5 - g)$

red balls

Red Box

9 balls

$(8 - g)$ green balls

$\left.\begin{array}{l} 6 - (5 - g) \\ = 1 + g \end{array}\right\}$ (red balls)

So, the number of red balls in the green box + the number of green balls in the red box $= (1 + g) + g = (2g + 1)$.

Here $(2g + 1)$ is an odd number.

Now, g cannot exceed 5, because only 5 balls are put in red box and it is taken that g green balls are put in red box.

So, 2g + 1 cannot be greater than $2 \times 5 + 1 = 11$.

Even if $\quad g = 0,\ 2g + 1 = 1$

and hence, $\quad 1 \le 2g + 1 \le 11.$

The odd primes from 10 to 11 are 3, 5, 7 and 11.

So, the only composite odd number less than 11 is 9, since 1 is neither composite nor prime, $2g + 1$ can either be 9 or 1.

So, $2g + 1 = 1 \quad \Rightarrow \quad g = 0$

and $2g + 1 = 9 \quad \Rightarrow \quad g = 4$

Only for the value of $g = 0$ or 4, we get the number $2g + 1$ to be non-prime.

Thus, it implies that we should find the number of ways of drawing all 5 red (to put in red box) or 4 green and 1 red in the draw.

The number of ways of drawing 5 red out of 6 red and 0 green out of 8 green is

$$= {}^6C_5 - {}^8C_0.$$

The number of ways of drawing 4 green and 1 red balls is

$$= {}^8C_4 \times {}^6C_1$$

Total number of drawing 5 balls is ${}^{14}C_5$ and hence, the required probability is

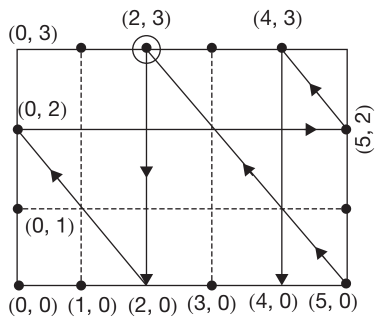
$$\frac{{}^6C_5 \times {}^8C_0 + {}^8C_4 \times {}^6C_1}{{}^{14}C_5}$$

$$= \frac{6 \times 1 + \dfrac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} \cdot 6}{\dfrac{14 \times 13 \times 12 \times 11 \times 10}{1 \times 2 \times 3 \times 4 \times 5}}$$

$$= \frac{6 + 420}{14 \times 13 \times 11} = \frac{426}{14 \times 13 \times 11} = \frac{213}{1001}.$$

Problem 40 *An oil vendor has three different measuring vessels A, B and C with capacities* 8 *litres,* 5 *litres and* 3 *litres. The vessel A is filled with oil, he wants to divide the oil into two equal parts, by pouring it from one container to another, without using any other measuring vessels other than the three. How can he do it?*

Solution: It is clear that, after pouring the oil several times into the different containers A, B and C, finally he should have 4 litres in vessel A and 4 litres in vessel B. Since C can hold a maximum of 3 litres only, this can be done by using a rectangular coordinate system. B can hold 0, 1, 2, 3, 4 and 5 litres and C can hold only 0, 1, 2 and 3 litres.



We represent the contents of B and C in a rectangular coordinate system using a 5×3 grid. Since no fraction is involved, we take only the 24 lattice points (i, j).

Here $i = 0, 1, 2, 3, 4, 5$; $j = 0, 1, 2, 3$ are used as follows:

In the horizontal lines (x-axis) are plotted (0, 0) to (5, 0) to represent the possibilities of different measures of oil that B can hold, and in the vertical line (y-axis), the points (0, 0) to (0, 3) are plotted to represent the possibilities of different measures of oil that C can hold.

We do not fill both the vessels *B* and *C* with 5 litres and 3 litres, respectively (5, 3) at any stage, as this forces us to use vessel *A* again. Vessel *A* is filled with 8 litres in the beginning.

To start with, filling the oil in vessel *B* from vessel *A* represents the point (5, 0). This is shown by the arrow from (0, 0) to (5, 0) and this is followed by (2, 3) (by pouring oil from *B* to *C*, *B* now has 2 litres and *C* has 3 litres). This is followed by (2, 0) (by pouring oil from *C* to *A*, *C* is empty and *A* has 3 + 3 = 6 litres). Now, (follow the arrows) (0, 2) (by pouring oil from B to C). This is followed by (5, 2) (by pouring 5 litres from *A* into *B*) and (5, 2) is followed by (4, 3) [by pouring 1 litre from *B* to *C*, as *C* can hold one more litre and hence (5 – 1, 2 + 1) = (4, 3) is reached].

Now, we finally get (4, 0) from (4, 3) by pouring 3 litres of oil from *C* into *A*.

Now, *B* has 4 litres and *A* has 4 litres.

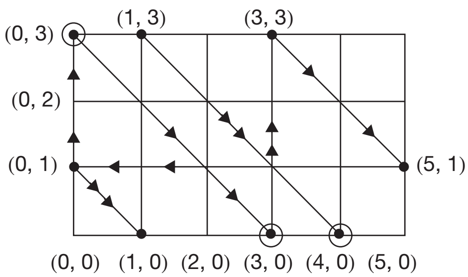
Thus in seven stages (minimum), we accomplish this task.

The above schematic representation can be given in a tabular column also as follows:

Stage	8 L Vessel	5 L Vessel	3 L Vessel
0 (initial)	8	0	0
1	3	5	0
2	3	2	3
3	6	2	0
4	6	0	2
5	1	5	2
6	1	4	3
7	4	4	0

We have several other methods, but the one given above is the best solution. Since in this case, we accomplish the task in the minimum number of steps. We give here a diagrammatic representation as well as a tabular column for yet another solution.

Here we have



$$
\begin{aligned}
& (1) - (0, 3) \\
\rightarrow\ & (2) - (3, 0) \\
\rightarrow\ & (3) - (3, 3) \\
\rightarrow\ & (4) - (5, 1) \\
\rightarrow\ & (5) - (0, 1) \\
\rightarrow\ & (6) - (1, 0) \\
\rightarrow\ & (7) - (1, 3) \\
\rightarrow\ & (8) - (4, 0)
\end{aligned}
$$

In this case we accomplish the task in 8 stages (8 > 7!).

Stage	8 L	5 L	3 L
(Initial) 0	8	0	0
1	5	0	3
2	5	3	0
3	2	3	3
4	2	5	1
5	7	0	1
6	7	**1**	0
7	4	**1**	3
8	4	4	0

Problem 41 *Consider a square array of dots, coloured red or blue, with* 20 *rows and* 20 *columns. Whenever two dots of the same colour are adjacent in the same row or column; they are joined by a segment of their common colour. Adjacent dots of unlike colours are joined by a black segment. There are* 219 *red dots,* 39 *of them on the border of the array, not at the corners. There are* 237 *black segments. How many blue segments are there?*

Solution: In each row, there are 19 segments (Since there are 20 points in each row).

There are 20 rows and hence there are $20 \times 19 = 380$ horizontal segments.

Similarly, there are $20 \times 19 = 380$ vertical segments (There are 20 columns with 19 segments in each column).

Therefore, the total number of segments = 760.

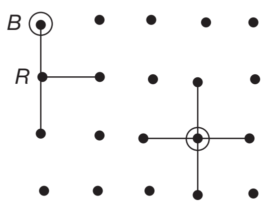
Number of black segments = 237.

Number of segments which are either blue or red = 523.

Let r denote the number of red segments and each red segment has 2 red points as the end point of the segment and each black segment has one end point coloured blue and the other end point coloured red.

So, the total number of times a red dot becomes an end point of a segment is

$$= 2 \times r + 237 = 2r + 237 \qquad (1)$$



There are altogether 219 red dots and of these, 39 are on the border.

So, the number of red dots in the interior is 180.

Each red dot on the border accounts for 3 segments (Since none of the red dots is on the corner).

So, the number of segments for which each red point on the border becomes the end points 3.

So, the total number of segments to which the 39 border red dots are end points $39 \times 3 = 117$.

Each of the 180 red points on the interior becomes the end point for 4 segments.

So, the total number of segments for which the 180 red points are the end points $= 180 \times 4 = 720$.

So the total number of times a red dot becomes an end point, *i.e.*, total number of red ends

$$= 117 + 720 = 837 \qquad (2)$$

Hence, Eqs. (1) and (2) represent the same number, the result,

$$\therefore\ 2r + 237 = 837$$

$$\therefore\ r = 300.$$

i.e., the number of red segments = 300

and the number of blue segments = 523 − 300 = 223.

Problem 42 *Suppose on a certain island there are* 13 *grey,* 15 *brown and* 17 *crimson chameleons. If two chameleons of two different colours meet, they both change to the third colour.* (*For example, when a grey and brown pair meet, then both would change to crimson*). *This is the only time they change colour. Is it possible for all chameleons eventually to be of the same colour?*

Solution: We will write the number of grey, brown and crimson chameleons as triples (g, b, c). An encounter of grey and brown changes the count (g, b, c) to $(g, b, c) + (-1, -1, 2)$.

Similarly, the other encounters will lead to changes $(-1, 2, -1)$ and $(2, -1, -1)$ in the count of grey, brown and crimson chameleons. Let there be m encounters of $(-1, -1, 2)$ kind, n encounters of $(-1, 2, -1)$ kind and l encounters of $(2, -1, -1)$ kind leading to all chameleons of the same colour, *i.e.*, the final triples will be either (45, 0, 0) or (0, 45, 0) or (0, 0, 45). Hence, we get the following equations if we end up in the triple (45, 0, 0), *i.e.*, all grey chameleons.

$$(13, 15, 17) + m(-1, -1, 2) + n(-1, 2, -1) + l(2, -1, -1) = (45, 0, 0)$$

$$\therefore\ -m - n + 2l = 32$$

$$-m + 2n - l = -15$$

$$2m - n - l = -17$$

These three equations are consistent, but of rank < 3. Hence, they have infinity of solutions given by

$$m = l - \frac{49}{3} \quad \text{and} \quad n = l - \frac{47}{3}$$

Note that we will never get all the three m, n, l to be integers in these solutions. Hence, the equations even though they are consistent, they are of no use to us as we want l, m, n to be positive integers.

Similarly, when the terminal triple is either (0, 45, 0) or (0, 0, 45), we get systems of equations which do have an infinity of solutions but which do not provide integer solutions. Hence, no sequence of encounters will even lead to all chameleons to be of the same colour.

Aliter 1: For this solution we use very elementary modulo arithmetic. Note that our initial configuration (13, 15, 17) when taken modulo 3 is (1, 0, 2). Let us see the effect of each of the encounters modulo 3 on (1, 0, 2). Consider encounter 1 leading to the change $(-1, -1, 2)$. This leads to the new configuration $(1, 0, 2) + (-1, -1, 2)$ (modulo 3) = $(0, -1, 4)$ (modulo 3) = (0, 2, 1) (modulo 3). Note that one of the components of the triple (original as well as the resultant) was divisible by 3, one left a remainder of 1, and the third left a remainder of 2 when divided by 3. Similarly, using encounters $(-1, 2, -1)$, we get (0, 2, 1) modulo 3 and using $(2, -1, -1)$, we get $(3, -1, 1)$(modulo 3) = (0, 2, 1) (modulo 3). Whatever be the encounter, the resultant triple has the same configuration, one component divisible by 3, one leaves a remainder of 1 and the other leaves a remainder of 2 when divided by 3. So, the successive encounters lead to the triples (0, 2, 1), (2, 1, 0), (1, 0, 2), (0, 2, 1) and so on. But if all chameleons must be of the same colour, we must end with (45, 0, 0) or (0, 45, 0) or (0, 0, 45). Taking modulo 3, this implies that we have to arrive at (0, 0, 0) modulo 3. But we will never arrive at a triple where every component is divisible by 3 by our above discussion. Hence, the chameleons can never be of the same colour.

Aliter 2: Let us use weights for each colour; 0 for grey, 1 for brown and 2 for crimson. The value of a triple (g, b, c) is calculated as $(0 \times g + 1 \times b + 2 \times \mathrm{c})$ modulo 3. For the initial configuration the value is $(0 \times 13 + 1 \times 15 + 2 \times 17)$ modulo 3 = 1 (modulo 3). Let us now see how each of the encounters affects the value. In the case $(-1, -1, 2)$ the value is changed by $-1 \times 0 + (-1) \times 1 + 2 \times 2 = 3 \pmod 3 = 0 \pmod 3$, *i.e.*, no change. Similarly, for the other two encounters $(-1, 2, -1)$ and $(2, -1, -1)$, the value is changed by 0 (mod 3) only. Hence, the value remains the same after any number of encounters in any order. But the value of the final required configurations namely, (45, 0, 0), (0, 45, 0) or (0, 0, 45) is 0 (mod 3). But the original value, namely, 1 (mod 3) does not change by the encounters and hence, can never reach 0 (mod 3). Hence, the chameleons cannot all end up with the same colour.

Aliter 3: We will enumerate all possible triples that we can arrive at due to these encounters and check whether we can ever arrive at (45, 0, 0), (0, 45, 0) or (0, 0, 45). Instead of 1 grey and 1 brown becoming 2 crimson, we will take the general case of r grey and r brown becoming $2r$ crimsons. Similarly for the other encounters as follows:

	G	B	C	Changes	Due to Encounters	
Initial stage	13	15	17	–13	–13	+ 26
1	0	2	43	4	–2	–2
2	4	0	41	–4	8	–4
3	0	8	37	+6	–8	–8
4	16	0	29	–16	32	–16
5	0	32	13	26	–13	–13
6	26	19	0	–19	–19	38
7	7	0	38	–7	14	–7
8	0	14	31	28	–14	–14
9	28	0	17	–17	34	–17
10	11	34	0	–11	–11	22
11	0	23	22	44	–22	–22
12	44	1	0	–1	–1	2
13	43	0	2	–2	4	–2
14	41	4	0	–4	–4	8
15	37	0	8	–8	16	–8
16	29	16	0	–16	–16	32
17	13	0	32	–13	26	–13
18	0	26	19	38	–19	–19

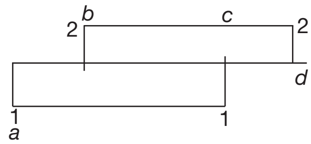
19	38	7	0	–7	–7	14
20	31	0	14	–14	28	–14
21	17	28	0	–17	–17	34
22	0	11	34	34	–11	–11
23	22	0	23	–22	44	–22
24	0	44	1	2	–1	–1
25	2	43	0	–2	–2	4
26	0	41	4	8	–4	.4
27	8	37	0	–8	–8	16
28	0	29	16	32	–16	–16
29	32	13	0	–13	–13	26
30	19	0	26	–19	38	7
31	0	38	7	14	–7	–7
32	14	31	0	–14	–14	28
33	0	17	28	34	–17	–17
34	34	0	11	–11	22	–11
35	23	22	0	–22	–22	44
36	1	0	44	–1	2	–1
37	0	2	43			

In the 37th stage we get back to (0, 2, 43), the same as we got in the first stage. Note that, at no stage did we get 2 components to be equal. Thus, it starts recurring and we will never reach the configurations (0, 0, 45), (0, 45, 0) or (45, 0, 0). Hence, the result.

Problem 43 *During a certain lecture each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that at some moment, any three of them were sleeping simultaneously. Assume that no one was sleeping before the lecture.* **[USA MO, 1986]**

Solution: Here we use proof by contradiction.

That is, we assume that there is no moment when any three of the mathematicians were sleeping simultaneously. Since every pair of mathematicians had some common time interval when both of them were sleeping, there are ${}^5C_2 = 10$ non-overlapping time intervals, (Non-overlapping because at no point of time did three of them sleep simultaneously by our assumption) one interval of common dozing for each of the ten pairs. Each such interval is started by a moment when one of the mathematicians in the pair fell asleep. Each of the 5 mathematicians fell asleep twice.

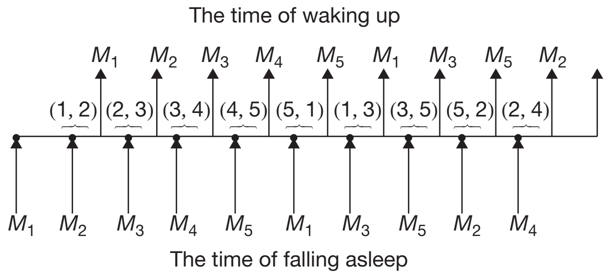


∴ There are exactly 10 such moments such that each moment initiated a different interval (as we have to account for 10 non-overlapping intervals). Let us now consider the first common dozing interval, say, that of mathematicians 1 and 2. The moment b starts the common interval. But note that moment a is already used up and does not start any other common dozing interval.

∴ We are left with 8 moments and 9 common dozing intervals which have to start at these 8 moments which is impossible. Hence it is not possible that all the 10 intervals are non-overlapping and hence, in an interval, there will be 3 mathematicians sleeping simultaneously.

Aliter: Let the 5 mathematicians be m_1, m_2, m_3, m_4 and m_5. Let the 10 pairs be (m_1, m_2), (m_1, m_3), (m_1, m_4), (m_1, m_5), (m_2, m_3), (m_2, m_4), (m_2, m_5), (m_3, m_4), (m_3, m_5) and (m_4, m_5).

If these pairs have 10 non-overlapping time intervals when each pair sleeps, then each mathematician sleeps with 4 of his colleagues in turn. But each mathematician can sleep for only 2 stretches. Therefore, we form the time interval as follows: We will represent the mathematicians m_1, m_2, m_3, m_4, m_5 on a line segment showing the moment they fall asleep and the moment they wake up. We will show that the hypothesis is not satisfied (each pair sleeping in a common interval), if we do not allow three of them to sleep during one time interval.



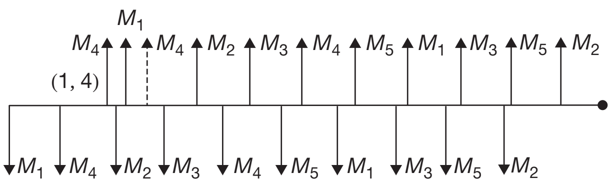
Explanation of the diagram: After representing the mathematicians M_1, M_2, M_3, M_4, M_5 and showing the time of their falling asleep, after the 5th mathematicians falls asleep, M_1 goes to sleep for his second nap. After M_1 starts sleeping for the second time, M_2 cannot come for his second nap, as every pair should occur exactly once and we had M_1 and M_2 sleeping simultaneously at the initial stage itself. So, the points, showing the other four mathematicians to follow M_1 for their second nap, should be M_3, M_5, M_2 and M_4 in that order.

Now each mathematicians appears twice, and we have the pairs (M_1, M_2), (M_2, M_3), (M_3, M_4), (M_4, M_5) (M_5, M_1), (M_1, M_3), (M_3, M_5), (M_5, M_2) and (M_2, M_4).

Here these pairs common sleep period is shown as the ordered pairs of their subscripts (1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 3), (3, 5), (5, 2) and (2, 4).

Thus, we have just nine pairs, sleeping simultaneously and the pair (1, 4) did not sleep simultaneously.

In the diagram, when M_4 appears for the second time, he sleeps along with M_2. So, if we replace M_4 between M_1 and M_2 in the figure, so that M_4's waking up moment is shown after M_2 starts sleeping but before M_3 starts sleeping as in the following figure.



Since both M_1 and M_4 wake up after M_2 falls asleep, both M_1 and M_4 sleep simultaneously with M_2 and the time interval between M_2 falling asleep and M_4 getting up (or M_2 getting up as M_4 may get up after M_2 gets up but before M_3 falls asleep shown by the dotted arrow) shown as (1, 4, 2) is the moment, when all the three M_1, M_4 and M_2 sleep simultaneously. Hence, the statement is proved.

Problem 44 *A difficult mathematical competition consisted of a Part I and a Part II within combined total of 28 problems. Each contestant solved 7 problems altogether. For each pair of problems there were exactly two contestants who solved both of them. Prove that there was a contestant who in Part I solved either no problem or at least 4 problems.*

Solution: We will find the total number of contestants.

Since for each pair of problems there were exactly two contestants, let us assume that an arbitrary problem p_1 was solved by r contestants. Each of these r contestants solved 6 more problems, solving $6r$ more problems in all counting multiplicities. Since every problem, other than p_1 was paired with p_2 and was solved by exactly two contestants, each of the remaining 27 problems (*i.e.*, other than p_1) is counted twice among the problems solved by the r contestants, *i.e.*,

$$6r = 2 \times 27$$

or $$r = 9.$$

Therefore, an arbitrary problem p_1 is solved by 9 contestants. Hence, in all we have $\dfrac{9 \times 28}{7} = 36$ contestants, as each contestant solves 7 problems.

For the rest of the proof, let us assume the contrary, that is, every contestant solved either 1, 2 or 3 problems in Part I.

Let us assume that there are n problems in Part I and let x, y, z be the number of contestants who solved 1, 2 and 3 problems in Part I.

Since every one of the contestants solves either 1, 2 or 3 problems in Part I, we get

$$x + y + z = 36 \tag{1}$$

$$x + 2y + 3z = 9n \tag{2}$$

(Since each problem was solved by 9 contestants.)

Since every contestant among y solves a pair of problems in Part I and every contestant among z solves 3 pairs of problems in Part I and as each pair of problems was solved by exactly two contestants, we get the following equations:

$$y + 3z = 2.{}^nC_2 = 2 \cdot \frac{n(n-1)}{2} = n(n-1) \tag{3}$$

From Eqs. (1), (2) and (3), we get

$$z = n^2 - 10n + 36$$

and, $$y = -2n^2 + 29n - 108 = -2\left(n - \frac{29}{4}\right)^2 - \frac{23}{8} < 0.$$

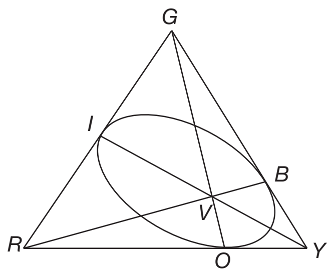
As $y < 0$ is not an acceptable result, our assumption is wrong.

Hence, there is at least one contestant who solved either no problem from Part I or solved at least 4 problems from Part I.

Problem 45 *There are certain number of balls and they are painted with the following conditions:*

(i) *Every two colours appear on exactly one ball.*
(ii) *Every two balls have exactly one colour in common.*
(iii) *There are four colours such that any three of them appear on one ball.*
(iv) *Each ball has three colours.*

Find the number of balls and the number of colours used.



Solution: Let us represent each of the balls by a line segment with three points to show the three colours.

Thus, *ROY* is a ball with three colours red, orange and yellow. We have to have three more balls such that on each of them one of the colours should be red, orange or yellow. So, next draw lines through R, O, Y to meet at a common point G standing for green colour. But the balls with colours RG, OG and YG must have a third colour in them say, Indigo (I), Violet (V) and Blue (B). Thus, we have 7 balls and 7 colours, in all.

7 colours R, O, Y, G, I, V, B and 7 balls

1. *ROY*, 2. *RIG*, 3. *RVB*, 4. *OVG*, 5. *YBG*, 6. *YVI*, 7. *IBO*.

Clearly, any pair of the above 7 balls have exactly one colour in common (satisfying condition 2). Each of the balls contribute 3 pairs of colours. In all, we have 21 pairs of colours in all the 7 balls. Now, 7 colours lead to $\frac{7\times 6}{2} = 21$ pairs of colours and each pair of colours is found in exactly one ball (satisfying condition 1). Each ball has 3 colours (condition 4 satisfied). Now, consider the four colours G, R, Y, V. No three of these colours are found on a ball (condition 3 is satisfied).

Thus, the total number of colours is 7 and the total number of balls is also 7.

Problem 46 *It is proposed to partition the set of positive integers into two disjoint subsets A and B. Subject to the following conditions:*

(i) 1 *is in A.*

(ii) *No two distinct members of A have a sum of the form*

$$2^k + 2 (k = 0, 1, 2, \ldots).$$

(iii) *No two distinct members of B have a sum of the form*

$$2^k + 2 (k = 0, 1, 2, \ldots).$$

Show that this partitioning can be carried out in a unique manner and determine the subsets to which 1987, 1988, 1989, 1997, 1998 *belong.*

Solution: Since it is given that $1 \in A$, $2 \notin A$. For if $2 \in A$, then $2^0 + 2 = 3$ is generated by 2 members of A violating the condition for the partitioning.

$$\therefore 2 \in B$$

Similarly, $3 \notin A$ as $1 + 3 = 4 = 2^1 + 2$

$$\therefore 3 \in B$$

But $4 \notin B$. For if $4 \in B$, then $2^2 + 2 = 4 + 2 = 6$ is generated by two members of B.

$\therefore$ The partitioning for the first few positive integers is

$A = \{1, 4, 7, 8, 12, 13, 15, 16, 20, 23, \ldots\}$

$B = \{2, 3, 5, 6, 9, 10, 11, 14, 17, 18, 19, 21, 22, \ldots\}$

Suppose $1, 2, \ldots, n - 1$ (for $n \geq 3$) have already been assigned to $A \cap B$ in such a way that no two distinct members of A or B have a sum $= 2^l + 2$ ($l = 0, 1, 2, \ldots$)

Now, we need to assign n to A or B.

Let k be a positive integer such that $2^{k-1} + 2 \leq n < 2^k + 2$. Then, assign '$n$' to the complement of the set to which $2^k + 2 - n$ belongs. But for this, we need to check that whether $2^k + 2 - n$ has already been assigned or not. Now as $n \geq 2^{k-1} + 2 > 2^{k-1} + 1$

$$2n > 2^k + 2$$

$$\therefore n > 2^k + 2 - n$$

Since all numbers below n have been assumed to be assigned to either A or B, $2^k + 2 - n$ has already been assigned and hence n is also assigned uniquely.

For example, consider $k = 1$

$$3 = 2^0 + 2 \le n < 2^1 + 2 = 4.$$

Consider $n = 3$, $4 - n = 1$ Now $1 \in A$ (given)

$$\therefore 3 \in B$$

Consider $k = 2$

$$\therefore 2^{2-1} + 2 \le n < 2^2 + 2 = 6$$

$$4 \le n < 6$$

When $n = 4$, as $6 - n = 2 \in B$, we assign 4 to A.

When $n = 5$ as $6 - 5 = 1 \in A$, we assign 5 to B.

Since the set to which n gets assigned is uniquely determined by the set to which $2^k + 2 - n$ belongs, the partitioning is unique.

Looking at the pattern of the partitioning of the initial set of positive integers, we conjecture the following:

1. $n \in A$ if $4 \mid n$
2. $n \in B$ if $2 \mid n$ but $4 \nmid n$
3. If $n = 2^r \cdot k + 1$, $r \ge 1$, k odd, then $n \in A$ if k is of the form $4m - 1$ and $n \in B$ if k is of the form $4m + 1$.

Proof of the conjecture: We note that $1, 4 \in A$ and $2, 3 \in B$. If $2^{k-1} + 2 \le n < 2^k + 2$ and all numbers less than n have been assigned to A or B and satisfy the above conjectures, then if $4 \mid n$, as $2^k + 2 - n$ is divisible by 2 but not by 4, $2^k + 2 - n \in B$. Hence, $n \in A$. Similarly, if 2 divides n but not 4, then $2^k + 2 - n$ is divisible by 4 and hence, is in A.

$$\therefore n \in B.$$

If $n = 2^r \cdot k + 1$ where $r \ge 1$, k odd and $k = 4m - 1$, then

$$2^k + 2 - n = 2^k - 2^r \cdot k + 1 = 2^r(2^{k-r} - k) + 1$$

where clearly $2^{k-r} - k$ is odd and equals 1 (mod 4).

$$\therefore 2^k + 2 - n \in B.$$

Hence, $n \in A$. Similarly, it can be shown that if $n = 2^r \cdot k + 1$, where $k \equiv 1 \pmod 4$, then $n \in B$. Thus, the conjecture is proved.

Now, 1988 is divisible by 4.

$$\therefore 1988 \in A$$

$$1987 = 2^1 \cdot 993 + 1 \quad \text{where } 993 = 1 \pmod 4$$

$$\therefore 1987 \in B$$

$$1989 = 2^2 \cdot 497 + 1 \quad \text{where } 497 = 1 \pmod 4$$

$$\therefore 1989 \in B$$

$$2 \mid 1998 \text{ but } 4 \nmid 1998$$

$$\therefore 1998 \in B$$

$$1997 = 2^2 \cdot 499 + 1 \quad \text{where } 499 = 3 \pmod 4$$

$$\therefore 1997 \in A.$$

Check Your Understanding

1. Given $p, q \in \mathbb{N}$, prove that $\sum_{k=1}^{q-1}\left\lfloor \frac{kp}{q} \right\rfloor = \sum_{k=1}^{p-1}\left\lfloor \frac{kq}{p} \right\rfloor$

2. Prove that $\sum_{d|n}\phi(d) = n$, where $\phi(d)$ = number of positive integers coprime with d and less than or equal to d.

3. Prove that $\sum_{k=1}^{n}\tau(k) = \sum_{k=1}^{n}\left\lfloor \frac{n}{k} \right\rfloor$ where $\tau(k)$ is number of divisors of k.

4. Prove that $\frac{{}^{2n}C_n}{n+1} = {}^{2n}C_n - {}^{2n}C_{n-1}$ and hence or otherwise, deduce that ${}^{2n}C_n$ is always divisible by $(n + 1)$.

5. Prove that $\sum_{P\subseteq X}\sum_{Q\subseteq X}|P\cap Q| = n4^{n-1}$, where X is a set of n elements.

6. Let n and r be integers with $0 \le r \le n$. Find a simple expression for $S_r = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^r\binom{n}{r}$.

7. Let n be positive integer not less than 3. Find a direct combinational interpretation of the identity $\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$.

8. Find the number of functions $f: \{1, 2, 3, \ldots, n\} \to \{1947, 1951, 2018, 2020\}$ such that $f(1) + f(2) + \cdots + f(n)$ is odd.

9. Let n be a positive integer. Prove that the binomial coefficients $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \cdots, \binom{n}{n-1}$ are all even if and only if n is a power of 2.

10. Find all $n \in \mathbb{N}$, such that $\binom{n}{r}$ is odd $\forall r = 0, 1, 2, \ldots, n$.

11. Delete 1 0 1 digits from the number 1 3 5 7 9 11 13 15 17 19 … 109 111 in such a way that the remaining number is

(i) as small as possible,

(ii) as big as possible.

12. You are given 7 sheets of paper and you cut any number of these into 7 small pieces. Out of the total sheets you get, you again cut some into 7 pieces and you continue the process. At every stage you count the total number of sheets you have. Show that you will never get 605 pieces.

13. During election campaign, n different kinds of promises are made by various political parties, $n > 0$. No two parties have exactly the same set of promises. While several parties may make the same promise, every pair of parties have atleast one promise in common. Prove that there can be at most 2^{n-1} parties.

14. The number 3 can be expressed as an ordered sum of one or more positive integers in four ways as follows:
$3, 1 + 2, 2 + 1, 1 + 1 + 1$.
Show that the positive integer n can be so expressed in 2^{n-1} ways.

15. Let n be any natural number. Find the sum of the digits appearing in the integers $1, 2, 3, \ldots, 10^n - 2, 10^n - 1$.

16. Let $f(n)$ denote the number of solutions (x, y) of $x + 2y = n$ for which both x and y are non-negative integers. Show that $f(0) = f(1) = 1, f(n) = f(n - 2) + 1, n = 2, 3, 4, \ldots$. Find a simple explicit formula for $f(n)$.

17. At a party, there are more than 3 people. Every four of the people have the property that one of the four is acquainted with the other three. Show that with the possible exception of three of the people, every one at the party is acquainted with all of the others at the party.

18. What is the least number of plane cuts required to cut a block of size $a \times b \times c$ into abc unit cubes if piling is permitted?

19. In a mathematical competition, a contestant can score 5, 4, 3, 2, 1, or 0 points for each problem. Find the number of ways he can score a total of 30 points for 7 problems.

20. Every person, who has ever lived has upto this moment, made a certain number of hand-shakes. Prove that the number of people who have made an odd number of handshakes is even.

21. Show that among any seven distinct positive integers not greater than 126, one can find two of them, say, x and y satisfying the inequalities $1 < \frac{x}{y} \le 2$.

22. Given a set of $(n + 1)$ positive integers none of which exceeds $2n$, show that atleast one member of the set must divide another member of the set.

23. There are six closed discs in a plane such that none contains the centre of any other disc (even on the boundary). Show that they do not have a common point.

24. Prove that if 5 pins are stuck on to a piece of cardboard in the shape of an equilateral triangle of side length 2, then some pair of pins must be within distance 1 of each other.

25. Given any $(n + 2)$ integers show that for some pair of them either their sum or their difference is divisible by $2n$.

26. Two players, play the game. The first player selects any integer from 1 to 11 inclusive. The second player adds any positive integer from 1 to 11 inclusive to the number selected by the first player. They continue in this manner alternatively. The player who reaches 56 wins the game. Which player has the advantage?

27. You are given 6 congruent balls two each of colours red, white and blue and informed that one ball of each colour weighs 15 gram, while the other weighs 16 grams. Using an equal arm balance only twice, determine which three are the 16 gram balls.

28. Find the number of integers in the set $\{1, 2, \ldots, 10^3\}$ which are not divisible by 5 nor by 7 but are divisible by 3.

29. Find the number of integers in the set $\{1, 2, \ldots, 120\}$ which are divisible by exactly m of the integers 2, 3, 5, 7 where $m = 0, 1, 2, 3, 4$.

30. For how many paths consisting of a sequence of horizontal and/or vertical line segments with each segment connecting a pair of adjacent letters in the diagram below is the word MATHEMATICS spelled out as the path is traversed from beginning to end.

M
M A M
M A T A M
M A T H T A M
M A T H E H T A M
M A T H E M E H T A M
M A T H E M A M E H T A M
M A T H E M A T A M E H T A M
M A T H E M A T I T A M E H T A M
M A T H E M A T I C I T A M E H T A M
M A T H E M A T I C S C I T A M E H T A M

31. A group of 100 students took examination in English, Science and Mathematics. Among them, 92 passed in English, 75 in Science and 63 in Mathematics; at most 65 passed in English and Science, at most 54 in English and Mathematics and at most 48 in Science and Mathematics. Find the largest possible number of the students that could have passed in all the three subjects.

32. Lines $L_1, L_2, \ldots, L_{100}$ are distinct. All lines L_{4n}, n being positive integer are parallel to each other. All lines L_{4n-3}, n a positive integer pass through a given point A. Find the maximum number of points of intersection of pairs of lines from the complete set $(L_1, L_2, \ldots, L_{100})$.

33. How many integers with four different digits are there between 1,000 and 9999 such that the absolute value of the difference between the first digit and the last digit is 2?

34. A multi set is an ordered collection of elements, where elements can repeat. For example, $\{a, a, b, c, c\}$ is a multiset of size five. Discover the number of multisets of size four, which can be constructed from the given 10 distinct elements.

35. Find the number of numbers from 1 to 10^{100}, having the sum of their digits equal to 3.

36. Two students from Standard XI and several students from Standard XII participated in a chess tournament. Each participant played with every other once only. In each game, the winner has received one point, the loser zero and for the game drawn, both the players got 0.5 points each. The two students from Standard XI together scored 8 points and the scores of all the participants of Standard XII are equal.

(i) How many students of Standard XII participated in the tournament?

(ii) What was the equal score in Standard XII?

37. Show that an equilateral triangle, cannot be covered completely by two smaller equilateral triangles.

38. The diagonal connecting two opposite vertices of a rectangular parallelepiped is $\sqrt{73}$ units. Prove that if the squares of the edges of the parallelepiped are integers, then its volume cannot exceed 120.

39. In a group of 7 people, the sum of the ages of the members is 332 years. Prove that three members can be chosen, so that the sum of their ages, is not less than 142 years.

40. Ten students solved a total of 35 problems in a Mathematics contest; each problem was solved by exactly one student. There is one student who solved exactly one problem, at least one student who solved exactly two problems and at least one student who solved exactly three problems. Prove that, there is also at least one student, who has solved at least 5 problems.

41. Let T be the set of triplets (a, b, c) of integers, such that $1 < a < b < c < 6$. For each triplet (a, b, c) consider the number $a \times b \times c$. Add all these numbers corresponding to the triplets in T. Prove that the resulting sum is a multiple of seven.

42. There are 9 cells in a 3×3 square, when these cells are filled by numbers -1, 0, 1. Prove that, of the 8 sums obtained, at least two sums are equal.

43. How many 6-digit numbers are there such that
 (i) The digits of each number are all from the set $\{1, 2, 3, 4, 5\}$
 (ii) Any digit that appears in the number appears at least twice.
 (Example: 225252 is admissible while 222133 is not).

44. Show that, in any group of 5 students there are two students who have identical number of friends within the group.

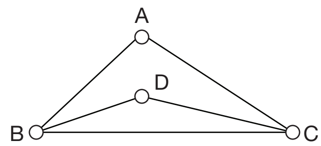
45. Given 11 different natural numbers, none greater than 20. Prove that, two of these can be chosen, one of which divides the other.

46. Find the number of 6-digit natural numbers, such that the sum of their digits is 10 and each of the digits 0, 1, 2, 3, occurs at least once in them.

47. Prove that, among 18 consecutive 3-digit numbers, there is at least one number, which is divisible by the sum of the digits.

48. A rectangle with sides $2m-1$ and $2n-1$ is divided into squares of unit length by drawing parallel lines to the sides. Find the number of rectangles possible with odd side lengths.

49. A road network as shown in the figure connect four cities. In how many ways can you start from any city (say A) and come back to it without travelling on the same road more than once?



50. Consider the lines $x=k$ and $y=k$, $k\in\{1, 2, \ldots, 9\}$. The number of non-congruent rectangles, whose sides are along these lines, is ______.

51. A point P, is at a distance of 12 cm from the centre of a circle of radius 13 cm. Find the number of chords of the circle passing through P which have integral lengths.

52. Let P_n denotes the number of ways of selecting 3 people out of 'n' sitting in a row, if no two of them are consecutive and Q_n is the corresponding figure when they are in a circle. If $P_n - Q_n = 6$, then find the value of n.

53. Take a ΔABC. Take n points of sub-division on side AB and join each of them to C. Likewise, take n points of sub-division on side AC and join each of them to B. Into how many parts is ΔABC divided?

54. Each side of an equilateral ΔABC is divided into 6 equal parts. The corresponding points of subdivision are joined. Find the number of equilateral triangles oriented the same way as ΔABC.

55. Let $n = 10^6$. Evaluate $\sum_{d/n} \log_{10} d$.

56. Let $n = 180$. Find the number of positive integral divisors of n^2, which do not divide n.

57. Show that the number of positive integral divisors of 111 … 1(2010 times) is even.

58. How many unordered pairs $\{a, b\}$ of positive integers a and b are there such that LCM (a, b)= 1,26,000?
 (***Note:*** An unordered pair $\{a, b\}$ means $\{a, b\} = \{b, a\}$)

59. The sum of the factors of 7!, which are odd and are of the form $3t+1$ where t is a whole number, is ______.

60. Consider a set $\{1, 2, 3,\ldots, 100\}$. Find the number of ways in which a number can be selected from the set so that it is of the form x^y, where $x, y \in N$ and ≥ 2, is ______.

Challenge Your Understanding

1. Let A, B be disjoint finite sets of integers with the following property.
If $x \in (A \cup B)$, then either $x + 1 \in A$ or $x - 2 \in B$.
Prove that $n(A) = 2n(B)$ [*i.e.*, $|A| = 2|B|$].

2. Find all positive integers k, for which the set $A = \{1996, 1996 + 1, 1996 + 2, \ldots, 1996 + k\}$ with $k + 1$ elements can be partitioned into two subsets B and C such that the sum of the elements of B = sum of the elements of C.

3. Suppose you and your husband attended a party with three other married couples. Several hand-shakes took place. No one shook hands with himself or (herself) or with his (or her) spouse, and no one shook hands with other more than once. After all the hand-shaking was completed, suppose you asked each person including your husband, how many hands he or she had shaken? Each person gave a different answer.
(i) How many hands did you shake?
(ii) How many hands did your husband shake?

4. Let $S = \{1, 2, \ldots, 100\}$ and A be any subset of S containing 53 members. Show that A has two numbers a, b such that $a - b = 12$. Construct a subset B of S with 52 numbers such that for any two numbers a, b of B, $|a - b| \neq 12$.

5. Let A be any set of 19 distinct integers chosen from the AP 1, 4, 7, 10, …, 100. Show that A must contain at least two distinct integers whose sum is 104. Find a set of 18 distinct integers from the same progression such that the sum of no two distinct integers from the set equals 104.

6. In a room containing N people $N > 3$, at least one person has not shaken hands with every one else in the room. What is the maximum number of people in the room that could have shaken hands with every one else?

7. A positive integer n has the decimal representation $n = d_1 d_2 \ldots d_m$.
(i) n is called ascending if $0 < d_1 \leq d_2 \leq \ldots \leq d_m$
(ii) n is called strictly ascending if $0 < d_1 < d_2 < \ldots < d_m$.
Find the total number of type (i) and type (ii) numbers, which are less than 10^9.

8. Let N(k) = $\{1, 2, \ldots, k\}$. Find the number of:
(i) functions from $N(n)$ to $N(m)$.
(ii) one-to-one functions from $N(n)$ to $N(m)$, $n \leq m$.
(iii) strictly increasing functions from $N(n)$ to $N(m)$, $n \leq m$.
(iv) non- decreasing functions from $N(n)$ to $N(m)$.

9. Let $n = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7^4$. Find the number of positive integral divisors of n which are greater than $\sqrt{n}$.

10. Let $m = \sum_{i=0}^{k} m_i p^i, n = \sum_{i=0}^{k} n_i p^i$; $m_i, n_i \in \{0, 1, 2, \ldots, p - 1\}$ and p is a prime number, prove that $\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i}$ (mod p).

11. Let $T(n)$ denote the number of non-congruent triangles with integer side lengths and perimeter n.
Thus $T(1) = T(2) = T(3) = T(4) = 0$, while $T(5) = 1$. Prove that
(i) $T(2006) < T(2009)$
(ii) $T(2005) = T(2008)$.

12. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be distinct points in a plane. Let D and d be the longest and the shortest distances respectively between pairs of points among them. Prove that, $\frac{D}{d} \geq \sqrt{3}$.

13. Several football teams enter a tournament, in which, each team play every other team exactly once. Show that, at any moment, during the tournament, there will be two teams, which have played up to that moment, an identical number of games.

14. Given 7-element of set $A = \{a, b, c, d, e, f, g\}$. Find a collection T of 3-element subsets of A, such that each pair of elements from A, occurs exactly in one of the subsets of T.

15. In how many different ways, can the digits 1 through 5, be arranged to form a five digit number, in which, the digits, alternately rise and fall? These numbers are called Mountain Numbers; for example, 13254 is a Mountain Number while 12354 is not.

16. If A is a 50 element subset of the set $\{1, 2, 3, \ldots, 100\}$ such that, no two numbers from A, add upto 100, show that A contains a square.

17. Show that, there exist two powers of 1999, whose difference is divisible by 1998.

18. If the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are divided into three groups, show that, the product of the numbers in one of the groups, exceeds 71.

19. Show that, there exists a power of 3 which ends in the digits 001.

20. If 181 square integers are given, prove that, one can find a subset of 19 numbers among these such that, the sum of these elements is divisible by 19.

21. Given any 13 distinct real numbers, prove that, there are two of them, say x and y, such that, $0 < \dfrac{x-y}{1+xy} < 2-\sqrt{3}$.

22. Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person A phones to person B and tells B everything what he knows, while B tells A nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything?

23. Consider a rectangular array of dots with an even number of rows and an even number of columns. Colour the dots, each one red or blue, subject to the condition that a each row, half the dots are red and the other half are blue and in each column also, half the, dots are red and the other half are blue. Now, if two points are adjacent and like coloured, join them by an edge of their colour. Show that the number of blue segments is equal to the number of red segments.

24. Teams $T_1, T_2, \ldots, T_n$ take part in a tournament in which every team plays every other team just once. One point is awarded for each win and it is assumed that there are no draws. Let $s_1, s_2, \ldots, s_n$ denote the total scores of $T_1, T_2, \ldots, T_n$ respectively. Show that for $1 < k < n$, $s_1 + s_2 + \cdots + s_k \le n_k - \dfrac{1}{2}\, k(k+1)$.

25. Seventeen people correspond by mail with one another each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of the topics. Prove that there are atleast three people who write to one another about the same topic.

26. No matter which 55 positive integers one may select from 1, 2, 3, …, 100. Prove that there will be some two that differ by 9, some two that differ by 10, some two that differ by 12, some two that differ by 13, but surprisingly their need not be any two that differ by 11.

27. There is a $2n \times 2n$ array (matrix) consisting of 0's and 1's and there are exactly $3n$ zeroes. Show that it is possible to remove all the zeroes by deleting some n rows and some n columns.

28. Let $a(n)$ denote the number of ways of expressing the positive integer n as an ordered sum of l's and 2's, e.g., $a(5) = 8$ because $5 = 1 + 1 + 1 + 1 + 1 = 2 + 1 + 1 + 1 = 1 + 2 + 1 + 1 = 1 + 1 + 2 + 1 = 1 + 1+1 + 2 = 2 + 2+1 = 2 + 1 + 2 = 1 + 2 + 2$. Let $b(n)$ denote the number of ways of expressing n as an ordered sum of integers greater than 1, for example, $b(7) = 8$ because $7 = 3 + 2 + 2 = 2 + 3 + 2 = 2 + 2 + 3 = 3 + 4 = 4 + 3 = 2 + 5 = 5 + 2 = 7$. Prove that $a(n) = b(n + 2)$ for $n = 1, 2, \ldots$.

29. A pack of 13 distinct cards is shuffled in some particular manner and then repeatedly in exactly the same manner. What is the maximum number of shuffles required for the cards to return to their original positions?

30. Each of n boys attends a school-gathering with both his parents. In how many ways can the $3n$ people be divided into groups of three such that each group contains a boy, a male parent and a female parent, and no boy is with both his parents in his group?

31. A permutation $a_1, a_2, \ldots, a_n$ are $1, 2, 3, \ldots, n$ is said to be good if and only if $(a_j - j)$ is constant for all j, $1 \le j \le n$. Determine the number of good permutations for $n = 1999$, $n = 2000$.

32. An international society has its members from six different countries. The list of members contains 1978 names numbered 1, 2, 3, …, 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country. **[IMO, 1978]**

33. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that

$$a_{2n-1} = 0,\ a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}),\ n = 1, 2, 3, \ldots, \text{ where } x = 2 + \sqrt{2}, y = 2 - \sqrt{2}.$$

Here a path of n jumps is a sequence of vertices $(P_0, \ldots, P_n)$ such that

(i) $P_0 = A, P_n = E$.

(ii) for every i, $0 \le i \le n - 1$, P_i is distinct from E.

(iii) for every i, $0 \le i \le n - 1$, P_i and $P_i + 1$ are adjacent. **[IMO, 1979]**

34. Let n and k be given relatively prime natural numbers $k < n$. Each number in the set $M = \{1, 2, \ldots, n - 1\}$ is coloured either blue or white. It is given that

(i) for each $i \in M$ both i and $(n - i)$ have the same colour;

(ii) for each $i \in M$, $i \ne k$, both i and $(f - k)$ have the same colour.

Prove that all numbers in M have the same colour. **[IMO, 1985]**

35. $2 \times 2 \times n$ hole in a wall is to be filled with $2n$, $1 \times 1 \times 2$ bricks. In how many different ways can this be done if the bricks are indistinguishable?

36. Let $P_1, P_2, \ldots, P_n$ be distinct two element subsets of the set of elements $\{a_1, a_2, \ldots, a_n\}$ such that if $P_i \cap P_j \ne \phi$, then (a_i, a_j) is one of the P's. Prove that each of the a_s appears in exactly two of the P's.

37. Ten airlines serve a total of 1983 cities. There is direct service without a stop over between any two cities and if an airline offers a direct flight from A to B, it also offers a direct flight from B to A. Prove that at least one of the airlines provides a round trip with an odd number of landings.

38. Five students A, B, C, D, E took part in a contest. One prediction was that the contestants could finish in the order A B C D E. This prediction was very poor. In fact, no contestant finished in the position predicted and no two contestants predicted to finish consecutively did so. A second prediction had the contestants finishing in the order D A E C B. This prediction was better. Exactly two of the contestants finished in the places predicted and two disjoint pairs of students predicted to finish consecutively actually did so. Determine the order in which the contestants finished.

39. Suppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections these perpendiculars can have.

40. In a plane, a set of n points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let d be the length of the segment we define a diameter of the set to be any connecting segment of length d. Prove that the number of diameters of the given set is at most n.

41. In a mathematical contest, the three problems A, B and C were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A, the number who solved problem B, was twice the number who solved C. The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A. How many students solved only problem B?

42. In a sports contest, there were m medals awarded on n successive days ($n > 1$), on the first day, one medal and $\frac{1}{7}$ of the remaining $(m - 1)$ medals were awarded on the second day, two medals and $\frac{2}{7}$ of the now remaining medals were awarded; and so on. On the nth and last day, the remaining n medals were awarded. How many days did the contest last, and how many medals were awarded altogether?

43. Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of given points.

44. A certain organization has n members and it has $(n + 1)$ three member committees, no two of which have identical membership. Prove that there are two committees which share exactly one member. **[USA MO, 1979]**

45. In a party with 1982 persons, among any group of four there is at least one person who knows each of the other three. What is the minimum number of people in the party who know everyone else? **[USA MO, 1982]**

46. On an infinite chess board, a game is played as follows: At the start n^2 pieces are arranged on the chess board in $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square immediately beyond the piece who has been jumped over is then removed. Find those values of n for which the game will end with only one piece remaining on chess board. **[IMO, 1993]**

47. Find the number of ways in which one can place the numbers $1, 2, \ldots, n^2$ on square of $n \times n$ chess board, one on each such that the numbers in each row and each column are in AP (assume $n \geq 3$). **[INMO, 1992]**

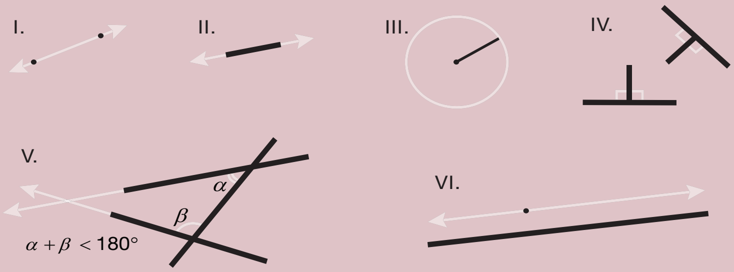
48. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured, the set of coloured edges necessarily contains a triangle all of whose edges have the same colour. **[IMO, 1992]**

49. Nine mathematicians meet at an international conference and discover that among any three of them, at least two speak a common language. If each of the mathematicians can speak utmost three languages, prove that there are atleast three of the mathematicians who can speak the same language. **[USA MO, 1979]**

50. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an AP? Justify your answer. **[IMO, 1983]**

Chapter 8

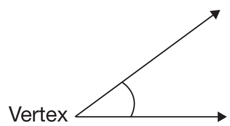
I. A straight line segment can be drawn joining any two points.
II. Any straight line segment can be extended indefinitely in a straight line.
III. Given any straight lines segment, a circle can be drawn having the segment as radius and one endpoint as center.
IV. All right angles are congruent.
V. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the parallel postulate.
VI. Given any straight line and a point not on it, there exists 'one and only one straight line' which passes through that point and never intersects the first line, no matter how far they are extended. This statement is equivalent to the fifth of Euclid's postulates, called parallel postulate.



Euclid of Alexandria

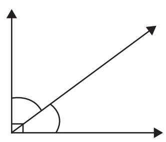
Geometry

8.1 Angle



An angle is the figure formed by two rays, called the sides of the angle and sharing a common endpoint, called the vertex of the angle.

8.1.1 Complementary Angles

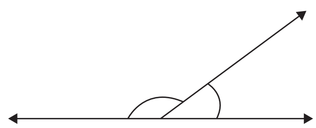


Complementary angles are angle pairs whose measures add up to one right angle (1/4 turn, 90°, or $\pi/2$ radians). If the two complementary angles are adjacent their non-shared sides form a right angle.

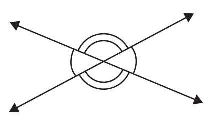
8.1.2 Supplementary Angles

Two angles whose measures add up to a straight angle (1/2 turn, 180°, or π radians) are called supplementary angles.

If the two supplementary angles are adjacent (*i.e.*, have a common vertex and share just one side), their non-shared sides form a straight line. Such angles are called a linear pair of angles.

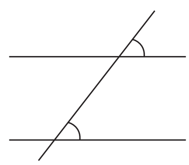


8.1.3 Vertically Opposite Angles (VOA)



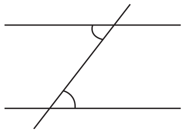
A pair of angles opposite to each other, formed by two intersecting straight lines that form an 'X'-like shape, are called vertical angles or opposite angles or vertically opposite angles. They are abbreviated as vert. opp. $\angle$s. They are always equal.

8.1.4 Corresponding Angles Postulate or CA Postulate



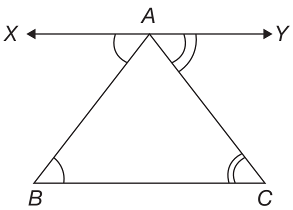
If two parallel lines are cut by a transversal, then corresponding angles are congruent

8.1.5 Alternate Interior Angles Theorem or AIA Theorem



If two parallel lines are cut by a transversal, then alternate interior angles are congruent to each other.

8.1.6 Angle Sum Theorem



Sum of all the angles of a triangle is 180°.

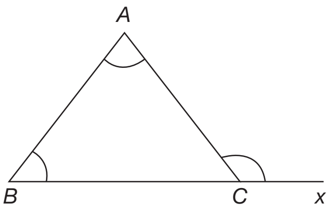
Construction: Draw a line XY through the vertex A and parallel to base BC.

$\angle XAB = \angle ABC$

(alternate interior angles between two parallels)

Similarly $\angle YAC = \angle ACB$

Now, $\angle ABC + \angle BAC + \angle ACB = \angle XAB + \angle BAC + \angle YAC = 180°$



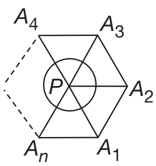
Corollary 1: Exterior angle of a triangle is equal to sum of two opposite interior angles.

Construction: Extend BC to point X such that C lies in between B and X.

Proof: Exterior angle at vertex C is

$\angle ACX = 180° - \angle ACB = \angle BAC + \angle ABC$

(using previous theorem)

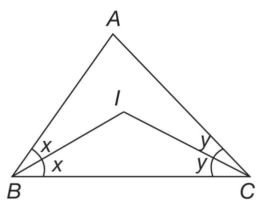


Corollary 2: In any n sided convex polygon sum of all angles is $(n - 2) \times 180°$ and also sum of all exterior angles (taken in one direction, *i.e.*, either clockwise or counter clockwise) in any convex polygon is 360°.

Construction: Take a point P inside the polygon Join it with all the vertices.

Proof: As there are n triangle having P as common vertex, sum of all angles of all triangles is $n \times 180°$. Now remove from it sum of angles at vertex P which is 360°. Hence sum of all interior angles of the polygon is $n \times 180° - 360° = (n-2) \times 180°$

Example 1 *If the bisectors of $\angle ABC$ and $\angle ACB$ of a triangle meet at a point I. then prove that* $\angle BIC = 90° + \frac{1}{2}\angle A$.



Solution:

Given: In ΔABC, BI, CI bisects $\angle B$ and $\angle C$

To Prove: $\angle BIC = 90° + \frac{1}{2}\angle A$

Proof: In ΔABC, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow \angle A + 2x + 2y = 180°$

$$\Rightarrow x + y = 90° - \frac{1}{2}\angle A \qquad (1)$$

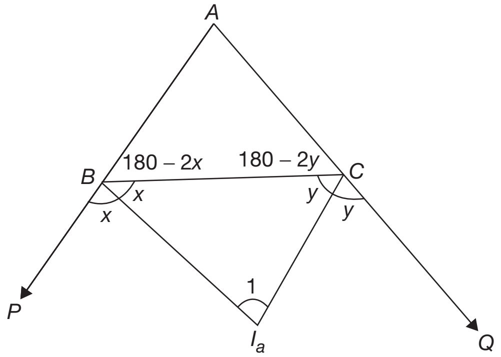
In ΔIBC, $\angle I + x + y = 180°$

$$\Rightarrow \angle I + 90° - \frac{1}{2}\angle A = 180° \qquad \text{(From Eq. (1))}$$

$$\Rightarrow \angle I = 90° + \frac{1}{2}\angle A.$$

Example 2 *The sides AB and AC of a triangle ABC are produced to P and Q respectively. If the bisectors of $\angle PBC$ and $\angle QCB$ intersects at I_a then prove that* $\angle BI_aC = 90° - \frac{1}{2}\angle A$

Solution:



BI_a bisects $\angle PBC$ and CI_a bisects $\angle QCB$

Let $\angle I_aBP = \angle I_aBC = x$ and $\angle I_aCB = \angle I_aCQ = y$

$\angle ABC = 180° - 2x$ and $\angle ACB = 180° - 2y$

In ΔABC, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow \angle A + 180° - 2x + 180° - 2y = 180°$

$\Rightarrow x + y = 90° + \frac{1}{2}\angle A$ (1)

In ΔBI_aC, $x + y + \angle I_a = 180°$

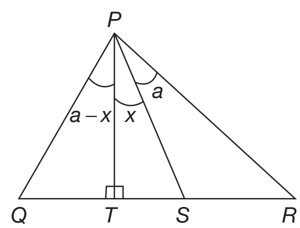
$\Rightarrow 90° + \frac{1}{2}\angle A + \angle I_a = 180°$ (From Eq. (1))

$\Rightarrow \angle I_a = 180° - 90° - \frac{1}{2}\angle A$

$\Rightarrow \angle BI_aC = 90° - \frac{1}{2}\angle A.$

Example 3 *PS is the bisector of $\angle QPR$ and $PT \perp QR$ show that $\angle TPS = \frac{1}{2}(\angle Q - \angle R)$ Where $\angle Q < \angle R$.*

Solution:



Let $\angle QPS = \angle SPR = a$ and $\angle TPS = x$

$\therefore \angle QPT = a - x$

In ΔPTR, by using exterior angle property

$\angle QTP = 90° = a + x + \angle R$ (1)

In ΔPTQ, by using exterior angle property

$\angle PTR = 90° = a - x + \angle Q$ (2)

$\therefore$ From Eq. (1) and Eq. (2)

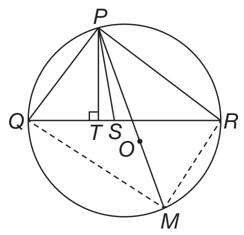
$a + x + \angle R = a - x + \angle Q$

$\Rightarrow 2x = \angle Q - \angle R$

$\Rightarrow x = \frac{1}{2}(\angle Q - \angle R).$

Note: If $\angle R > \angle Q$, then $x = \frac{1}{2}(\angle R - \angle Q)$

Example 4 *In above question if PM is the circum-diameter of ΔPQR then prove that PS bisects $\angle TPM$.*



Solution:

Construction: Join QM, RM

Proof: Since POM is a diameter, $\angle PRM = 90°$

$\Rightarrow \angle QRM = 90° - \angle R$

$\Rightarrow \angle QPM = \angle QRM = 90° - \angle R$ (Angle in same segment)

$\Rightarrow \angle TPM = \angle QPM - \angle QPT$

$= (90° - \angle R) - (90° - \angle Q)$

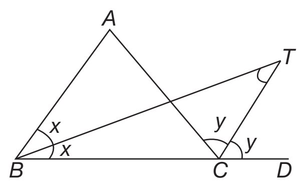
$\Rightarrow \angle TPM = \angle Q - \angle R$

Since $\angle TPS = \frac{1}{2}(\angle Q - \angle R)$ (From Previous problem)

$\therefore \angle SPM = \frac{1}{2}(\angle Q - \angle R)$

$\therefore$ PS bisects $\angle TPM$.

Example 5 *Prove that the angle between internal bisector of one base angle and the external bisector of the other base angle of a triangle is equal to one half of the vertical angle.*



Solution:

Given: BT bisects $\angle ABC$ and CT bisects $\angle ACD$

To prove: $\angle BTC = \frac{1}{2}\angle A$

Proof: In ΔABC, by using exterior angle property of a triangle

$\angle ACD = \angle ABC + \angle A$

$\Rightarrow 2y = 2x + \angle A$

$\Rightarrow \angle y = \angle x + \frac{1}{2}\angle A$ (1)

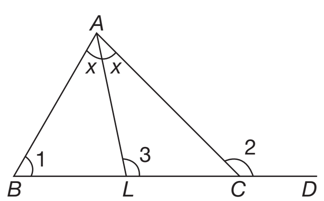
In ΔTBC, by using exterior angle property

$\angle y = \angle x + \angle T$ (2)

$\therefore$ From Eqs. (1) and (2), we get, $\angle x + \angle T = \angle x + \frac{1}{2}\angle A$

$$\Rightarrow \angle T = \frac{1}{2}\angle A.$$

Example 6 *The side BC of ΔABC is produced, such that D is on ray BC. The bisector of $\angle A$ meets BC in L as shown in the figure. Prove that $\angle ABC + \angle ACD = 2\angle ALC$.*



Solution:

In ΔABC, by using

Exterior angle property

$\angle 2 = \angle 1 + 2\angle x$

Adding $\angle 1$ to both sides

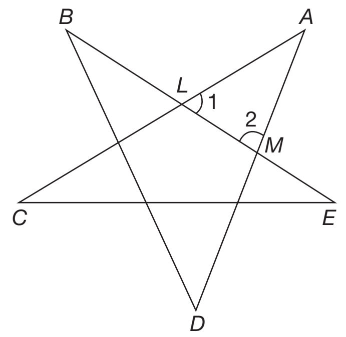
$\angle 1 + \angle 2 = \angle 1 + \angle 1 + 2\angle x$

$$= 2\angle 1 + 2\angle x = 2(\angle 1 + \angle x)$$

$$\Rightarrow \angle 1 + \angle 2 = 2\angle 3$$

$$\therefore \angle ABC + \angle ACD = 2\angle ALC.$$

Example 7 *The given figure shows a five point star. Find sum of the angle $\angle A + \angle B + \angle C + \angle D + \angle E$.*



Solution: Let BE intersects AC and AD at L and M respectively

Now, in ΔMBD, by using exterior angle property $\angle 2 = \angle B + \angle D$ (1)

Similarly, in $\Delta LCE, \angle 1 = \angle C + \angle E$ (2)

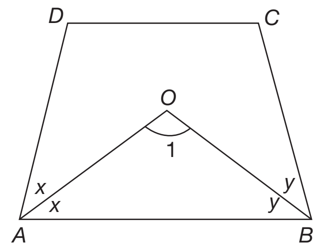
In ΔALM, $\angle A + \angle 1 + \angle 2 = 180°$

$\Rightarrow \angle A + \angle C + \angle E + \angle B + \angle D = 180°$ (From Eqs. (1) and (2))

Or $\angle A + \angle B + \angle C + \angle D + \angle E = 180°$

Note: In n point star sum of all the angles at its vertices is $(n-4)\times 180°$.

Example 8 *In a quadrilateral ABCD, AO and BO are the bisectors of $\angle A$ and $\angle B$ respectively, prove that $\angle AOB = \frac{1}{2}(\angle C + \angle D)$.*



Solution:

In quadrilateral $ABCD$, $\angle A + \angle B + \angle C + \angle D = 360°$

$\Rightarrow 2x + 2y + \angle C + \angle D = 360°$

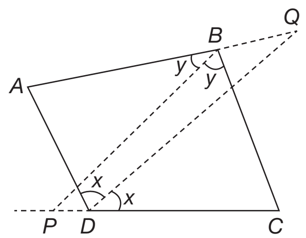
$\Rightarrow x + y = 180° - \frac{1}{2}(\angle C + \angle D)$

In ΔAOB, $x + y + \angle 1 = 180°$

$\Rightarrow 180° - \frac{1}{2}(\angle C + \angle D) + \angle 1 = 180°$

$\Rightarrow \angle 1 = \frac{1}{2}(\angle C + \angle D).$

Example 9 *In the figure bisectors of $\angle B$ and $\angle D$ of quadrilateral ABCD meets CD and AB produced at P and Q respectively. Prove that $\angle P + \angle Q = \frac{1}{2}(\angle ABC + \angle ADC)$.*



Solution:

Let $\angle ABP = \angle PBC = y = \frac{1}{2}\angle B$

and $\angle ADQ = \angle QDC = x = \frac{1}{2}\angle D$

$\Rightarrow \angle PDQ = 180° - x$ and $\angle PBQ = 180° - y$ (1)

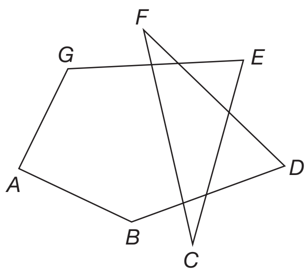
In quadrilateral $PDQB, \angle P + \angle PDQ + \angle Q + \angle QBP = 360°$

$\Rightarrow \angle P + 180° - x + \angle Q + 180° - y = 360°$ (From Eq. (1))

$\angle P + \angle Q = x + y = \frac{1}{2}(\angle B + \angle D).$

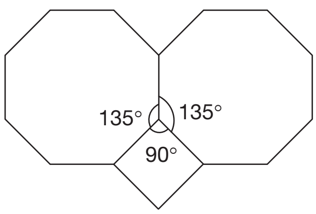
Build-up Your Understanding 1

1. Of the three angles of a triangle, one is twice the smallest and another is three times the smallest. Find the angles.
2. Can two internal angle bisector in a triangle be perpendicular?
3. If the angles of a triangle are in the ratio 5 : 7 : 6, determine the three angles.
4. The difference between two angles of a triangles is 24°. All angles are numerically double digits. Find the number of possible values of third angle.
5. In ΔABC, the angle bisectors of the exterior angles of $\angle A$ and $\angle B$ intersect opposite sides CB produced and AC produced at D and E respectively, and $AD = AB = BE$. Then find angle A.
6. Prove that, in n point star sum of all the angles at its vertices is $(n-4)\times 180°$.
7. In a regular polygon an interior angle is four times bigger than corresponding external angle. Find the number of sides of the polygon.
8. The interior angle of a n sided regular polygon is 48° more than the interior angle of a regular hexagon. Find n.
9. The interior angles of a polygon are in Arithmetic Progression. The smallest interior angle is 120° and common difference is 5°. Find the number of sides.
10. If in a convex polygon, the sum of all interior angles excluding one is 2210°, then find the excluded angle and number of sides of the polygon.
11. In a convex polygon the sum of all interior angles is less than 2017°. Find the maximum number of sides.
12. If all exterior angles of a polygon are obtuse then find the number of sides of the polygon.



13. In the adjacent diagram, Find $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G$.
14. There are four points A, B, C, D on the plane, such that any three points are not collinear. Prove that in triangles ABC, ABD, ACD, BCD there is at least one triangle which has an interior angle not greater than 45°.
15. Prove that a convex polygon cannot have more than three acute internal angles.
16. In ΔABC, $AB = AC$. D is a point on BC such that $AB = CD$. E on AB such that $DE \perp AB$. Prove that $2\angle ADE = 3\angle B$.
17. Given a quadrilateral $ABCD$, E is a point on AD. F is a point inside $ABCD$ such that CF, EF bisects $\angle ACB$ and $\angle BED$ respectively. Prove that

$$\angle CFE = 90° + \frac{1}{2}(\angle CAD + \angle CBE).$$



18. Two regular octagons and one square completely cover the part of a plane around a point without any overlapping shown in the figure. Find all the other possible combinations of three regular polygons, two of which are congruent and one different.
19. Three regular polygons have one vertex in common and just fill the whole space at that vertex. If the number of sides of the polygons are a, b, c respectively, prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$. Also find all possible (a, b, c) with $a \le b \le c$.
20. Quadrilateral $ABCD$ has $\angle BDA = \angle CDB = 50°$, $\angle DAC = 20°$ and $\angle CAB = 80°$. Find angles $\angle BCA$ and $\angle DBC$.

8.2 Congruent Triangles

Two triangles are congruent if and only if one of them can be made to superpose on the other so as to cover it exactly.

Some of the following congruencies are often used:

8.2.1 Side Angle Side (SAS) Congruence Postulate

Two triangles are congruent, if two side and the included angle of one are equal to the corresponding sides and the included angle of the other triangle.

8.2.2 Angle Side Angle (ASA) Congruence Postulate

Two triangles are congruent, if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

8.2.3 Angle Angle Side (AAS) Congruence Postulate

If any two angles and a non-included side of one triangle are equal to the corresponding angle and side of another triangle, then the two triangles are congruent.

8.2.4 Side Side Side (SSS) Congruence Postulate

Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

8.2.5 Right Angle Hypotenuse Side (RHS) Congruence Postulate

Two right triangles are congruent, if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other triangle.

Example 10 *In the adjacent diagram it is given that* $AB = CF$, $EF = BD$ *and* $\angle AFE = \angle CBD$. *Prove that* $\Delta AFE \cong \Delta CBD$

Solution:

We have, $AB = CF$

$\Rightarrow AB + BF = CF + BF$

$\Rightarrow AF = CB$ (1)

In ΔAFE and ΔCBD

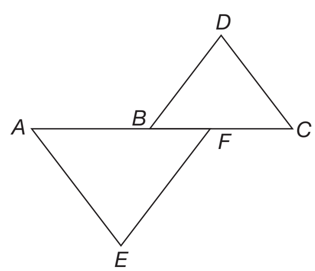
$AF = CB$ (From Eq. (1))

$\angle AFE = \angle CBD$ (Given)

$FE = BD$ (Given)

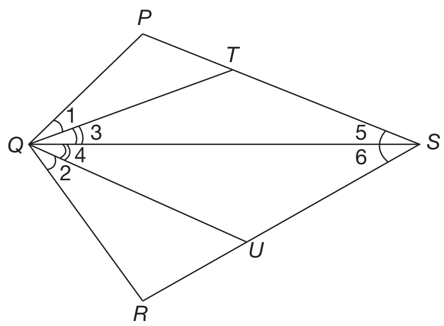
So by SAS congruence, we have

$\Delta AFE \cong \Delta CBD$



Example 11 *In the figure PQRS is a quadrilateral and T and U respectively are points on PS and RS, such that* $PQ = RQ$, $\angle PQT = \angle RQU$ *and* $\angle TQS = \angle UQS$. *Prove that* $QT = QU$.

Solution:



Proof: In ΔPQS and ΔRQS

We have $PQ = RQ$

$\angle 1 = \angle 2$

$\angle 3 = \angle 4$

$\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$

$\Rightarrow \angle PQS = \angle RQS$

$QS = QS$ (Common)

$\Rightarrow \Delta PQS \cong \Delta RQS$ (By SAS)

$\Rightarrow \angle 5 = \angle 6$ (CPCT-Corresponding parts of congruent triangles)

In ΔTQS and ΔUQS

$\angle 3 = \angle 4$

$QS = QS$

$\angle 5 = \angle 6$

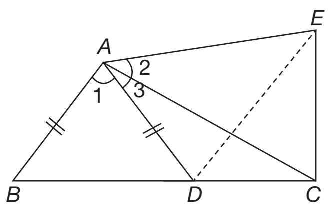
$\therefore$ By ASA congruences

$\Delta TQS \cong \Delta UQS$

$\Rightarrow QT = QU$ (CPCT).

Example 12 *In the figure* $AC = AE$, $AB = AD$ *and* $\angle BAD = \angle EAC$, *prove that* $BC = DE$.

Solution:



Construction: Join DE

Proof: In ΔABC and ΔADE

$AB = AD$ (Given) (1)

Also $\angle 1 = \angle 2$

$\Rightarrow \angle 1 + \angle 3 = \angle 2 + \angle 3$

$\Rightarrow \angle BAC = \angle DAE$ (2)

Also $AC = AE$ (Given) (3)

Using Eqs. (1), (2) and (3) and applying SAS congruences

$\Delta ABC \cong \Delta ADE$

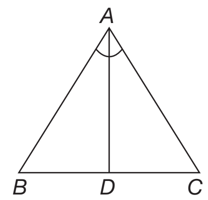
$\Rightarrow BC = DE$ (CPCT).

Example 13 *Prove that angles opposite to two equal sides of a triangle are equal.*

Solution:

Given: In ΔABC, $AB = AC$.

To prove: $\angle B = \angle C$



Construction: Draw the bisector AD of $\angle A$ which meets BC at D.

Proof: In $\angle BAD$ and $\angle CAD$

$AB = AC$ (Given)

$\angle BAD = \angle CAD$ (Construction)

$AD = AD$ (Common)

$\therefore$ By SAS congruences

$\Delta BAD \cong \Delta CAD$

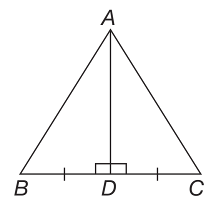
$\Rightarrow \angle ABD = \angle ACD$ (CPCT)

Hence proved.

Note: If two angles of a triangle are equal, then sides opposite to them are also equals (proof is left for the reader).

Example 14 *If the altitude from one vertex of a triangle bisects the opposite side, then prove that triangle is an isosceles.*

Solution:



Given: In ΔABC, $AD \perp BC$ and $BD = DC$

To prove: $AB = AC$

Proof: In ΔADB and ΔADC

$AD = AD$ (Common)

$\angle ADB = \angle ADC = 90°$

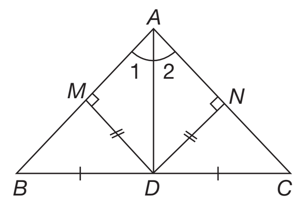
$DB = DC$ (Given)

$\therefore$ By SAS congruences, $\Delta ADB \cong \Delta ADC$

$\Rightarrow AB = AC$ (CPCT).

Example 15 *If the bisector of the vertical angle of a triangle bisects the base of the triangle, then prove that the triangle is isosceles.*

Solution:



Given: AD bisects $\angle BAC$ of ΔABC and $BD = DC$

To prove: $AB = AC$

Construction: Draw $DM \perp AB$, $DN \perp AC$

Proof: In ΔAMD and ΔAND

$\angle AMD = \angle AND = 90°$

$\angle 1 = \angle 2$ (Given)

$AD = AD$ (Common)

$\therefore$ By AAS congruence

$\Delta AMD \cong \Delta AND$

$\Rightarrow DM = DN$ (CPCT)

In ΔMDB and ΔNDC

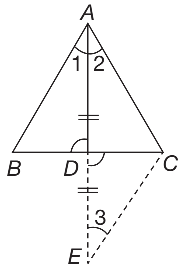
$\angle DMB = \angle DNC = 90°$

$DM = DN$ (Proved above)

$DB = DC$ (Given)

$\therefore$ By RHS congruence, $\Delta MDB \cong \Delta NDC$

$\Rightarrow \quad \angle MBD = \angle NCD$



Aliter:

Construction: Produce AD to E such that $AD = DE$ join EC

Proof: In ΔADB and ΔEDC

$AD = ED$ (Construction)

$\angle ADB = \angle EDC$ (VOA)

$DB = DC$ (Given)

$\therefore$ By SAS congruences, $\Delta ADB \cong \Delta EDC$

$\Rightarrow AB = EC$ (CPCT)

$\Rightarrow \angle 1 = \angle 3$, *i.e.*, $\angle BAD = \angle CED$ (CPCT)

But $\angle 1 = \angle 2$ (Given)

$\Rightarrow \angle 2 = \angle 3$

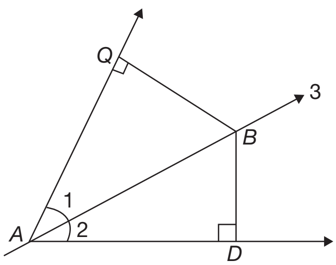
$\Rightarrow AC = CE$

But $CE = AB$

$\Rightarrow AC = AB$

Hence proved.

Example 16 *Line 'l' is the bisector of $\angle A$ and B is any point on l. BP and BQ are perpendiculars from B to the arms of A. Prove that $BP = BQ$ or B is equidistant from the arms of $\angle A$.*



Solution:

In ΔAPB and ΔAQB, we have

$\angle APB = \angle AQB = 90°$

$\angle 1 = \angle 2$ (Given l is the angle bisector)

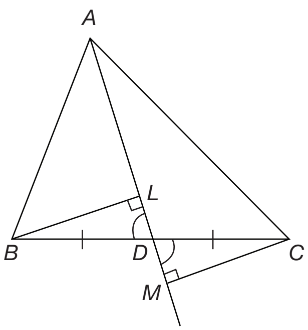
$AB = AB$ [Common]

$\therefore$ By AAS congruences, $\Delta APB \cong \Delta AQB$

$\Rightarrow PB = QB$.

Note: Each point on the angle bisector is equidistant from the arms of an angle.

Example 17 *In the figure AD is a median and BL, CM are perpendiculars drawn from B and C respectively on AD and AD produced. Prove that $BL = CM$*



Solution:

Proof: In ΔBDL and ΔCDM

$\angle BLD = \angle CMD = 90°$

$\angle BDL = \angle CDM$ (VOA—Vertically Opposite Angle)

$BD = CD$ (Given)

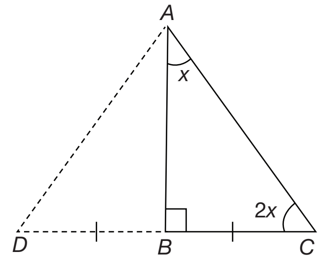
$\therefore$ By AAS congruences

$\Delta BDL \cong \Delta CDM$

$\Rightarrow BL = CM$ (CPCT).

Note: In this figure $BLCM$ will be a parallelogram.

Example 18 *In a right angled triangle, if one acute angle is double of another. Prove that the hypotenuse is double the smallest side.*



Solution:

Given: In ΔABC, $\angle B = 90°$ and $\angle ACB = 2\angle CAB$

To prove: $AC = 2BC$

Construction: Produce CB to D such that $CB = BD$. Join AD.

Proof: In ΔABD and ΔABC

$AB = AB$ (Common)

$\angle ABD = \angle ABC = 90°$

$BD = BC$ (Construction)

$\therefore$ By SAS congruences, $\Delta\, ABD \cong \Delta\, ABC$

$\Rightarrow \angle ADB = \angle ACB = 2x\ (CPCT)$

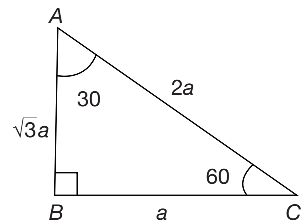
And $\angle BAD = \angle BAC = x$

$\angle DAC = \angle ACD = \angle CDA$

$\therefore$ ΔADC is an equilateral triangle

$\Rightarrow AC = DC = DB + BC = BC + BC$

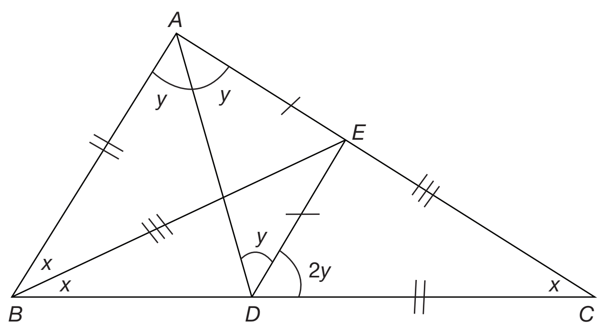
$\Rightarrow AC = 2BC.$



Note: In 30°–60°–90° triangle, sides are $a, \sqrt{3}a, 2a$ respectively.

Example 19 *ABC is a triangle in which* $\angle B = 2\angle C$. *D is a point on BC such that AD bisects* $\angle BAC$ *and* $AB = CD$. *Prove that* $\angle BAC = 72°$.

Solution:



Given: In ΔABC, $\angle B = 2\angle C$

AD bisects $\angle BAC$

$AB = CD.$

To prove: $\angle BAC = 72°$

Construction: Draw BE the angle bisector of $\angle ABC$ which meets AC at E. Join DE.

Proof: Let $\angle C = x$ then $\angle B = 2x$ and $\angle BAD = \angle CAD = y$

Since BE bisects $\angle ABC$

$\therefore \angle ABE = \angle EBC = x$

Then in ΔBEC,

$\angle EBC = \angle ECB = x$

$\therefore BE = CE$

In ΔABE and ΔDCE,

$AB = DC$ (Given)

$\angle ABE = \angle DCE = x$

$BE = CE$ (Proved above)

$\therefore$ By SAS congruence

$\Delta ABE \cong \Delta DCE$

$\therefore AE = DE$ (CPCT)

$\angle BAE = \angle CDE = 2y$

Since $AE = DE$ $\therefore \angle EAD = \angle EDA = y$

In ΔABD,

$3y = 2x + y$ (Exterior angle property)

$\therefore 2y = 2x$

$y = x$

In ΔABC, by ASP (Angle Sum Property) of a triangle

$\angle A + \angle B + \angle C = 180°$

$\Rightarrow 2y + 2x + x = 180°$

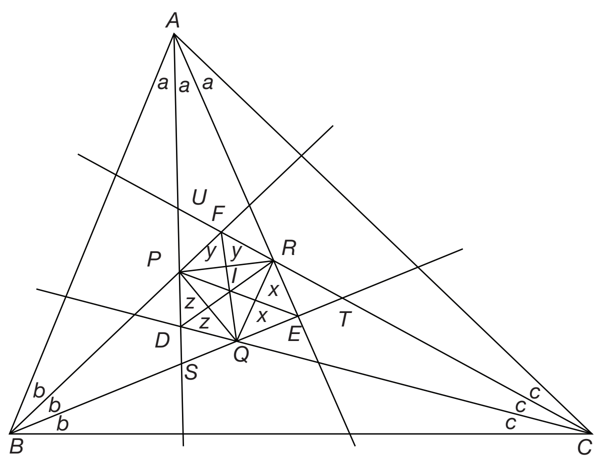
$\Rightarrow 5y = 180°$ (As $x = y$)

$\Rightarrow y = 36°$

$\Rightarrow \angle BAC = 2y = 72°$

Hence proved.

Example 20 *Prove that in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.*



Solution:

Given: AP, AR trisects angle A; BP, BQ trisects angle B; CQ, CR trisects angle C;

To Prove: ΔPQR is an equilateral triangle.

Proof: In ΔABC $3a + 3b + 3c = 180°$

$\Rightarrow a + b + c = 60°$ (1)

In ΔABE, AP, BP are the angle bisectors

$\therefore$ P is the incentre of ΔAEB

$\therefore$ PE bisects $\angle AEB \Rightarrow \angle PEA = \angle PEB = x$ (say)

Similarly Q is the incentre of ΔBFC and R is the incentre of ΔADC.

QF bisects $\angle BFC \Rightarrow \angle BFQ = \angle CFQ = y$ (say)

RD bisects $\angle ADC \Rightarrow \angle RDA = \angle RDC = z$ (say)

Also in ΔAEB, $2a + 2b + 2x = 180°$

$\Rightarrow a + b + x = 90°$

$\Rightarrow 60° - c + x = 90°$ (From Eq. (1))

$\Rightarrow x = 30° + c$ (2)

Similarly, $y = 30° + a$

and $z = 30° + b$

In ΔAPB,

$\angle APB = 180° - (a + b)$

$= 180° - (60° - c)$ (From Eq. (1))

$\Rightarrow \angle APB = 120° + c$

$\Rightarrow \angle BPS = \angle APF = 180° - \angle APB = 60° - c$

In ΔBPS,

$\angle PSQ = 60° - c + b$ (Exterior angle property)

In ΔPSE,

$\angle SPE + (60° - c + b) + x = 180°$

$\Rightarrow \angle SPE + 60° + b - c + 30° + c = 180°$ (From Eq. (2))

$\Rightarrow \angle SPE = 90° - b$ (3)

In ΔPDI,

$90° - b + z + \angle PID = 180°$

$\Rightarrow 90° - b + 30° + b + \angle PID = 180°$

$\Rightarrow \angle PID = 60°$

Similarly $\angle DIQ = 60°$

So $\angle PIQ = 120°$

Similarly $\angle QIR = 120°$

$\angle PIR = 120°$

In ΔPID and ΔQID

$\angle PID = \angle QID = 60°$

$\angle IPD = \angle IQD = 90° - b$ (From Eq. (3))

$ID = ID$

$\therefore$ By AAS congruency

$\Delta PID \cong \Delta QID$

$\Rightarrow PI = QI$ and $PD = QD$ (CPCT)

DI is the $\perp$ bisector of PQ

As DIR is a straight line, DR is the $\perp$ bisector of PQ

$\Rightarrow PR = QR$ (4)

Similarly PE is the $\perp$ bisector of QR

$\Rightarrow PQ = PR$ (5)

From Eqs. (4) and (5)

$PQ = QR = PR$

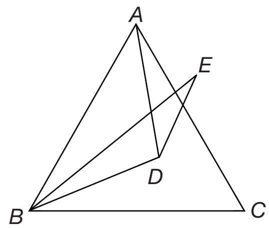
Frank Morley

9 Sep 1860–17 Oct 1937
Nationality: English

Note: The above problem, is known as **Morley's trisector theorem**, was discovered in 1899 by Anglo-American mathematician Frank Morley. It has various generalizations; in particular, if all of the trisectors are intersected, one obtains four other equilateral triangles.

Build-up Your Understanding 2

1. 'O' is the circumcentre of ΔABC. M is the mid-point of the median through A. Join OM and produce it to N such that $OM = MN$. Show that, N lies on the altitude through A.
2. In a given quadrilateral $ABCD$, $AB = AD$, $\angle BAD = 60°$, $\angle BCD = 120°$. Prove that $BC + DC = AC$.
3. Given that ΔABC is an isosceles right triangle with $AC = BC$ and $\angle ACB = 90°$. D is a point on AC and E is on the extension of BD such that $AE \perp BE$.
   If $AE = 1/2\ BD$, prove that BD bisects $\angle ABC$.
4. In the figure point D is an interior point of equilateral triangle ABC. It is given that $DA = DB$. Point E is also given so that $\angle DBE = \angle DBC$ and $BE = AB$. Find $\angle E$.



5. In the figure, given that in ΔABC, $AB = AC$, D is on AB and E is on the extension of AC such that $BD = CE$. The segment DE intersects BC at G. Prove that $DG = GE$ (see Figure 8.1).

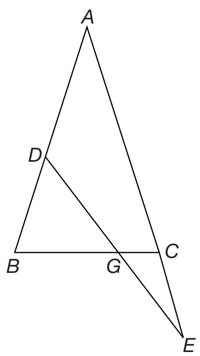
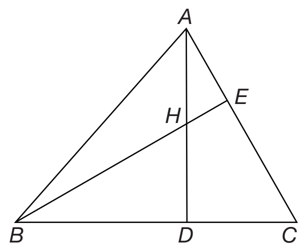
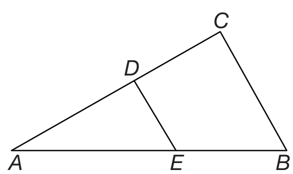


Figure 8.1

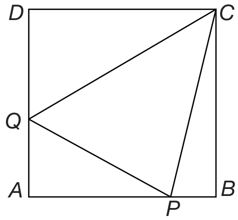
6. Given BE and CF are the altitudes of the ΔABC. P, Q are on BE and the extension of CF respectively, such that $BP = AC$, $CQ = AB$, Prove that $AP \perp AQ$.
7. In the square $ABCD$, E is the mid-point of AD, BD and CE intersect at F. Prove that $AF \perp BE$.
8. In figure, AD, BE are the altitudes of ΔABC with orthocentre H, which lies in the interior of the triangle. If $BH = AC$, Find $\angle B$.



9. Triangle ABC is a right triangle with $\angle A = 30°$ and $\angle C = 90°$. Segment DE is perpendicular to AC at D and $AD = CB$ as indicated in the figure.
   Find DE, if $DE + AC = 4$.



10. Each side of square $ABCD$ has length 1 unit. Points P and Q belong to AB and DA, respectively. Find $\angle PCQ$ if the perimeter of ΔAPQ is 2 units. The square is shown in the figure.



11. As shown in the figure, in ΔABC, D is the mid-point of BC, $\angle EDF = 90°$, DE intersects AB at E and DF intersects AC at F. Prove that $BE + CF > EF$ (see Figure 8.2).

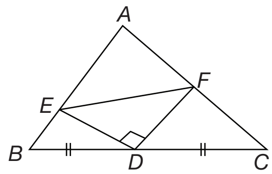
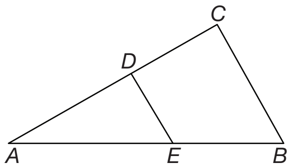


Figure 8.2

12. Given that ABC is an equilateral triangle of side 1, ΔBDC is isosceles with $DB = DC$ outward of ΔABC and $\angle BDC = 120°$. If points M and N are on AB and AC respectively such that $\angle MDN = 60°$, find the perimeter of ΔAMN.



13. In the equilateral ΔABC, the points D and E are on AC and AB respectively, such that BD and CE intersect at P, and the area of the quadrilateral $ADPE$ is equal to area of ΔBPC, find $\angle BPE$.

14. In the figure, ΔABD and BEC are both equilateral with A, B, C being collinear, M and N are midpoints of AE and CD respectively, AE intersects BD at G and CD intersects BE at H. Prove that (i) ΔMBN is equilateral, (ii) $GH \parallel AC$ (see Figure 8.4).

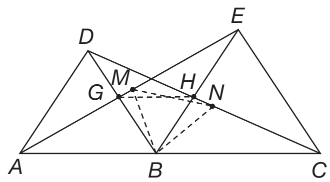


Figure 8.4

15. Squares $ABDE$ and $BCFG$ are drawn outside of triangle ABC. Prove that triangle ABC is isosceles if DG is parallel to AC. **[Leningrad MO, 1988]**

16. Given that ΔABC is right angled isosceles triangle with $\angle ACB = 90°$. D is the mid-point of BC, CE is perpendicular to AD, intersecting AB and AD at E and F respectively. Prove that $\angle CDF = \angle BDE$.

17. In an isosceles triangle ABC, $AB = BC$, $\angle B = 20°$. M, N are on AB and BC respectively such that $\angle MCA = 60°$, $\angle NAC = 50°$. Find $\angle NMC$ in degrees.

[Moscow MO, 1952]

18. Isosceles triangle ABC is shown in the figure. In that triangle, $\angle A = \angle B = 80°$ and cevian AM is drawn to side BC so that $CM = AB$. Find $\angle AMB$ (see Figure 8.5).

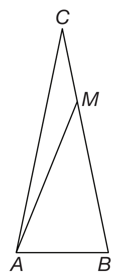


Figure 8.5

19. In ΔABC, $\angle ABC = \angle ACB = 80°$. The point P is on AB such that $\angle BPC = 30°$. Prove that $AP = BC$.

20. In ΔABC, $\angle C = 48°$. D is any point on BC, such that $\angle CAD = 18°$ and $AC = BD$. Find $\angle ABD$.

21. D is an inner point of an equilateral ΔABC satisfying $\angle ADC = 150°$. Prove that the triangle formed by taking the segments AD, BD, CD as its three sides is a right triangle. **[North Europe MO, 2003]**

22. In the isosceles right triangle ABC of the figure, $\angle A = 90°$ and $AB = AC$. Suppose that D is the interior point of the triangle, so that $\angle ABD = 30°$ and $AB = DB$. Prove that $AD = CD$ (see Figure 8.6).

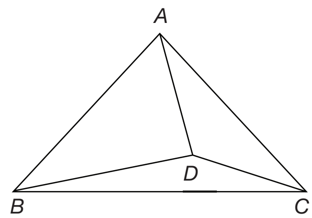
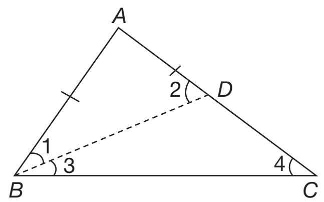


Figure 8.6

8.3 Triangle Inequality

8.3.1 Theorem 1



If two sides of a triangle are unequal, the longer sides has greater angle opposite to it.

Given: In ΔABC, $AC > AB$

To prove: $\angle B > \angle C$

Construction: Mark a point D on AC such that $AB = AD$. Join BD.

Proof: In ΔABD, $AB = AD$

$\Rightarrow \angle 1 = \angle 2$

In ΔBDC, by exterior angle property

$\angle 2 = \angle 3 + \angle 4$

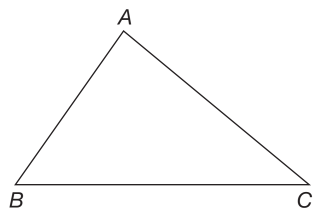
$\Rightarrow \angle 2 > \angle 4$

$\Rightarrow \angle 1 > \angle 4$ (As $\angle 1 = \angle 2$)

$\Rightarrow \angle 1 + \angle 3 > \angle 1 > \angle 4$

$\Rightarrow \angle ABC > \angle ACB$.

8.3.2 Theorem 2



(Converse of theorem 1) In a triangle, the greater angle has the longer side opposite to it.

Given a ΔABC in which $\angle ABC > \angle ACB$

To prove: $AC > AB$.

Proof: In ΔABC, we have the following three possibilities

(i) $AC = AB$ (ii) $AC < AB$ (iii) $AC > AB$

Out of these there are three possibilities among those exactly one must be true.

Case 1: When $AC = AB$

$\Rightarrow \angle B = \angle C$ (Angles opposite to equal sides are equal)

But it is given that $\angle B > \angle C$

$\therefore$ Which is a contradiction and hence $AC \neq AB$.

Case 2: When $AC < AB$

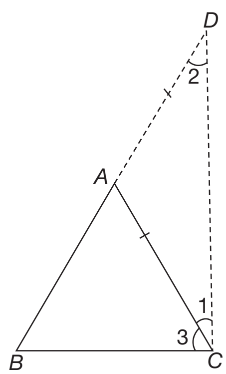
Then $\angle ABC < \angle ACB$ ($\because$ Longer side has the greater angle opposite to it)

But it is given that $\angle B > \angle C$ which is again a contradiction

Thus we are left with the only possibility

$AC > AB$ which must be true and hence $AC > AB$.

8.3.3 Theorem 3



The sum of any two sides of a triangle is greater than the third side.

Given: ΔABC

To prove: $AB + AC > BC$, $AB + BC > AC$ and $AC + BC > AB$

Construction: Produced side BA to D such that $AD = AC$. Join CD

Proof: In ΔACD,

$AC = AD$

$\therefore \angle 1 = \angle 2$

Also $\angle 1 + \angle 3 > \angle 1$

$\therefore \angle 1 + \angle 3 > \angle 2$

In ΔBCD, $\angle C > \angle D$

$\therefore$ In ΔBCD, $BD > BC$ (The side opposite to greater angle is longer)

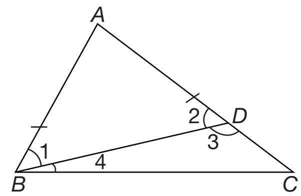
$BA + AD > BC$

$\Rightarrow BA + AC > BC$ (As $AD = AC$)

Similarly we can prove others.

Corollary: The difference of any two sides of a triangle is less than the third side.

Proof:



To prove: $AC - AB < BC$; $BC - AC < AB$; $BC - AB < AC$

Let if possible $AC > AB$

Take a point D on AC such that $AD = AB$, join BD.

Since $AD = AB$

So $\angle 1 = \angle 2$

Also $\angle 2 + \angle 3 = 180°$ and $\angle 1 + \angle 4 < 180°$

$\Rightarrow \angle 1 + \angle 4 < \angle 2 + \angle 3$

$\Rightarrow \angle 4 < \angle 3$ (As $\angle 1 < \angle 2$)

$\Rightarrow BC > CD = AC - AD$

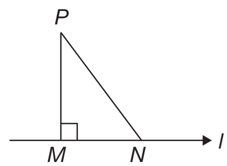
$BC > AC - AB$ (As $AD = AB$ by construction)

$\Rightarrow \quad AC - AB < BC$

Similarly we can prove others.

8.3.4 Theorem 4

Of all the line segments that can be drawn to a given line, from a point not lying on it the perpendicular line segment is the shortest.



Given: A straight line l and a point P not lying on l. $PM \perp l$ and N is any point on l other than M.

To prove: $PM < PN$

Proof: ΔPMN, $\angle M = 90°$

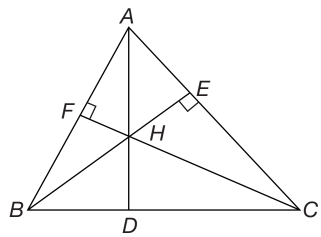
So, $\angle N < 90°$

$\Rightarrow \angle N < \angle M$

$\Rightarrow PM < PN$ (Side opposite to greater angle is larger)

Thus PM is the shortest of all line segments from P on line 'l'

Example 21 *Show that the sum of the three altitudes of a triangle is less than the sum of three sides of the triangle.*



Solution:

Given: In ΔABC, $AD \perp BC$, $BE \perp AC$, $CF \perp AB$

To prove: $AD + BE + CF < AB + BC + CA$

Proof: Since in a right angled triangle, hypotenuse is the longest side.

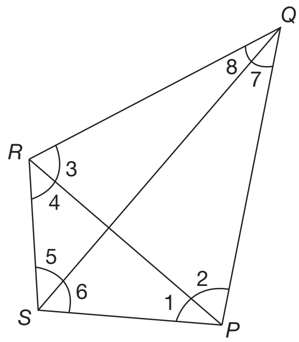
In ΔABD, $AB > AD$

In ΔBCE, $BC > BE$

In ΔACF, $AC > CF$

Adding all we get $AB + BC + CA > AD + BE + CF$.

Example 22 *In the figure PQRS is a quadrilateral. PQ is its longest side and RS is its shortest side. Prove that $\angle R > \angle P$ and $\angle S > \angle Q$.*



Solution:

Given: In quadrilateral $PQRS$, PQ is the longest side and RS is the shortest

To prove: (i) $\angle R > \angle P$ (ii) $\angle S > \angle Q$

Construction: Join PR and SQ

Proof:

(i) In ΔPQR, $PQ > QR$ (as PQ is the longest side)

$\Rightarrow \angle 3 > \angle 2$ (1)

In ΔPRS $RS < PS$ (as RS is the shortest side)

$\Rightarrow \angle 4 > \angle 1$ (2)

Adding Eqs. (1) and (2)

$\angle 3 + \angle 4 > \angle 1 + \angle 2$

$\Rightarrow \angle R > \angle P$

(ii) In ΔPQS, $PQ > PS$ (As PQ is the longest side)

$\Rightarrow \angle 6 > \angle 7$ (3)

In ΔQRS, $QR > RS$ (As RS is the shortest side)

$\Rightarrow \angle 5 > \angle 8$ (4)

Adding Eqs. (3) and (4) we get $\angle 5 + \angle 6 > \angle 7 + \angle 8$

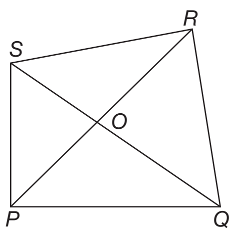
$\Rightarrow \angle S > \angle Q$.

Example 23 *In quadrilateral PQRS, diagonals intersect at O. Show that*

(i) $PQ + QR + RS + SP > PR + QS$

(ii) $PQ + QR + RS + SP < 2\,(PR + QS)$

Solution:



Proof: Since the sum of any two sides of a triangle is greater than the third side

(i) $\therefore$ In ΔPQR, $PQ + QR > PR$

In ΔQRS, $QR + RS > QS$

In ΔPRS, $RS + SP > PR$

In ΔPSQ, $PS + PQ > QS$

Adding all four we get, $2(PQ + QR + RS + SP) > 2(PR + QS)$

$\Rightarrow PQ + QR + RS + SP > PR + QS$

(ii) In ΔOPQ, $OP + OQ > PQ$

In ΔORQ, $OQ + OR > RQ$

In ΔORS, $OR + OS > RS$

In ΔOPS, $OS + OP > PS$

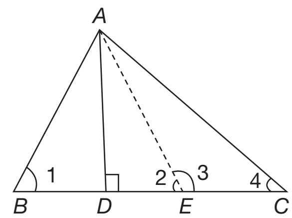
Adding all the above four inequality we get,

$2(OP + OR + OQ + OS) > PQ + QR + RS + SP$

$\Rightarrow 2(PR + QS) > PQ + QR + RS + SP$.

or $PQ + QR + RS + SP < 2(PR + QS)$

Example 24 *In ΔABC, $AD \perp BC$ if $DC > DB$ prove that $AC > AB$.*



Construction: Take a point E on DC such that $DB = DE$. Join AE.

Proof: since in ΔABE, $AB = AE$

$\therefore \angle 1 = \angle 2$

Now $\angle 2 + \angle 3 = 180°$ (Linear pair)

Also $\angle 1 + \angle 4 < 180°$

$\Rightarrow \angle 1 + \angle 4 < \angle 2 + \angle 3$

$\Rightarrow \angle 4 < \angle 3$ $(\because \angle 1 = \angle 2)$

In ΔAEC, $AC > AE$

$\Rightarrow AC > AB$ (As $AE = AB$).

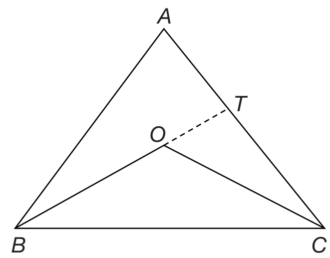
Example 25 *O is any point in the interior of $\angle ABC$. Prove that*

(i) $AB + AC > OB + OC$

(ii) $AB + BC + CA > OA + OB + OC$

(iii) $OA + OB + OC > \frac{1}{2}(AB + BC + CA)$

Solution:



Constructions: Produce BO to cut AC at T.

Proof: In $\angle ABT$, since sum of any two sides is greater than the third side

(i) $\therefore AB + AT > BT$

$\Rightarrow AB + AT > BO + OT$ (1)

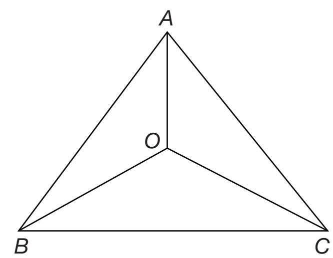
In ΔOTC,

$OT + TC > OC$ (2)

Adding Eqs. (1) and (2), $AB + AT + OT + TC > BO + OT + OC$

$\Rightarrow AB + AC > OB + OC$

(ii) Join OA



Since $AB + AC > OB + OC$ (3)

Similarly $AB + BC > OA + OC$ (4)

$AC + BC > OA + OB$ (5)

Adding Eqs. (3), (4) and (5) we get, $2(AB + BC + AC) > 2(OA + OB + OC)$

$\Rightarrow AB + BC + CA > OA + OB + OC$

(iii) Since in ΔOBC, $OB + OC > BC$ (6)

Also in ΔOAC, $OC + OA > AC$ (7)

And In ΔOAB, $OA + OB > AB$ (8)

Adding Eqs. (6), (7) and (8), we get $2(OA + OB + OC) > (AB + BC + AC)$

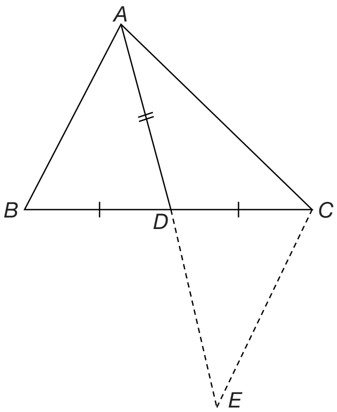
Note: (iii) is also true if O is any point in the plane of ΔABC but not (i).

Example 26 *Prove that any two sides of a triangle are together greater than twice the median drawn to the third side.*

Solution:

Given: In ΔABC, AD is a median

To prove: $AB + AC > 2AD$



Construction: Produce AD to E such that $AD = DE$, join CE.

Proof: In ΔADB and ΔEDC

$AD = ED$ (Construction)

$\angle ADB = \angle EDC$ (VOA)

And $BD = CD$ (Given)

$\therefore$ By SAS Congruency, $\Delta ADB \cong \Delta EDC$

$\therefore AB = EC$

In ΔACE,

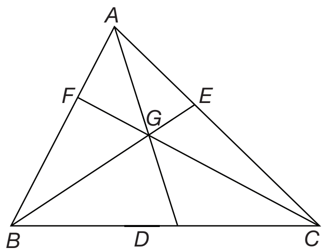
$AC + CE > AE$

$\Rightarrow AC + AB > AD + DE$ (As $CE = AB$)

$\Rightarrow AC + AB > 2AD$ (As $DE = AD$)

Example 27 *In ΔABC, If AD, BE, CF are the medians than prove that*

$$\frac{3}{4}(AB + BC + CA) < AD + BE + CF < AB + BC + CA$$



Solution:

Since by previous question

$AB + AC > 2AD$ (1)

Similarly

$AB + BC > 2BE$ (2)

$AC + BC > 2CF$ (3)

Adding Eqs. (1), (2) and (3), we get

$2(AB + BC + CA) > 2(AD + BE + CF)$

$\Rightarrow AB + BC + CA > AD + BE + CF$

Or $AD + BE + CF < AB + BC + CA$

Also in ΔGBC,

$GB + GC > BC$ (4)

Similarly, $GC + GA > AC$ (5)

And $GA + GB > AB$ (6)

Adding Eqs. (4), (5) and (6), we get

$2(GA + GB + GC) > AB + BC + CA$

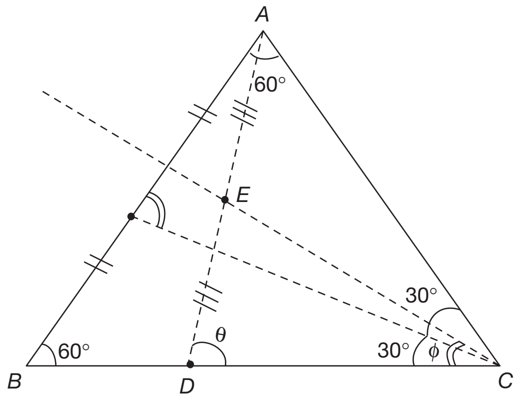
$\Rightarrow 2\left(\frac{2}{3}AD + \frac{2}{3}BE + \frac{2}{3}CF\right) > AB + BC + CA$ (As G being centroid of the triangle divides median in 2 : 1 ratio, See proof of it on pp. 8.27–8.28)

$\Rightarrow AD + BE + CF > \frac{3}{4}(AB + BC + CA)$

Thus $\frac{3}{4}(a + b + c) < m_a + m_b + m_c < a + b + c.$

Example 28 *Let ABC be an equilateral triangle. Let E be the mid-point of the segment AD, which is drawn through A to meet the side BC at D. Show that $AE < CE$.*

Solution:



Given: ΔABC is equilateral, $\angle A = \angle B = \angle C = 60°$; $AE = ED$

To prove: $AE < CE$ or $AE = ED < EC$ or $ED < EC$

Proof: $\angle B \le \theta < \pi - \angle C \Rightarrow 60° \le \theta < 180° - 60° = 120°$

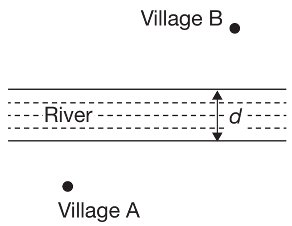
$\frac{\pi}{2} - \angle B < \phi < C \Rightarrow 30° \le \phi \le 60° \Rightarrow \phi < \theta \Rightarrow ED < EC$

$\Rightarrow AE < EC$ (As $ED = AE$)

Hence proved.

Build-up Your Understanding 3

1. Find the number of triangles with integral side lengths such that second largest side is 4 and only one side being largest.
2. Let each side of the triangle is a prime number and divisor of 2001. Find the number of such triangles.
3. Find the number of isosceles triangles with integral side lengths and having perimeter 144 and only one side being largest.
4. If a, b, c be the sides of a triangle prove that $\sqrt{a}, \sqrt{b}$, and $\sqrt{c}$ will also represents sides of a triangle.
5. Find a point P, inside a convex quadrilateral $ABCD$, such that $PA + PB + PC + PD$ is minimum.
6. Prove that in a convex quadrilateral $ABCD$,
   $\max\{AB + CD, AD + BC\} < AC + BD < AB + BC + CD + DA.$
   Also prove that, if $AB + BD \le AC + CD$, then $AB < AC$.
7. A line l is given in a plane and two points A and B are also in the same plane. Find P on the line such that $AP + PB$ is minimum. Give your answer in two cases separately A, B on same side of the line or on opposite side of the line.
8. A line l is given in a plane and two points A and B are also in the same plane such that AB not perpendicular to line l. Find P on the line such that $|AP - PB|$ is minimum. Give your answer in two cases separately A, B on same side of the line or opposite side of the line.
9. A line l is given in a plane and two points A and B are also in the same plane such that A and B are not at same distance from the line l. Find P on the line such that $|AP - PB|$ is maximum. Give your answer in two cases separately A, B on same side of the line or opposite side of the line.



10. Two villages A and B lie opposite side of a river whose banks are parallel lines. A bridge is to be built over the river, perpendicular to the bank. Where the bridge should be built so that the path from one village to other is as short as possible.

11. In an acute angle there is a fixed point A, locate points B and C, one on each arm of the angle such that perimeter of the ΔABC be minimum.

12. In the preceding problem if angle is 90°, then prove that perimeter of the ΔABC is always greater than twice the distance of A from the vertex of the right angle.

13. A line l is given in 3-D space and A and B are two fixed points in 3-D space. Find P on the line such that $AP + PB$ is minimum.

14. An ant sits on one vertex of a solid cube. Find the shortest path on the surface to reach opposite vertex.

15. An ant sits on the outside surface of a cylindrical drinking glass. There is a honey drop at some point on inside surface of the glass. Find the shortest possible length the ant must crawl to reach the point of honey drop.

16. An ant sits on the circumference of a right circular cone. Without changing its sense of motion about the axis of cone, it completes one round trip and reaches the starting point. Find the shortest possible path. The semi vertical angle of cone is less than 30°. Also discuss the case if semi vertical angle is more than 30°.

17. An ant sits at P, on the circumference of a right circular cone of semi-vertical angle such that $\theta < \sin^{-1}\left(\dfrac{1}{4}\right)$. Without changing its sense of motion about the axis of cone, it completes one round trip and reaches on the line OP where 'O' is the vertex of the cone. Find the shortest possible path.

18. P is a point inside the acute angle triangle ABC, prove that

$$\min\{PA, PB, PC\} + PA + PB + PC < AB + BC + CA$$

Evangelista Torricelli

15 Oct 1608–25 Oct 1647
Nationality: Italian

19. Let P be inside or on the triangle. Locate P such that $PA + PB + PC$ is maximum.

20. In a ΔABC with all angles smaller than 120°, locate a point P such that $PA + PB + PC$ is minimum.

Note: The point P is called **Torricelli's (or Fermat's) Point.**

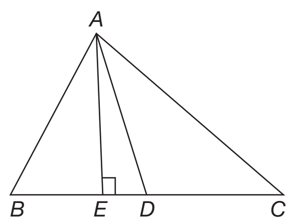
21. Let $ABCD$ and $PQRS$ be two convex quadrilaterals whose corresponding sides are equal. Prove that if $\angle A > \angle P$, then $\angle B < \angle Q$, $\angle C > \angle R$, and $\angle D < \angle S$.

22. The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the lengths of other three sides. Prove that two of the sides have the same length.

8.4 Ratio and Proportion Theorem (or Area Lemma)

If D is any point on the side BC of a triangle ABC then $[ABD] : [ADC] = BD : DC$.
Here $[XYZ]$ denotes area of ΔXYZ.

Construction: Draw $AE \perp BC$

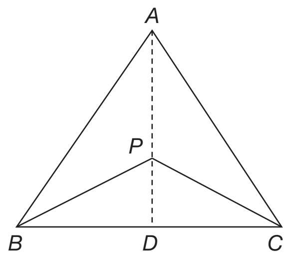


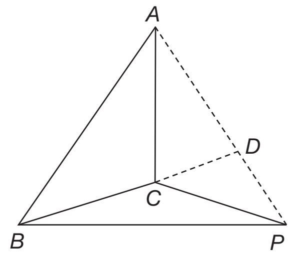
Proof: $\dfrac{[ABD]}{[ADC]} = \dfrac{\frac{1}{2} \times BD \times AE}{\frac{1}{2} \times DC \times AE}$

$$\therefore \quad \frac{[ABD]}{[ABC]} = \frac{BD}{DC}$$

Corollary: Let ΔABC be a triangle, $D \in BC$ (internally or externally) and $P \in AD$.

Then $\dfrac{DP}{DA} = \dfrac{[BCP]}{[BCA]}$.





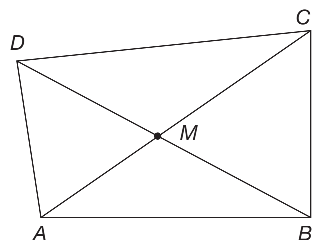
In other words, common base of two triangles, divides the line joining their third vertex, in the ratio of their areas.

Example 29 *ABCD is any quadrilateral. Diagonals AC and BD intersects at M. Prove that* $[AMD] \times [BMC] = [DMC] \times [AMB]$.

Solution: By Ratio proportion theorem

$$\frac{[AMD]}{[AMB]} = \frac{DM}{MB} \tag{1}$$

Also, $$\frac{[DMC]}{[AMC]} = \frac{DM}{MB} \tag{2}$$



Equating Eqs. (1) and (2)

$$\frac{[AMD]}{[AMB]} = \frac{[DMC]}{[BMC]}$$

$\Rightarrow [AMD] \times [BMC] = [AMB] \times [DMC]$.

Example 30 *D, E, F are points on the sides BC, CA, AB respectively of* ΔABC, *such that AD, BE, CF are concurrent at P, show that*

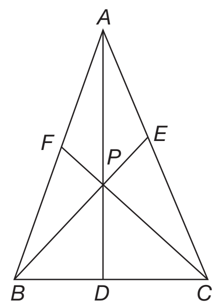
(i) $\dfrac{PD}{AD} + \dfrac{PE}{BE} + \dfrac{PF}{CF} = 1$

(ii) $\dfrac{AP}{AD} + \dfrac{BP}{BE} + \dfrac{CP}{CF} = 2$

(iii) $\dfrac{AP}{PD} = \dfrac{AF}{FB} + \dfrac{AE}{EC}$

Solution:

(i) Let $[BPC] = \Delta_1$, $[APC] = \Delta_2$, $[APB] = \Delta_3$ and $[ABC] = \Delta$



In ΔABD, $\dfrac{PD}{AD} = \dfrac{[BPD]}{[BAD]}$ also $\dfrac{PD}{AD} = \dfrac{[PDC]}{[CAD]}$

$$\therefore \quad \frac{PD}{AD} = \frac{[BPD]}{[BAD]} = \frac{[PDC]}{[CAD]} = \frac{[BPD]+[PDC]}{[BAD]+[CAD]} = \frac{[BPC]}{[ABC]}$$

$$\therefore \quad \frac{PD}{AD} = \frac{\Delta_1}{\Delta}$$

Similarly $\dfrac{PE}{BE}=\dfrac{\Delta_2}{\Delta}$ and $\dfrac{PF}{CF}=\dfrac{\Delta_3}{\Delta}$

$$\therefore \quad \frac{PD}{AD}+\frac{PE}{BE}+\frac{PF}{CF}=\frac{\Delta_1}{\Delta}+\frac{\Delta_2}{\Delta}+\frac{\Delta_3}{\Delta}=\frac{\Delta_1+\Delta_2+\Delta_3}{\Delta}=\frac{\Delta}{\Delta}=1$$

$$\Rightarrow \frac{PD}{AD}+\frac{PE}{BE}+\frac{PF}{CF}=1$$

(ii) Now, $\dfrac{AP}{AD}=\dfrac{[APB]}{[ADB]}=\dfrac{[APC]}{[ADC]}=\dfrac{[APB]+[APC]}{[ADB]+[ADC]}$

$$\Rightarrow \frac{AP}{AD}=\frac{[APB]+[APC]}{[ABC]}=\frac{\Delta_3+\Delta_2}{\Delta}$$

Similarly $\dfrac{BP}{BE}=\dfrac{\Delta_1+\Delta_3}{\Delta}$ and $\dfrac{CP}{CF}=\dfrac{\Delta_1+\Delta_2}{\Delta}$

$$\therefore \quad \frac{AP}{AD}+\frac{BP}{BE}+\frac{CP}{CF}=\frac{\Delta_3+\Delta_2}{\Delta}+\frac{\Delta_1+\Delta_3}{\Delta}+\frac{\Delta_2+\Delta_1}{\Delta}=\frac{2(\Delta_1+\Delta_2+\Delta_3)}{\Delta}=2\frac{\Delta}{\Delta}=2$$

$$\Rightarrow \frac{AP}{AD}+\frac{BP}{BE}+\frac{CP}{CF}=2$$

(iii) Since $\dfrac{AP}{PD}=\dfrac{[APB]}{[BPD]}=\dfrac{[APC]}{[PDC]}=\dfrac{[APB]+[APC]}{[BPD]+[PDC]}$

$$\frac{AP}{PD}=\frac{[APB]+[APC]}{[BPC]}=\frac{[APB]}{[BPC]}+\frac{[APC]}{[BPC]}$$

$$\Rightarrow \frac{AP}{PD}=\frac{\Delta_3}{\Delta_1}+\frac{\Delta_2}{\Delta_1} \qquad (1)$$

Also $\dfrac{AE}{EC}=\dfrac{[ABE]}{[CBE]}=\dfrac{[APE]}{[CPE]}=\dfrac{[ABE]-[APE]}{[CBE]-[CPE]}=\dfrac{[APB]}{[CBP]}=\dfrac{\Delta_3}{\Delta_1}$

Similarly $\dfrac{AF}{FB}=\dfrac{[AFC]}{[BFC]}=\dfrac{[AFP]}{[BFP]}=\dfrac{[AFC]-[AFP]}{[BFC]-[BFP]}=\dfrac{[APC]}{[BPC]}=\dfrac{\Delta_2}{\Delta_1}$

$$\Rightarrow \frac{\Delta_3}{\Delta_1}+\frac{\Delta_2}{\Delta_1}=\frac{AE}{EC}+\frac{AF}{FB} \qquad (2)$$

$$\Rightarrow \quad \frac{AP}{PD}=\frac{AE}{EC}+\frac{AF}{FB} \text{ (From Eqs. (1) and (2))}$$

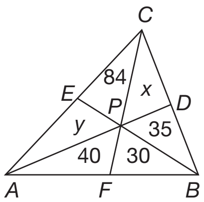
Note: Result (iii) is known as ***van Aubel's theorem***.

Henricus Hubertus van Aubel

20 Nov 1830–3 Feb 1906
Nationality: Belgian

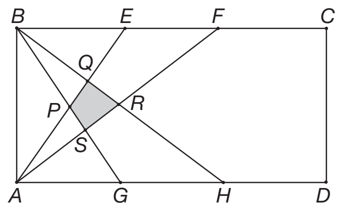
Build-up Your Understanding 4

1. Let ABC be a triangle and D, E are points on the segment BC, CA respectively, such that $AE = \lambda AC$ and $BD = \mu BC$. Let AD, BE intersects at F. Find, in terms of λ and μ, the ratio $AF : FD$.
2. In ΔABC, $AB = AC = 115$, $AD = 38$, and $CF = 77$ where D lies on AB and F lies on AC produced. DF intersects BC at E. Compute $\dfrac{[CEF]}{[DBE]}$.
3. As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of these triangles are as indicated. Find the area of the triangle ABC. **[AIME, 1985]**



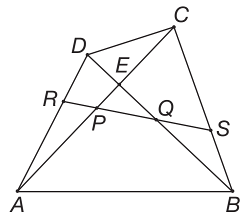
4. In ΔABC, E, F, G are points on AB, BC, CA respectively such that $AE : EB = BF : FC = CG : GA = 1 : 3$. K, L, M are the intersection points of the lines AF and CE, BG and AF, CE and BG, respectively. Suppose the area of ΔABC is 1; find the area of ΔKLM.

5. Suppose P, Q are two points on the same side of the line AB. R is a point on the segment PQ such that $PR = \lambda PQ$. Prove that $[ABR] = (1 - \lambda)\,[ABP] + \lambda[ABQ]$.



6. In rectangle $ABCD$, G and H are trisection points of AD, and E and F are trisection points of BC. If $AB = 360$ and $BC = 450$, compute the area of $PQRS$.

7. Let D, E, F be points on the sides BC, CA, AB respectively such that $\dfrac{BD}{DC} = \dfrac{CE}{EA} = \dfrac{AF}{FB} = \dfrac{m}{n}$. Prove that if AD, BE, and CF are joined, then they will form a triangle by their intersections, whose area is to that of the triangle ABC as $(m-n)^2 = m^2 + mn + n^2$.

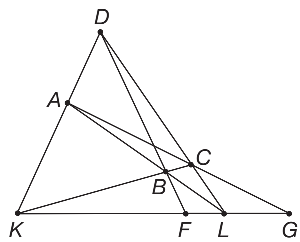


8. In the figure $ABCD$ is a convex quadrilateral. AC and BD intersect at E. P, Q are the mid-points of AC and BD respectively. Given that $AE = \lambda AC$ and $BE = \mu BD$.

(i) Find the ratios $AR : RD$ and $BS : SC$ (in terms of λ and μ).

(ii) Suppose the area of $ABCD$ is 1. What is the area of $ABSR$?

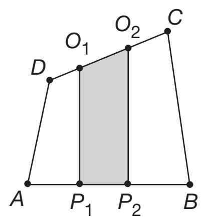
9. Given non-collinear points A, B, C, segment BA is trisected by points D and E, and F is the mid-point of segment AC. DF and BF intersect CE at G and H, respectively. If $[EDG] = 18$, compute $[FGH]$.



10. In the figure there is a convex quadrilateral $ABCD$. The lines DA and CB intersect at K, the lines AB and DC intersect at L, the lines AC and KL intersect at G, the lines DB and KL intersect at F. Prove that $\dfrac{KF}{FL} = \dfrac{KG}{GL}$.

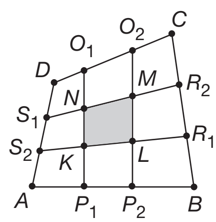
11. A given convex pentagon $ABCDE$ has the property that the area of each of the five triangles ABC, BCD, CDE, DEA, and EAB is unity. Show that all pentagons with the above property have same area, and calculate the area. Show, further that there are infinitely many non-congruent pentagons having the above property.

[USA MO, 1972]

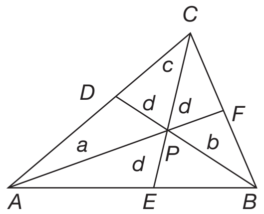


12. Given a convex quadrilateral $ABCD$. Let P_1, P_2 be the trisection points of the segment AB and Q_1, Q_2 be the trisection points of the segment CD as shown in the figure. Prove that $\dfrac{[P_1P_2Q_2Q_1]}{[ABCD]} = \dfrac{1}{3}$.

In the adjacent figure, we trisect BC, DA by the points R_1, R_2, S_1, S_2. Prove that $\dfrac{[KLMN]}{[ABCD]} = \dfrac{1}{9}$.



13. In trapezoid $ABCD$ with bases AB and CD, $AB = 14$ and $CD = 6$. Points E and F lie on AB, such that $AD \parallel CE$ and $BC \parallel DF$. Segments DF and CE intersect at G, and AG intersects BC at H. Compute $\dfrac{[CGH]}{[ABCD]}$.



14. Let P be an interior point of the triangle ABC and extend lines from the vertices through P to the opposite side. Let a, b, c and d denote the lengths of segments indicated in the figure. Find the product abc if $a + b + c = 43$ and $d = 3$.

[AIME, 1988]

15. Let P be an interior point of ΔABC. Let BP, CP meet AC, AB in E and F respectively. If $[BPF] = 4$, $[BPC] = 8$ and $[CPE] = 1$, find $[AFPE]$.

16. If S is the circumcentre of ΔABC, AS meets BC at M, BS meets CA at N and CS meets AB at P, prove that, $\dfrac{1}{AM} + \dfrac{1}{BN} + \dfrac{1}{CP} = \dfrac{2}{R}$, where R is the circumradius of the triangle.

17. P is in the interior of ΔABC. The lines AP, BP, CP meet the opposite sides BC, CA, AB in D, E, and F respectively.

(i) Prove that, $\dfrac{AP}{PD} + \dfrac{BP}{PE} + \dfrac{CP}{PF} \geq 6$.

(ii) When does the equality hold?

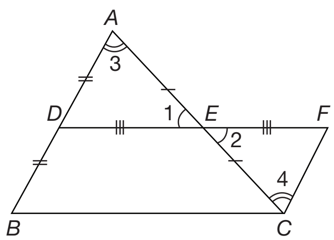
8.5 Mid-point Theorem

The line segment joining the mid-points of any two sides of a triangle is parallel to the third side and is equal to half of it.

Given: In ΔABC, D, E are the mid-points of AB and AC respectively

To prove: $DE \parallel$ BC and $DE = \dfrac{1}{2} BC$

Construction: Produce DE to F such that $DE = EF$. Join CF.



Proof: In ΔAED and ΔCEF,

$AE = CE$ (Given)

$\angle 1 = \angle 2$ (VOA)

$ED = EF$ (Construction)

$\therefore$ By SAS congruence $\Delta AED \cong \Delta CEF$

$\therefore$ $AD = CF$

but $AD = BD$

$\therefore$ $BD = CF$

Also $\angle 3 = \angle 4$ (CPCT)

$\Rightarrow AB \parallel CF$

In quadrilateral $BDFC$,

$BD = CF$ and $BD \parallel CF$

Since in a quadrilateral if one pair of opposite side is equal and parallel then it is a parallelogram.

$\therefore$ $BCFD$ is a parallelogram

$\therefore$ $DF = BC$ and $DF \parallel BC$

$\therefore$ $DE + EF = BC$ and $DE \parallel BC$

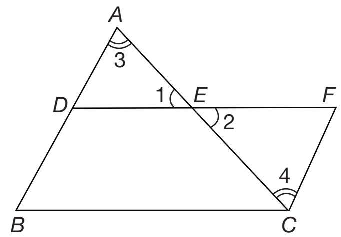
$\Rightarrow 2DE = BC$

$\Rightarrow DE = \dfrac{1}{2} BC$, $DE \parallel BC$.

8.5.1 Converse of Mid-point Theorem

The line drawn through the mid-point of one side of a triangle parallel to another side, bisects the third side.

Given: In ΔABC, D is mid-point of AB. $DE \parallel BC$
To prove: $AE = EC$
Construction: Draw $CF \parallel BA$
Which cuts DE *produced* at F.



Proof: Since $DF \parallel BC$ and $BD \parallel CF$
$\therefore$ $BDFC$ is a parallelogram

$$BD = CF$$
But $$BD = AD$$
$$\therefore AD = CF$$

In ΔEAD and ΔECF

$$\angle 1 = \angle 2 \quad \text{(VOA)}$$
$$\angle 3 = \angle 4 \quad \text{(Alternate interior angles)}$$
$$AD = CF$$

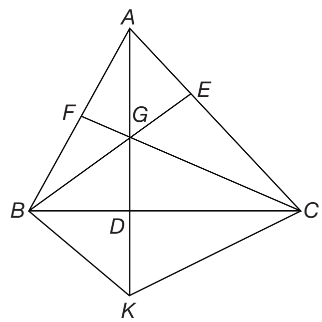
$\therefore$ By AAS congruence ΔEAD $\cong \Delta ECF$

$$AE = CE \quad \text{(CPCT)}$$

Hence proved.
Also $DE = FE$ (CPCT)
As $DF = BC$
$\Rightarrow DE + EF = BC$
$\Rightarrow 2DE = BC$
$\Rightarrow DE = \frac{1}{2}BC.$

Example 31 *Prove that in a triangle all the medians are concurrent and their point of intersection, i.e., centroid divides the median in the ratio* 2:1.

Solution:



Given: In ΔABC
Let BE, CF are the medians and let they intersect at G. Join AG and produce it to cut BC at D.
To prove: (i) $BD = DC$

(ii) $\dfrac{AG}{GD} = \dfrac{2}{1} = \dfrac{BG}{GE} = \dfrac{CG}{GF}$

Construction: Produce AD to K such that $AG = GK$.
Proof: In ΔABK,
F, G are the mid-points of AB and AK respectively.
$\therefore$ By mid-point theorem,

$$FG \parallel BK \text{ and } FG = \frac{1}{2}BK \quad (1)$$

$\Rightarrow GC \parallel BK$
In ΔAKC,
G, E are the mid-points of AK and AC respectively.
$\therefore$ By mid-point theorem,

$$GE \parallel KC,\ GE = \frac{1}{2}KC \quad (2)$$

$\Rightarrow BG \parallel KC$
Since in a quadrilateral $BGCK$
$BG \parallel KC$ and $GC \parallel BK$
$\therefore$ $BGCK$ is a parallelogram.

And in a parallelogram diagonals bisects each other

$\therefore$ $BD = DC$ and hence AD is a median

Also GD $= DK = x$

$\therefore$ $AG = GK = GD + DK = 2x$

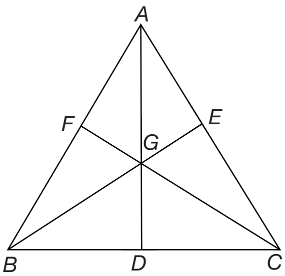
$$\therefore \quad \frac{AG}{GD} = \frac{2x}{x} = \frac{2}{1}$$

Also $GE = \frac{1}{2}KC = \frac{1}{2}BG$

$$\therefore \quad \frac{BG}{GE} = \frac{2}{1}$$

And $GF = \frac{1}{2}KB = \frac{1}{2}CG$

$$\therefore \quad \frac{CG}{GF} = \frac{2}{1}.$$



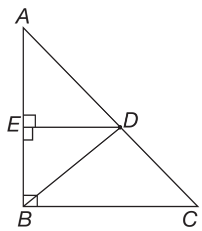
Note: In ΔABC, the mid-points of the sides BC, CA and AB are D, E and F respectively. The lines AD, BE and CF are called medians of the triangle ABC, the points of concurrency of three medians is called centroid and usually denoted by G.

$AG = \frac{2}{3}AD$; $BG = \frac{2}{3}BE$; $CG = \frac{2}{3}CF$ median of a triangle divides the triangle into two parts of equal areas

In adjacent diagram, area of all six triangles are equal, *i.e.*,

$$[BGD] = [CGD] = [CGE] = [AGE] = [AFG] = [BFG] = \frac{1}{6}[ABC]$$

Example 32 *Prove that the mid-point of the hypotenuse of a right angled triangle is equidistant from all its vertices.*



Solution: Given In ΔABC, $\angle B = 90°$, $AD = DC$

To prove: $BD = \frac{1}{2}AC$

Construction: Draw $DE \parallel CB$

Proof: In ΔABC, D is a mid-point of AC and $DE \parallel CB$

$\therefore$ By converse of mid-point theorem E is a mid-point of AB, *i.e.*, $AE = EB$

also $\angle E = 90°$ $\quad\therefore$ $DE \perp AB$

In ΔAED and ΔBED

$AE = BE$ (Proved above)

$\angle AED = \angle BED = 90°$

$ED = ED$ (Common)

$\therefore$By SAS congruence $\Delta AED \cong \Delta BED$

$\therefore$ $AD = BD$

but $AD = CD$

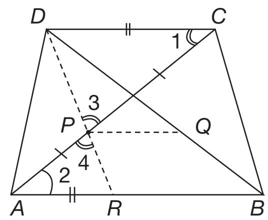
$$\therefore \quad BD = AD = CD = \frac{1}{2}AC.$$

Example 33 *Prove that the line segment joining the mid-points of the diagonals of a trapezium is parallel to each of the parallel sides and is equal to half the difference of these sides.*

Solution: Given In trapezium $ABCD$, $AB \parallel CD$, P and Q are the mid-points of diagonal AC and BD respectively

To prove: $PQ \parallel AB \parallel DC$ and $PQ = \frac{1}{2}(AB - DC)$

Construction: Join DP and produce it to cut AB at R.



Proof: In ΔCPD and ΔAPR

$\angle 1 = \angle 2$ (Alternate interior angles)

$CP = AP$ (As P is the mid-point of AC)

$\angle 3 = \angle 4$ (VOA)

$\therefore$ By ASA congruence $\Delta CPD \cong \Delta APR$

$\therefore$ $CD = AR$ and $DP = RP$

In ΔDRB

P and Q are the mid-points of DR and DB respectively

$PQ \parallel RB$ and $PQ = \frac{1}{2} RB$

$\Rightarrow PQ \parallel AB \parallel DC$ and $PQ = \frac{1}{2}(AB - AR)$ (As $RB = AB - AR$)

$\Rightarrow PQ = \frac{1}{2}(AB - CD)$. (As $AR = CD$)

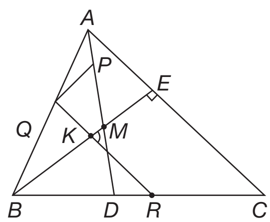
Example 34 *In the figure $BE \perp AC$. AD is any line from A to BC intersecting BE in H. P, Q and R are respectively the mid-points of AH, AB and BC. Prove that $\angle PQR = 90°$.*

Solution:

Given: In ΔABC, $BE \perp AC$. Q, R are the mid-points of AB, BC respectively AD is any line which cuts BE at H. P is a mid-point of AH.

To prove: $\angle PQR = 90°$

Construction: Join QR which cuts BE at K



Proof: Since In ΔABC, Q, R are the mid-points of AB, BC respectively.

$\therefore$ By mid-point theorem $QR \parallel AC$,

also, $\angle BEC = 90°$

$\therefore$ $\angle BKR = 90° = \angle HKR$

In ΔABH, Q and P are the mid-points of AB and AH respectively

$\therefore$ By mid-point theorem

$QP \parallel BH$

$\therefore$ $\angle PQR = \angle HKR = 90°$ (Corresponding angles)

$PQ \perp QR$.

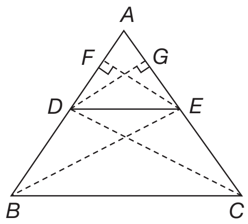
8.6 Basic Proportionality Theorem (Thales' Theorem)

Thales of Miletus

c. 624 BC–c. 546 BC
Nationality: Greek

If a line is drawn parallel to one side of a triangle intersecting the other two sides, at distinct points, then it divides the other two sides in the same ratio.

Given: In ΔABC, $DE \parallel BC$



To prove: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Construction: Draw $EF \perp AD$, and $DG \perp AE$. Join BE and CD

Proof: $\dfrac{[ADE]}{[BDE]} = \dfrac{AD}{DB}$ (1)

Also $\dfrac{[AED]}{[CED]} = \dfrac{AE}{EC}$ (2)

Since $DE \parallel BC$

$\therefore$ Triangles having same base and between the same parallel are equal in area

$\therefore$ $[BDE] = [CED]$

$\Rightarrow$ $\dfrac{[ADE]}{[BDE]} = \dfrac{[ADE]}{[CDE]}$ (3)

$\therefore$ From Eqs. (1), (2) and (3) we get $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.

Corollary: If in a triangle ABC, $DE \parallel BC$ intersects AB in D and AC in E, then

(i) $\dfrac{AB}{AD} = \dfrac{AC}{AE}$ (ii) $\dfrac{AB}{DB} = \dfrac{AC}{EC}$

(i) Since $\dfrac{AD}{DB} = \dfrac{AE}{EC}$ (by BPT)

$$\Rightarrow \frac{DB}{AD} = \frac{EC}{AE}$$

$$\Rightarrow 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$$

$$\Rightarrow \frac{AD + DB}{AD} = \frac{AE + EC}{AE}$$

$$\Rightarrow \frac{AB}{AD} = \frac{AC}{AE}$$

(ii) Again using $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Adding 1 to both sides

$$\frac{AD}{DB} + 1 = \frac{AE}{EC} + 1$$

$$\Rightarrow \frac{AD + DB}{DB} = \frac{AE + EC}{EC}$$

$$\frac{AB}{DB} = \frac{AC}{EC}.$$

Note: In ΔABC, if $DE \parallel BC$, we have

(i) $\dfrac{AD}{DB} = \dfrac{AE}{EC}$ (ii) $\dfrac{DB}{AD} = \dfrac{EC}{AE}$ (iii) $\dfrac{AB}{AD} = \dfrac{AC}{AE}$

(iv) $\dfrac{AD}{AB} = \dfrac{AE}{AC}$ (v) $\dfrac{AB}{DB} = \dfrac{AC}{EC}$ (vi) $\dfrac{DB}{AB} = \dfrac{EC}{AC}$

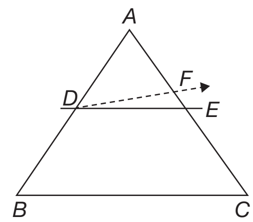
8.6.1 Converse of Basic Proportionality Theorem

If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side.

Given: In ΔABC

$$\frac{AD}{DB} = \frac{AE}{EC} \qquad (1)$$

To prove: $DE \parallel BC$



Proof: Let if possible

$DE \nparallel BC$

Let $DF \parallel BC$

Then by BPT in ΔABC

$$\frac{AD}{DB} = \frac{AF}{FC} \qquad (2)$$

$\therefore$ From Eqs. (1) and (2)

$$\frac{AE}{EC} = \frac{AF}{FC}$$

Adding 1 to both sides

$$\frac{AE}{EC} + 1 = \frac{AF}{FC} + 1$$

$$\frac{AE + EC}{EC} = \frac{AF + FC}{FC}$$

$$\frac{AC}{EC} = \frac{AC}{FC}$$

$$\Rightarrow \quad \frac{1}{EC} = \frac{1}{FC} \quad \Rightarrow \quad EC = FC$$

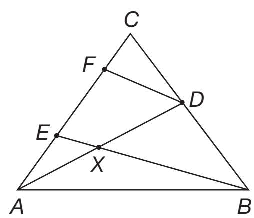
This is possible only if E and F coincides and thus $DE \parallel BC$.

Example 35 *In a triangle ABC, points D and E respectively divide the sides BC and CA in the ratio* $\frac{BD}{DC} = m$, *and* $\frac{AE}{EC} = n$. *The segments AD and BE intersect in a point X. Find the ratio* $\frac{AX}{XD}$.

Solution:

Given: In ΔABC, $\frac{BD}{DC} = \frac{m}{1}$; $\frac{AE}{EC} = \frac{n}{1}$ and AD, BE intersect at X.

To find: $\frac{AX}{XD}$.



Construction: Draw $DF \parallel BE$.

Proof: Since $DF \parallel BE$.

In ΔCEB

$$\therefore \text{ By BPT, } \frac{EF}{FC} = \frac{BD}{DC} = \frac{m}{1}$$

$$\Rightarrow \frac{EF}{EC} = \frac{m}{m+1}.$$

In ΔADF, $EX \parallel FD$

$\therefore$ By BPT

$$\frac{AX}{XD} = \frac{AE}{EF} = \frac{AE}{EC} \cdot \frac{EC}{EF} = \frac{n}{1} \cdot \frac{(m+1)}{m}$$

$$\therefore \quad \frac{AX}{XD} = \frac{n(m+1)}{m}.$$

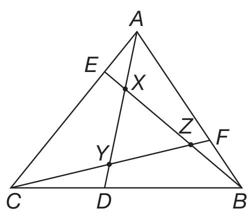
Note: $\dfrac{AX}{XD} = \dfrac{AE}{EC} \cdot \dfrac{BC}{BD}$ or $\dfrac{\frac{AE}{EC}}{\frac{BD}{BC}}$.

Example 36 *On the sides BC, CA, AB of ΔABC, points D, E, F are taken in such a way that* $\dfrac{BD}{DC} = \dfrac{CE}{EA} = \dfrac{AF}{FB} = \dfrac{2}{1}$. *Show that the area of the triangle determined by the lines AD, BE, CF is* $\dfrac{1}{7}$ *th of area of ΔABC.*

Solution:

Given: In ΔABC,

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = \frac{2}{1}$$



To prove: $[XYZ] = \dfrac{1}{7}[ABC]$

By previous question

$$\frac{AX}{XD} = \frac{\frac{AE}{EC}}{\frac{BD}{BC}} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$$

$$\therefore \quad \frac{AX}{AD} = \frac{3}{7}$$

Also, $\dfrac{[ABD]}{[ABC]} = \dfrac{BD}{BC} = \dfrac{2}{3}$

$$\therefore \quad [ABD] = \frac{2}{3}[ABC]$$

Now $\dfrac{[ABX]}{[ABD]} = \dfrac{AX}{AD} = \dfrac{3}{7}$

$$[ABX] = \frac{3}{7}[ABD] = \frac{3}{7} \times \frac{2}{3}[ABC]$$

$$\therefore \quad [ABX] = \frac{2}{7}[ABC]$$

Similarly $[BCZ] = \dfrac{2}{7}[ABC]$

$$[ACY]=\frac{2}{7}[ABC]$$

Thus $[XYZ] = [ABC] - ([ABX] + [BCZ] + [ACY])$

$$=\left(1-\left(\frac{2}{7}+\frac{2}{7}+\frac{2}{7}\right)\right)[ABC]$$

$$=\frac{1}{7}[ABC]$$

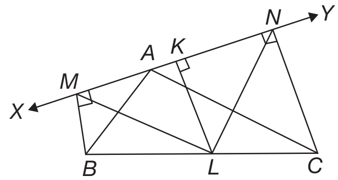
Aliter: See alternate of it in example 82 on Page 8.84.

Example 37 *In ΔABC, BM and CN are perpendiculars from B and C respectively on any line passing through A. If L is the mid-point of BC prove that ML = NL.*

Solution:

Given: ΔABC, XAY is any line passes through A. $BM \perp XY$ and $CN \perp XY$. And $BL = CL$, L is mid-point of BC.

To prove: $LM = LN$



Construction: Draw $LK \perp XAY$

Proof: Since perpendiculars drawn on the same line are parallel to each other

$\therefore BM \parallel LK \parallel CN$

Also by proportional intercept property

$$\frac{BL}{LC}=\frac{MK}{KN}$$

$$1=\frac{MK}{KN} \quad [\because \quad BL = LC]$$

$\Rightarrow MK = KN$

In ΔMKL and ΔNKL

$MK = NK$

$\angle MKL = \angle NKL = 90°$

$KL = KL$ (Common)

$\therefore$ By SAS congruence, ΔMKL $\cong \Delta NKL$

$\Rightarrow LM = LN.$ (CPCT)

Example 38 *Inscribe a square in a given triangle, so that, one side of the square may lie along a side of the triangle and the other two vertices lie on the other two sides (one in each) of the triangle. Justify your construction.*

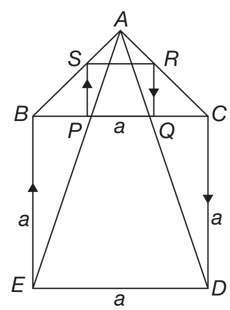
Solution: Let ABC be the triangle in which a square is to be inscribed as desired.

Construct a square $BCDE$ on the opposite side of $\angle A$.

Join AE and AD to cut BC at P and Q respectively.

Erect perpendiculars at P and Q to cut AB at S and AC at R, join SR.

Then PQRS is the square inscribed in ΔABC as desired.



Proof: $PQRS$ is a right angled trapezium (or right trapezoid) by construction and by application of Thales' Theorem we will show that $PQ = PS = QR$ to prove $PQRS$ is a square.

Consider $\Delta\, AED$, where $PQ \parallel ED$.

$$\therefore \quad \frac{AP}{AE} = \frac{PQ}{ED} \tag{1}$$

$$PQ = \left(\frac{AP}{AE}\right)a \text{ (As } PQ = a) \tag{2}$$

Consider ΔAEB, where $PS \parallel BE$.

Here $\dfrac{PS}{BE} = \dfrac{AP}{AE}$

$$\Rightarrow \quad PS = \left(\frac{AP}{AE}\right)BE = \left(\frac{AP}{AE}\right)a \tag{3}$$

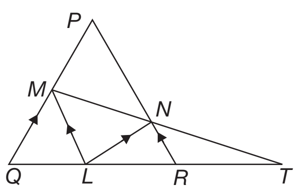
Consider ΔACD, where $QR \parallel CD$.

Here $\dfrac{QR}{CD} = \dfrac{AQ}{AD} = \dfrac{AP}{AE}$ (From Eq. (1))

$$\Rightarrow \quad RQ = \left(\frac{AP}{AE}\right)a \tag{4}$$

From Eqs. (2), (3) and (4), we see $PQ = PS = RQ \Rightarrow PQRS$ is a square.

Example 39 *L is a point on the side QR of ΔPQR. LM, LN are drawn parallel to PR and QP meeting QP, PR at M and N respectively. MN produced meets QR produced in T. Prove that LT is the geometric mean between RT and QT.*



Solution:

In ΔMLT, $NR \parallel ML$

$$\therefore \frac{TR}{TL} = \frac{TN}{TM} \quad \text{(BPT)} \tag{1}$$

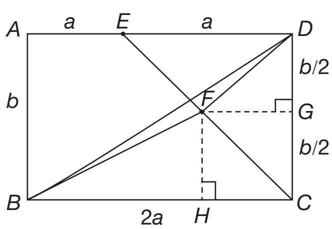
In ΔTQM,

$$\frac{TL}{TQ} = \frac{TN}{TM} \quad \text{(BPT)} \tag{2}$$

By equating Eqs. (1) and (2) we get, $\dfrac{TR}{TL} = \dfrac{TL}{TQ} \quad \Rightarrow \quad TL^2 = TR \cdot TQ$

That is, TL is the geometric mean between TR and TQ.

Example 40 *ABCD is a rectangle, E is the mid-point of AD. F is the mid-point of EC.* $[ABCD] = 120$ *cm*^{2}*; find* $[BDF]$.



Solution:

Construction: Draw perpendicular from F to CD and BC to meet them at G and H respectively.

Let the sides of the rectangle have lengths $2a$ and b. Now [ABCD] = 120

That is, $(2a) \cdot (b) = 120 \Rightarrow ab = 60$ (1)

Because of Thales theorem, $FG \parallel ED$ in ΔCED and F being the midpoint of CE, G will be the midpoint of DC; also $CG = GD = \dfrac{b}{2}$

Now $[BDF]=[BDC]-[DFC]-[FBC]$

$$=60-\left(\frac{1}{2}\times b\times\frac{a}{2}\right)-\left(\frac{1}{2}\times 2a\times\frac{b}{2}\right)$$

$$\therefore [BDF]=60-\frac{1}{4}ab-\frac{1}{2}ab=60-\frac{60}{4}-\frac{60}{2}=15\text{ cm}^2.$$

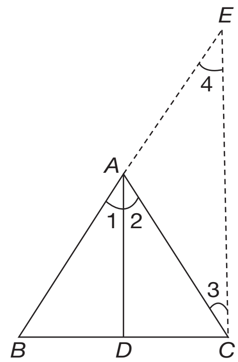
8.6.2 Internal Angle Bisector Theorem

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle

Given: In ΔABC, AD bisects $\angle BAC$

To prove: $\dfrac{AB}{AC}=\dfrac{BD}{DC}$

Construction: Draw $CE \parallel DA$ which cuts BA produced at E.



Proof: Since $AD \parallel EC$

$\therefore \angle 1=\angle 4$ (Corresponding angles)

$\angle 2=\angle 3$ (Alternate interior angles)

But $\angle 1=\angle 2$ (Given)

$\Rightarrow \angle 3=\angle 4 \Rightarrow AC=AE$

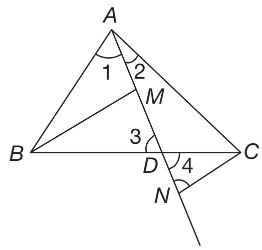
In ΔBCE, $AD \parallel EC$,

$\therefore$ By BPT

$$\frac{BA}{AE}=\frac{BD}{DC}$$

$$\Rightarrow \frac{AB}{AC}=\frac{BD}{DC}. \quad \text{(As } AE=AC\text{)}$$

Aliter:



Construction: Draw $BM \perp AD$

$CN \perp AD$ (AD produced)

In ΔAMB and ΔANC

$\angle 1=\angle 2$ (Given)

$\angle AMB=\angle ANC=90°$

$\therefore$ By AA similarity

$\Delta AMB \sim \Delta ANC$

$$\therefore \frac{AB}{AC}=\frac{BM}{CN} \quad (1)$$

In ΔMDB and ΔNDC

$\angle 3=\angle 4$ (VOA)

$\angle DMB=\angle DNC=90°$

$\therefore$ By AA similarity

$\Delta MDB \sim \Delta NDC$

$$\therefore \frac{BM}{CN}=\frac{BD}{CD} \quad (2)$$

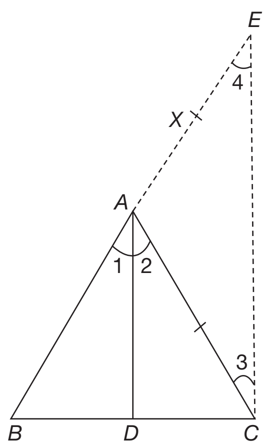
$\therefore$ From Eqs. (1) and (2)

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

Note: In ΔABC, if AD is the bisector of $\angle A$, then $\frac{[ABD]}{[ACD]} = \frac{AB}{AC}$.

8.6.3 Converse of Internal Angle Bisector Theorem

If a line through one vertex of a triangle divides the opposite sides in the ratio of other two sides, then the line bisects the angle at the vertex.



Given: In ΔABC, $\frac{AB}{AC} = \frac{BD}{DC}$

To prove: AD bisects $\angle A$

Construction: Produce BA to E such that $AE = AC \cdot$ Join EC.

Proof: Since $AE = AC$

$\therefore \angle 3 = \angle 4$

Since $\frac{AB}{AC} = \frac{BD}{DC}$

$\Rightarrow \frac{AB}{AE} = \frac{BD}{DC}$ (As $AC = AE$)

$\therefore$ By converse of BPT, In ΔBCE, we have $AD \parallel EC$

$\therefore \angle 1 = \angle 4$ (Corresponding angles)

$\angle 2 = \angle 3$ (Alternate interior angles)

But $\angle 3 = \angle 4$

$\Rightarrow \angle 1 = \angle 2.$

Hence AD bisects the angle $\angle A$.

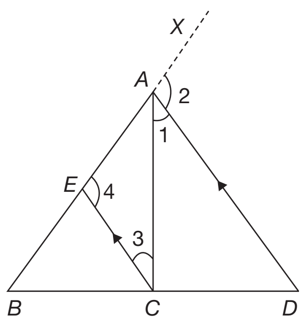
8.6.4 External Bisector Theorem

The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.

Given: In ΔABC, in which AD is the bisector of the exterior angle $\angle A$ and intersects BC produced in D.

To prove: $\frac{BD}{CD} = \frac{AB}{AC}$.

Construction: Draw $CE \parallel DA$, meeting AB in E.



Proof: Since AD bisects $\angle CAX$

$\therefore \angle 1 = \angle 2$

also, $AD \parallel EC$

$\angle 3 = \angle 1$ (Alternate interior angles)

$\angle 4 = \angle 2$ (Corresponding angles)

Since $\angle 1 = \angle 2 \Rightarrow \angle 3 = \angle 4$

$AC = AE.$

In ΔBAD, $CE \parallel DA$

$\therefore$ By BPT

$$\frac{AB}{AE} = \frac{DB}{DC}$$

$$\Rightarrow \frac{AB}{AC} = \frac{BD}{DC}. \qquad \text{(As } AE = AC\text{)}$$

Note: This result is not true for isosceles triangle because in that case exterior angle bisector is parallel to the base.

8.6.5 Converse of External Angle Bisector Theorem

If a line through one vertex of a triangle divides the opposite sides externally in the ratio of other two sides, then the line bisects the external angle at the vertex.

Prove of the theorem is left as an exercise.

Example 41 *ABCD is a quadrilateral in which $AB = AD$. The bisector of $\angle BAC$ and $\angle CAD$ intersect the sides BC and CD at the points E and F respectively. Prove that $EF \parallel BD$.*

Solution:

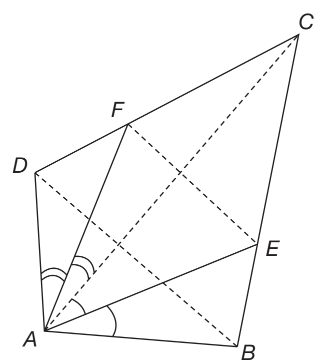
Given: In quadrilateral $ABCD$, $AB = AD$, AE bisects $\angle BAC$, AF bisects $\angle CAD$

To prove: $EF \parallel BD$

Construction: Join BD and EF

Proof: In ΔABC, since AE bisects $\angle BAC$

So by internal angle bisector theorem



$$\frac{BE}{EC} = \frac{AB}{AC} \qquad (1)$$

In ΔADC, AF bisects $\angle CAD$

$\therefore$ By internal angle bisector theorem

$$\frac{DF}{FC} = \frac{AD}{AC}$$

$$\Rightarrow \frac{DF}{FC} = \frac{AB}{AC} \text{ (As } AD = AB\text{)} \qquad (2)$$

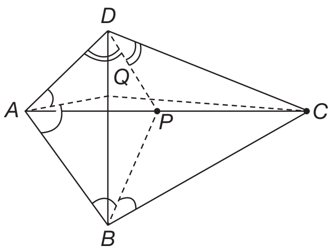
$\therefore$ From Eqs. (1) and (2)

$$\frac{BE}{EC} = \frac{DF}{FC}$$

$\therefore$ By converse of BPT in ΔBCD

$BD \parallel EF$.

Example 42 *In a quadrilateral ABCD, if bisector of the $\angle ABC$ and $\angle ADC$ meet on the diagonal AC, prove that the bisector of $\angle BAD$ and $\angle BCD$ will meet on the diagonal BD.*



Solution:

Given: $ABCD$ is a quadrilateral in which the bisectors of $\angle ABC$, $\angle ADC$ meet on the diagonal AC at P.

Construction: Let the bisector of $\angle BAD$ meet on the diagonal BD at Q. Join CQ.
To prove: Bisectors $\angle BAD$ and $\angle BCD$ meet on the diagonal BD. Which is equivalent to prove that CQ bisects $\angle BCD$.

Proof: Since in ΔABC, BP bisects $\angle ABC$

$\therefore$ By internal angle bisector theorem, $\dfrac{AB}{BC} = \dfrac{AP}{PC}$ (1)

Similarly in ΔADC, $\dfrac{AD}{DC} = \dfrac{AP}{PC}$ (2)

$\therefore$ From Eqs. (1) and (2), we get $\dfrac{AB}{BC} = \dfrac{AD}{DC}$

$\Rightarrow \dfrac{AB}{AD} = \dfrac{BC}{DC}$ (3)

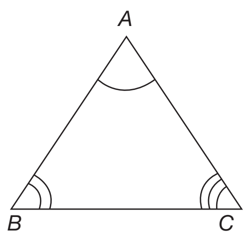
In ΔABD, AQ bisects $\angle BAD$

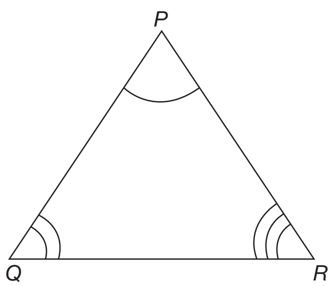
$\therefore$ By internal angle bisector theorem $\dfrac{AB}{AD} = \dfrac{BQ}{QD}$ (4)

From Eqs. (3) and (4), we get $\dfrac{BQ}{QD} = \dfrac{BC}{CD}$

$\therefore$ By converse of internal angle bisector theorem, CQ bisects $\angle BCD$.

8.7 Similar Triangles





Two triangles are similar if and only if
1. their corresponding angles are equal
2. their corresponding sides are proportional

Note: If ΔABC and ΔPQR are directly similar then

$\angle A = \angle P$, $\angle B = \angle Q$ and $\angle C = \angle R$ also, $\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{AC}{PR}$.

We have following criterion for similarity:

8.7.1 SSS Similarity (Side Side Side Similarity)

If in two triangles the sides of one triangle are proportional to those of the other then the corresponding angles of the two triangles are equal, *i.e.*, in the figure on p. 8.49 (Similar Triangles) if $\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{AC}{PR}$ then $\angle A = \angle P$; $\angle B = \angle Q$ and $\angle C = \angle R$.

8.7.2 AAA Similarity (Angle Angle Angle Similarity)

If in two triangle the angles of one triangle are equal to those of the other, then sides opposite to those angles are proportional. In the figure on p. 8.49 (Similar Triangles) if $\angle A = \angle P$; $\angle B = \angle Q$; $\angle C = \angle R$

Then $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$.

AA similarity also sufficient for the triangle to be similar

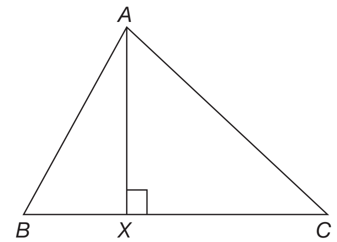
8.7.3 SAS Similarity (Side Angle Side Similarity)

If in two triangles, one angle of one triangle is equal to one angle of the other triangle and the sides containing these angle are proportional, then two triangles are similar.

In the the figure on p. 8.49 (Similar Triangles) if $\angle A = \angle P$ and $\frac{AB}{PQ} = \frac{AC}{PR}$, then $\Delta ABC \sim \Delta PQR$.

8.7.4 Area Ratio Theorem for Similar Triangles

The ratio of the areas of two similar triangles are equal to the ratio of the squares of any two corresponding sides.



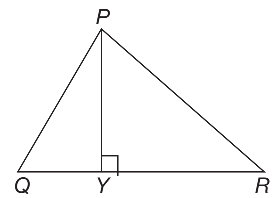
Given: $\Delta ABC \sim \Delta PQR$

That is, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$ (1)

And $\angle A = \angle P,\ \angle B = \angle Q,\ \angle C = \angle R$

To prove: $\frac{[ABC]}{[PQR]} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$

Construction: Draw $AX \perp BC$, $PY \perp QR$



Proof: $\frac{[ABC]}{[PQR]} = \frac{\frac{1}{2} \times BC \times AX}{\frac{1}{2} \times QR \times PY} = \left(\frac{BC}{QR}\right) \cdot \left(\frac{AX}{PY}\right)$ (2)

In ΔABX and ΔPQY

$\angle ABX = \angle PQY$

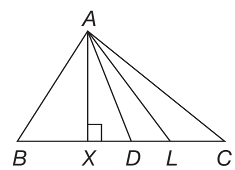
$\angle AXB = \angle PYQ = 90°$

$\therefore$ By AA similarity

$\Delta ABX \sim \Delta PQY$

$\therefore \frac{AB}{PQ} = \frac{AX}{PY}$ (3)

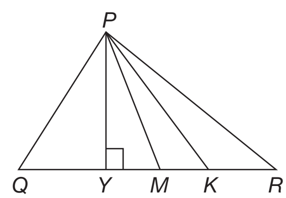
$\therefore$ From Eqs. (1), (2) and (3)



$\frac{[ABC]}{[PQR]} = \left(\frac{AB}{PQ}\right)\left(\frac{AB}{PQ}\right) = \frac{AB^2}{PQ^2}$

$\Rightarrow \frac{[ABC]}{[PQR]} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$. (Using Eq. (1))

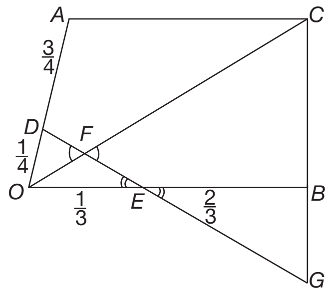
Note: In ΔABC and ΔPQR if AD, PM are the medians, AX, PY are the altitudes and AL, PK are the angle bisectors and Δ_1 and Δ_2 be their areas respectively then the following results also hold true, if $\Delta ABC \sim \Delta PQR$.



$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{AX}{PY} = \frac{AD}{PM} = \frac{AL}{PK} = \frac{AB+BC+CA}{PQ+QR+PR} = \frac{\sqrt{\Delta_1}}{\sqrt{\Delta_2}}.$$

Example 43 *Given a parallelogram OBCA, a straight line is constructed such that, it cuts off $\frac{1}{3}$ part of OB and $\frac{1}{4}$ part of OA. Find the fraction of length this line cuts off from the diagonal OC.*

Solution:



Construction: Extend the line to meet CB extended at G.

$\Delta OFD \sim \Delta CFG$ and $\Delta OED \sim \Delta BEG$

$$\therefore \frac{CF}{OF} = \frac{CG}{OD} = \frac{CB+BG}{OD} = \frac{CB}{OD} + \frac{BG}{OD} = \frac{OA}{OD} + \frac{BE}{OE} = 4+2=6$$

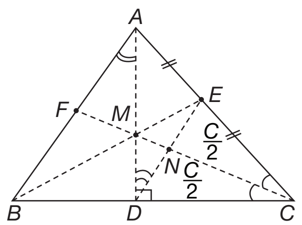
Thus $\frac{OF}{CF} = \frac{1}{6} \quad \Rightarrow \frac{OF}{OC} = \frac{1}{7}$

Thus the line cuts OC at F in the ratio of $OF : FC = 1 : 6$

That is, $\frac{1}{7}$ part of OC.

Example 44 *Let A, B, C be an acute angled triangle in which, D, E, F are points on BC, CA, AB respectively, such that $AD \perp BC$, $AE = EC$, CF bisects $\angle C$ internally. Suppose CF meets AD and DE in M and N respectively. If $FM = 2$, $MN = 1$, $NC = 3$, show that the perimeter and area of this triangle are equal numerically.*

Solution:



$FN = FM + MN = 2 + 1 = 3$ and $NC = 3$

$\therefore FN = NC \Rightarrow N$ is the mid-point of CF.

Also E is the mid-point of $AC \Rightarrow NE \parallel AF$ (By mid-point theorem)

$\therefore DE \parallel AB$

$\therefore BD = DC$ (by converse of mid-point theorem)

Thus AD is both altitude and median to BC

$\therefore \Delta ABC$ is isosceles $\Rightarrow AB = AC$ (1)

Also AD is the angle bisector of $\angle A$

$\therefore \Delta AMF \sim \Delta DMN$ (AA)

$$\therefore \frac{AM}{MD} = \frac{FM}{MN} = \frac{2}{1}$$

This proves that M is the centroid of ΔABC (as AD is median)

Thus CF is both angle bisector and median to ΔABC

i.e., ΔABC is isosceles $\Rightarrow AC = BC$. (2)

$\therefore AB = AC = BC$ (From Eqs. (1) and (2))

$\therefore \Delta ABC$ is equilateral.

Let the side of the equilateral triangle be 'a'.

CF, being the altitude,

$$CF = 6 \quad \Rightarrow \quad \frac{\sqrt{3}}{2}a = 6 \quad \Rightarrow \quad a = 4\sqrt{3}$$

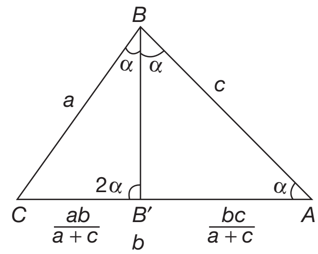
$\therefore$ Perimeter $= 3 \times 4\sqrt{3} = 12\sqrt{3}$

$$\text{Area} = \left(\frac{\sqrt{3}}{4}\right)(4\sqrt{3})(4\sqrt{3}) = 12\sqrt{3}$$

Thus area and perimeter are equal numerically.

Example 45 *Show that there is a unique triangle, whose side lengths are consecutive integers and one of whose angles is twice the other.*

Solution:



Let $\angle B = 2\alpha$

The bisector of $\angle B$ intersects AC at B', so that, $CB' = \dfrac{ab}{a+c}$ and $AB' = \dfrac{bc}{a+c}$

Now $\Delta ABC \sim \Delta BB'C$

$\therefore \dfrac{BC}{B'C} = \dfrac{AC}{BC} \Rightarrow BC^2 = AC \cdot B'C$

That is, $a^2 = (b)\left(\dfrac{ab}{a+c}\right)$ or $a^2 = \dfrac{ab^2}{a+c}$

i.e., $a(a+c) = b^2$ (1)

According to our assumption of the angles, $b > a$ holds.

$\therefore$ Either $b = (a+1)$ or $b = (a+2)$ (as a, b, c are consecutive)

In the first case, *i.e.*, $b = a + 1 \Rightarrow b^2 = a(a+c)$

$\Rightarrow (a+1)^2 = a(a+c)$, *i.e.*, $a^2 + 2a + 1 = a^2 + ac$

$\Rightarrow 2a+1 = ac \Rightarrow a \mid 1 \Rightarrow a = 1 \Rightarrow c = 3$ and $b = 2$

Which is impossible, thus $b \neq a+1$.

Then, let $b = a + 2$ then $c = a + 1$, now $(a+2)^2 = a(a+a+1) = 2a^2 + a$

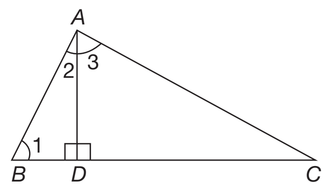
$\Rightarrow a^2 - 3a - 4 = 0$

$\therefore a = -1$ or 4, but $a \neq -1$ (reject)

$\therefore a = 4$; thus $b = 6$ and $c = 5$.

$\therefore$ There is only one triangle satisfying the conditions of the problem, *i.e.*, the triangle whose measures are 4, 5, and 6.

Example 46 *If a perpendicular AD is drawn from the right angled vertex A of a right angled triangle ABC to the hypotenuse BC then prove that triangles on both sides of the perpendicular are similar to the whole triangle and to each other. Also prove that* $BA^2 = BD \cdot BC$, $CA^2 = CD \cdot CB$ *and* $DA^2 = DB \cdot DC$



Given: In ΔABC, $\angle A = 90°$ and $AD \perp BC$

To prove: (i) $\Delta BDA \sim \Delta BAC$, (ii) $\Delta CDA \sim \Delta CAB$ and (iii) $\Delta BDA \sim \Delta ADC$

Proof:

(i) In ΔBDA and ΔBAC

$\angle DBA = \angle ABC$ (Common)

$\angle BDA = \angle BAC = 90°$

$\therefore$ By AA similarity

$\Delta BDA \sim \Delta BAC$

$\dfrac{BD}{BA} = \dfrac{BA}{BC} \Rightarrow BA^2 = BD \cdot BC$

(ii) In ΔCDA and ΔCAB

$\angle DCA = \angle ACB$ (Common)

$\angle CDA = \angle CAB = 90°$

$\therefore$ By AA similarly

$\Delta CDA \sim \Delta CAB$

$\dfrac{CD}{CA} = \dfrac{CA}{CB}$

$\therefore CA^2 = CD \cdot CB$

(iii) Since $\angle 1 + \angle 2 = 90°$

also $\angle 2 + \angle 3 = 90°$

$\therefore \angle 1 + \angle 2 = \angle 2 + \angle 3$

$\Rightarrow \angle 1 = \angle 3$

$\therefore$ In ΔBDA and ΔADC

$\angle 1 = \angle 3$ (Proved above)

$\angle BDA = \angle ADC = 90°$

$\therefore$ By AA similarity

$\Delta BDA \sim \Delta ADC$

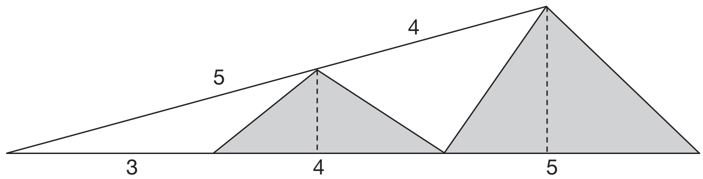
$$\frac{BD}{AD} = \frac{AD}{CD}$$

$AD^2 = BD \cdot CD.$

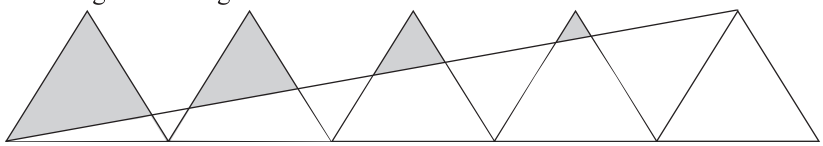
Note: $\dfrac{AB^2}{AC^2} = \dfrac{BD \cdot BC}{CD \cdot CB} = \dfrac{BD}{CD}.$

Build-up Your Understanding 5

1. In the given figure, what is the ratio of the areas of the two shaded triangles?



2. In the given figure, what is the ratio of the shaded area to the area of one of the five congruent triangles?



3. In ΔABC, BE and CF are the angular bisector of $\angle B$ and $\angle C$ meeting at I. Prove that $\dfrac{AF}{FI} = \dfrac{AC}{CI}$.

4. If the bisector of $\angle A$ in ΔABC meets BC at D, prove that $BD = \dfrac{ac}{b+c}$ and $DC = \dfrac{ab}{b+c}$.

5. P is any point within ΔABC and Q is a point outside ΔABC such that $\angle CBQ = \angle ABP$ and $\angle BCQ = \angle BAP$. Show that the triangles PBQ and ABC are similar.

6. PM and PN are the perpendiculars from a point to two given straight line OA and OB. If $\dfrac{PM}{PN}$ is a constant, prove that the locus of P is a straight line through O.

7. From A perpendiculars AX, AY are drawn to the bisectors of the exterior angles of B and C of ΔABC. Prove that $XY \parallel BC$.

8. A straight line, perpendicular to AI, is drawn through the incentre I of ΔABC, meeting AB, AC in D and E respectively. Prove that $BD \cdot CE = ID^2$.

9. Prove that the feet of the four perpendiculars dropped from a vertex of a triangle upon the four bisectors of the other two angle are collinear.

10. In triangle ABC, X and Y be the feet of perpendiculars from vertex A to the internal angle bisector of $\angle B$ and $\angle C$ respectively. Line XY meets AB at P and AC at Q. If $AB = 7$ cm, $BC = 8$ cm and $CA = 5$ cm then find PQ and XY.

11. We are given a triangle with the following property: One of its angles is quadrisected (divided into four equal angles) by the altitude, the angle bisector, and the median from that vertex. This property uniquely determines the triangle (up to scaling). Find the measure of the quadrisected angle.

12. Show that the sum of the reciprocals of the internal bisectors of a triangle is greater than the sum of the reciprocals of the sides of the triangle.

13. The internal bisector of the $\angle B$ of ΔABC meets the sides $B'C'$ and $B'A'$ of the medial triangle in the points A'', C'' respectively. Prove that AA'', CC' are perpendicular to the bisector of $\angle B$ and that $B'A'' = B'C''$.

14. In ΔABC, D, E, F are points on the sides BC, CA, AB respectively. Also A, B, C are points on YZ, ZX, XY of ΔXYZ respectively for which $EF \parallel YZ$, $FD \parallel ZX$, $DE \parallel XY$. Prove that area of $[ABC]^2 = [DEF] \cdot [XYZ]$.

15. In ΔABC, find points X, Y, Z on AB, BC, CA such that $AXYZ$ is a rhombus. Show that $[AXYZ] \le \frac{1}{2}[ABC]$.

16. Points O and H are the circumcentre and orthocentre of acute triangle ABC, respectively. The perpendicular bisector of segment AH meets sides AB and AC at D and E, respectively. Prove that $\angle DOA = \angle EOA$.

17. Let A and B be two distinct point on the same side of a line l and let L and M be foot of perpendiculars to l from A and B respectively. Let AM and BL intersects each other at P and Q be the foot of perpendicular from P to l. Prove that $\frac{1}{PQ} = \frac{1}{AL} + \frac{1}{BM}$.

18. Let ABC be a triangle. Construct two parallelograms $BADE$ and $BCFG$ on sides BA and BC, respectively. Suppose DE, FG produced meet at H. Show that the sum of the areas of the parallelograms is equal to the area of the parallelogram $ACIJ$, with sides CI, AJ equal and parallel to BH.

19. Let M be the mid-point of the side AB of ΔABC. Let P be a point on AB, between A and M and Let MD be drawn parallel to PC and intersecting BC at D. If the ratio of $[BPD]$ to $[ABC]$ be x, show that, $x = \frac{1}{2}$, independent of the position of P.

20. The mid-point of the hypotenuse of a right angled triangle ABC, right angled at B is M. A line is drawn perpendicular to the hypotenuse through M, in such a way, that the portion of it lying inside the triangle is 3 cm long and outside the triangle, up to the other side is 9 cm. Find the length of the hypotenuse.

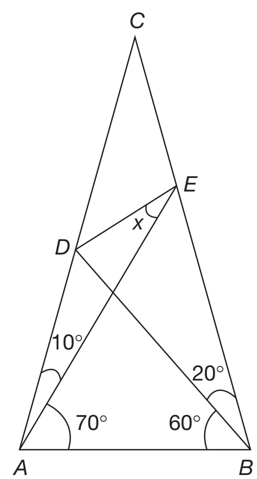
21. P, Q, and R are arbitrary points on the sides BC, CA, and AB respectively of triangle ABC. Prove that the three circumcentres of triangles AQR, BRP, and CPQ form a triangle similar to triangle ABC. **[British MO, 1984]**

22. OB is the perpendicular bisector of the segment DE. A is a point on OB. $AF \perp OB$, meeting OD at F. EF intersects OB at C. Prove that, OC is the harmonic mean between OA and OB.

23. The point P lies in the interior of ΔABC. A line is drawn through P, parallel to each side of a triangle. The line divides AB into three parts length (in that order); BC into three parts, length (in that order); CA into three parts length (in that order). Prove the following result: $a\,b\,c = a'b'c' = a''b''c''$.

24. Let the inscribed circle of triangle ABC touches side BC at D, side CA at E and side AB at F. Let G be the foot of the perpendicular from D to EF. Show that $\frac{FG}{EG} = \frac{BF}{CE}$.

25. Find the angle x in adjacent figure.



Baudhāyana Sulbsūtra

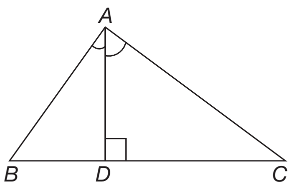
(Compiled around 8th to 7th centuries BCE)

दीर्घचतुरश्रस्याक्ष्णया रज्जुः पार्श्वमानी तिर्यग् मानी च यत् पृथग् भुते कुरुतस्तदुमयं करोति ॥

dirghachatursrasyākṣaṇayā rajjuh pārśvamāni, tiryagmāni, cha yatpṛthagbhūte kurutastadub-hayāṅ karoti.

A rope stretched along the length of the diagonal produces an area which the vertical and horizontal sides make together.

The lines are referring to a rectangle, It states that the square of hypotenuse equals the sum of the squarer of sides!



Pythagoras of Samos

c. 570 BC–c. 495 BC

Nationality: Greek

8.8 Baudhayana (Pythagoras) Theorem

In a right angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides

Given: In ΔABC, $\angle A = 90$

To prove: $BC^2 = AB^2 + AC^2$

Construction: Draw $AD \perp BC$

Proof: In ΔBDA and ΔBAC

$\angle DBA = \angle ABC$ (Common)

$\angle BDA = \angle BAC = 90°$

$\therefore$ By AA similarity

$\Delta BDA \sim \Delta BAC$

$$\therefore \quad \frac{BD}{BA} = \frac{BA}{BC}$$

$$\Rightarrow BA^2 = BD \cdot BC \quad (1)$$

In ΔCDA and ΔCAB

$\angle DCA = \angle ACB$ (Common)

$\angle CDA = \angle CAB = 90°$

$\therefore$ By AA similarity

$\Delta CDA \sim \Delta CAB$

$$\frac{CD}{CA} = \frac{CA}{CB}$$

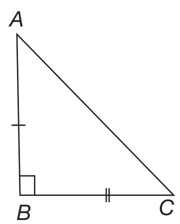
$$\Rightarrow CA^2 = CD \cdot CB \quad (2)$$

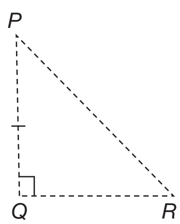
Adding Eqs. (1) and (2)

$$\begin{aligned} BA^2 + CA^2 &= BD \cdot BC + CD \cdot BC \\ &= BC \cdot (BD + CD) \\ &= BC \cdot BC \\ AB^2 + AC^2 &= BC^2. \end{aligned}$$

Note: $AB^2 + DC^2 = AC^2 + BD^2$

8.8.1 Converse of Baudhayana(or Pythagoras) Theorem





In a triangle if square of the longest side is equal to the sum of the squares of other two sides then angle opposite to the longest side is a right angle.

Given: In $\Delta ABC, AC^2 = AB^2 + BC^2$

To prove: $\angle ABC = 90°$

Construction: Construct a right angle triangle PQR right angled at Q and $PQ = AB$ and $QR = BC$.

Proof: Since in right angle triangle PQR, $\angle Q = 90°$

$\therefore$ By Baudhayana (or Pythagoras) theorem,

$$PR^2 = PQ^2 + QR^2$$

But $PQ = AB$ and $QR = BC$

$\therefore\ PR^2 = AB^2 + BC^2$ (1)

But it is given that

$$AC^2 = AB^2 + BC^2 \qquad (2)$$

$\therefore$ From Eqs. (1) and (2),

$$PR^2 = AC^2$$

$$\Rightarrow PR = AC$$

In ΔABC and ΔPQR

$$AB = PQ$$
$$BC = QR$$
$$AC = PR$$

$\therefore$ By SSS congruences

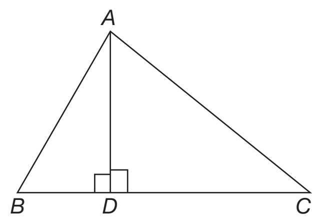
$$\Delta ABC \cong \Delta PQR$$

$\therefore\ \angle ABC = \angle PQR = 90°$

Some important result based on Baudhayana theorem:

8.8.2 Acute Angled Triangle Theorem

In ΔABC, if $\angle B < 90°$ and $AD \perp BC$, prove that $AC^2 = AB^2 + BC^2 - 2BD \cdot BC$



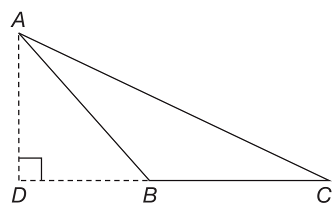
Proof: In ΔADC, by using Baudhayana (Pythagoras) theorem

$$AC^2 = AD^2 + DC^2$$
$$= AD^2 + (BC - BD)^2$$
$$= AD^2 + BD^2 + BC^2 - 2BD \cdot BC$$
$$\Rightarrow AC^2 = AB^2 + BC^2 - 2BD \cdot BC (\text{As } AD^2 + BD^2 = AB^2).$$

Corollary: Let AC be the largest side and $AC^2 < AB^2 + BC^2$ implies ΔABC is an acuteangle triangle.

8.8.3 Obtuse Angled Triangle Theorem

ΔABC is an obtuse triangle, obtuse angled at B. If $AD \perp CB$, prove that $AC^2 = AB^2 + BC^2 + 2BD \cdot BC$



Proof: In ΔADC, by using Baudhayana (Pythagoras) theorem

$$AC^2 = AD^2 + DC^2$$
$$= AD^2 + (DB + BC)^2$$
$$= AD^2 + DB^2 + BC^2 + 2\ BD \cdot BC$$
$$\Rightarrow AC^2 = AB^2 + BC^2 + 2BD \cdot BC \qquad (\text{As } AD^2 + DB^2 = AB^2)$$

Corollary: In ΔABC, $AC^2 > AB^2 + BC^2$ implies ΔABC is an obtuseangle triangle.

Apollonius of Perga

262 BC–190 BC
Nationality: Greek

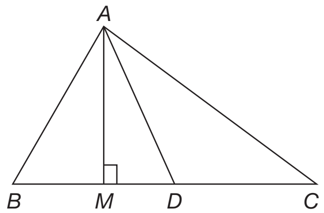
8.8.4 Apollonius Theorem

In any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.

Given: In ΔABC, AD is a median.

To prove: $AB^2 + AC^2 = 2AD^2 + 2BD^2$

$$\text{Or } AB^2 + AC^2 = 2AD^2 + \frac{1}{2}BC^2.$$



Construction: Draw $AM \perp BC$

Proof: InΔADB, $\angle D < 90°$

$\therefore$ By acute angled triangle theorem

$$AB^2 = AD^2 + BD^2 - 2DM \cdot BD$$

$$= AD^2 + BD^2 - DM \cdot BC \qquad (1)$$

(As $2BD = BC$)

In ΔADC, $\angle D > 90°$

$\therefore$By obtuse angled triangle theorem

$$AC^2 = AD^2 + DC^2 + 2DM \cdot DC$$

$$\therefore AC^2 = AD^2 + BD^2 + DM \cdot BC \qquad (2)$$

(As $2DC = BC$ and $DC = BD$)

$\therefore$ Eq. (1) + Eq. (2) gives,

$$AB^2 + AC^2 = 2AD^2 + 2BD^2 = 2AD^2 + 2\left(\frac{BC}{2}\right)^2 = 2AD^2 + \frac{2 \cdot BC^2}{4}$$

$$\Rightarrow AB^2 + AC^2 = 2AD^2 + \frac{1}{2}BC^2.$$

Matthew Stewarts

28 Jun 1717–23 Jan 1785
Nationality: Scottish

8.8.5 Stewart's Theorem

Let D be a point on side BC such that $BD = m$ and $DC = n$ and $AD = d$. Then $a(d^2 + mn) = b^2m + c^2n$.

Proof: WLOG (Without loss of generality) Let $\angle ADB < \angle ADC$

$\Rightarrow \angle ADB$ is acute and $\angle ADC$ is obtuse.

In ΔABD, by using acute angle theorem, we get

$$AB^2 = AD^2 + BD^2 - 2BD \cdot MD$$

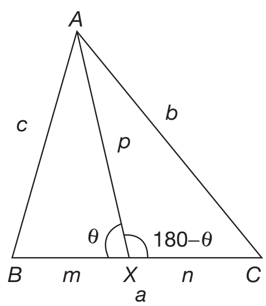
$$\Rightarrow c^2 = d^2 + m^2 - 2mx \qquad (1)$$

In ΔADC, by using obtuse angle theorem, we get

$$AC^2 = AD^2 + DC^2 + 2DC \cdot DM$$

$$\Rightarrow b^2 = d^2 + n^2 + 2nx \qquad (2)$$

$\therefore$ From $n \times$ Eq. (1) + $m \times$ Eq. (2), we get,



$$nc^2 + mb^2 = d^2(m+n) + mn^2 + m^2n$$

$$\Rightarrow b^2m + c^2n = d^2(m+n) + mn(m+n) = (d^2 + mn)(m+n)$$

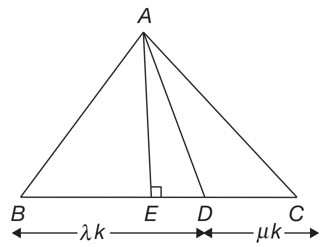
$$\Rightarrow b^2m + c^2n = (d^2 + mn) \cdot a$$

Notes:

1. A mnemonic of final result as 'man + dad = bmb + cnc' or 'A **man** and his **dad** put a **bomb** in the **sink**'.
2. Another version of Stewart's theorem is as follows:
   Let AD be of length d dividing BC into segments BD and DC *such that* $BD : DC = \lambda : \mu$. Then $\lambda AC^2 + \mu AB^2 = (\lambda+\mu)AD^2 + \lambda DC^2 + \mu BD^2$.

   Proof:

   To prove: $\lambda AC^2 + \mu AB^2 = (\lambda+\mu)AD^2 + \lambda DC^2 + \mu BD^2$.



   Now, $AB^2 = BE^2 + AE^2 = (\lambda k - ED)^2 + AE^2$

   $$\Rightarrow \mu AB^2 = \mu\lambda^2k^2 + \mu ED^2 - 2\mu\lambda k ED + \mu AE^2 \quad (1)$$

   Similarly $AC^2 = (\mu k + ED)^2 + AE^2$

   $$\Rightarrow \quad \lambda AC^2 = \lambda\mu^2k^2 + \lambda ED^2 + 2\mu\lambda k ED + \lambda AE^2 \quad (2)$$

   From adding Eqs. (1) and (2), we get

   $$\mu AB^2 + \lambda AC^2 = \mu BD^2 + \lambda CD^2 + (\mu+\lambda)ED^2 + (\mu+\lambda)AE^2$$

   $$\Rightarrow \lambda AC^2 + \mu AB^2 = \mu BD^2 + \lambda CD^2 + (\lambda+\mu)AD^2 \quad (\text{As } AE^2 + ED^2 = AD^2)$$

3. If AD is a median then $m = n = \frac{a}{2}$ and $AD = m_a$

   By applying Stewarts theorem we get $\frac{b^2a}{2} + \frac{c^2a}{2} = \left(m_a^2 + \frac{a^2}{4}\right)a$

   $$\Rightarrow b^2 + c^2 = 2m_a^2 + \frac{1}{2}a^2 \quad \text{(Apollonius theorem)}$$

   Or length of the median

   $$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4} \Rightarrow m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

   Similarly,

   $$m_b = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2} \text{ and } m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}.$$

4. If AD is the angle bisector,then $m = \frac{ca}{b+c} \quad n = \frac{ba}{b+c}$, $AD = t_a$

   By applying Stewarts theorem we get $\frac{b^2ca}{b+c} + \frac{c^2ba}{b+c} = \left(t_a^2 + \frac{a^2bc}{(b+c)^2}\right)a$

   $$\Rightarrow \frac{abc(b+c)}{(b+c)} = \left(t_a^2 + \frac{a^2bc}{(b+c)^2}\right)a$$

   $$\Rightarrow bc = t_a^2 + \frac{a^2bc}{(b+c)^2}$$

   $$\Rightarrow t_a^2 = bc - \frac{a^2bc}{(b+c)^2}$$

   $$\Rightarrow t_a^2 = bc\left[1 - \left(\frac{a}{b+c}\right)^2\right] \Rightarrow t_a = \sqrt{bc\left[1 - \left(\frac{a}{b+c}\right)^2\right]}.$$

Aliter 1: If AD is the angle bisector

$$[ABD]+[ACD]=[ABC]$$

$$\Rightarrow \frac{1}{2}t_a \cdot c\sin\frac{A}{2}+\frac{1}{2}t_a \cdot b\sin\frac{A}{2}=\frac{1}{2}bc\sin A$$

$$\Rightarrow \frac{1}{2}t_a \sin\frac{A}{2}(b+c)=\frac{1}{2}bc\cdot 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$\Rightarrow t_a=\frac{2bc}{b+c}\cos\frac{A}{2}$$

A special case: If $\angle A$ = 120° then the length of angle bisector AD is $t_a=\frac{2bc}{b+c}\cdot\cos 60°=\frac{bc}{b+c}$.

Aliter 2: If AD is the angle bisector of $\angle A$ in ΔABC and cuts the circumcircle at E. Then by using the result obtain in example 91 (on page 8.97), we get,

$$AD^2+BD\cdot DC=AB\cdot AC$$

$$\Rightarrow AD^2=AB\cdot AC-BD\cdot DC$$

$$=b\cdot c-\frac{ca}{b+c}\cdot\frac{ba}{b+c}$$

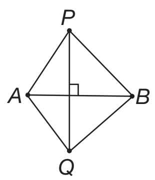
$$=bc-\frac{a^2bc}{(b+c)^2}$$

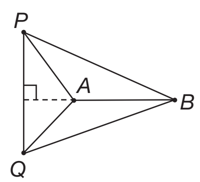
$$\Rightarrow AD^2=bc\left[1-\left(\frac{a}{b+c}\right)^2\right]$$

$$\Rightarrow t_a=\sqrt{bc\left[1-\left(\frac{a}{b+c}\right)^2\right]}.$$

8.8.6 Lemma

Let A, B, P, Q be four distinct points on a plane. Then $AB \perp PQ$ if and only if $PA^2 - PB^2 = QA^2 - QB^2$.





Proof: First we will assume $PA^2 - PB^2 = QA^2 - QB^2$ and we will prove $AB \perp PQ$.

Let foot of perpendicular from P and Q on AB be L and M respectively. Now we will prove $L = M$.

By Baudhayana theorem we have

$PA^2 = PL^2 + AL^2$ and $PB^2 = PL^2 + BL^2$

$\Rightarrow PA^2 - PB^2 = AL^2 - BL^2 = (AL + BL)(AL - BL) = AB(AB - 2BL)$ (1)

Similarly, $QA^2 - QB^2 = AB(AB - 2BM)$ (2)

Now From Eqs. (1) and (2), we get

$AB(AB - 2BL) = AB(AB - 2BM)$ (As $PA^2 - PB^2 = QA^2 - QB^2$)

$\Rightarrow BL = BM$

$\Rightarrow L = M$ (As L, M on AB and same side of B)

Now we will assume $AB \perp PQ$ and we will prove $PA^2 - PB^2 = QA^2 - QB^2$.

Let point of intersection of AB and PQ be L.

By Baudhayana theorem we have

$PA^2 = PL^2 + AL^2$ and $PB^2 = PL^2 + BL^2$

$\Rightarrow PA^2 - PB^2 = AL^2 - BL^2 = (QL^2 + AL^2) - (QL^2 + BL^2) = QA^2 - QB^2$.

Note: This Lemma is very useful when we need to prove two lines are perpendicular.

Example 47 *In a given triangle ABC, in the usual notation, it is given that, a, b, c are in geometric progression. Also it is true that,* $\log a - \log 2b$, $\log 2b - \log 3c$, $\log 3c - \log a$ *are in arithmetic progression. Prove that this triangle must be obtuse angled triangle.*

Solution:

a, b, c are in GP $\Rightarrow b^2 = ac$

$\log a - \log 2b$, $\log 2b - \log 3c$, $\log 3c - \log a$ are in AP

$\therefore 2(\log 2b - \log 3c) = (\log a - \log 2b) + (\log 3c - \log a)$

$\therefore 2\left(\log \frac{2b}{3c}\right) = \left(\log \frac{a}{2b} + \log \frac{3c}{a}\right)$

i.e., $\log\left(\frac{2b}{3c}\right)^2 = \log\left(\frac{\cancel{a}}{2b} \times \frac{3c}{\cancel{a}}\right) = \log \frac{3c}{a}$

$\therefore \frac{4b^2}{9c^2} = \frac{3c}{2b} \Rightarrow 8b^3 = 27c^3$

$\therefore 2b = 3c$ (Taking cube roots)

Also $4b^2 = 9c^2 \Rightarrow 4ac = 9c^2 \Rightarrow 4a = 9c$

Thus $4a = 6b = 9c = k$ (Say)

$\therefore a = \frac{k}{4}; b = \frac{k}{6}; c = \frac{k}{9}$

Here, $a^2 = \frac{k^2}{16}; b^2 = \frac{k^2}{36}; c^2 = \frac{k^2}{81}$

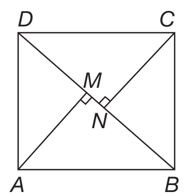
We see that, $\frac{k^2}{16} > \frac{k^2}{36} + \frac{k^2}{81} \Rightarrow a^2 > b^2 + c^2$

i.e., $\angle A > 90°$

$\therefore$ The triangle is obtuse.

Example 48 *ABCD is a rectangle. Points M and N are on BD such that* $AM \perp BD$ *and* $CN \perp BD$ *prove that* $BM^2 + BN^2 = DM^2 + DN^2$

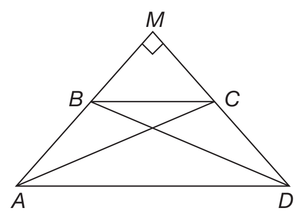
Solution:



$$BM^2 = AB^2 - AM^2$$
$$BN^2 = BC^2 - CN^2$$
$$\therefore BM^2 + BN^2 = (AB^2 - AM^2) + (BC^2 - CN^2)$$
$$= (DC^2 - CN^2) + (AD^2 - AM^2) \text{ (As } AB = DC, BC = AD\text{)}$$
$$\Rightarrow BM^2 + BN^2 = DN^2 + DM^2.$$

Example 49 *In a quadrilateral ABCD, given that* $\angle A + \angle D = 90°$ *prove that* $AC^2 + BD^2 = AD^2 + BC^2$.

Solution:



Construction: Produce AB and DC to cut at M

Since $\angle A + \angle D = 90°$

By ASP (angle sum property) of the triangle

In ΔAMD, $\angle M = 90°$

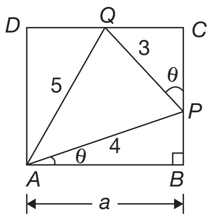
$\therefore$ In ΔAMC, $AC^2 = AM^2 + MC^2$

In ΔBMD,

$$BD^2 = BM^2 + MD^2$$
$$\therefore AC^2 + BD^2 = (AM^2 + MC^2) + (BM^2 + MD^2)$$
$$= (AM^2 + DM^2) + (MC^2 + MB^2)$$
$$\Rightarrow AC^2 + BD^2 = AD^2 + BC^2.$$

Example 50 *Let ABCD be a square. P and Q are any two points on BC and CD respectively. Such that AP = 4 cm, PQ = 3 cm, AQ = 5 cm. Find the side of the square.*

Solution:



Since $5^2 = 3^2 + 4^2$

i.e., $\angle 2 + \angle 3 = 180°$

$\angle 1 + \angle 4 < 180°$ By converse of Baudhayana (Or Pythagoras) theorem

$\angle APQ = 90°$

Let $\angle PAB = \theta$

$\Rightarrow \angle APB = 90° - \theta$

$\angle QPC = \theta.$

Let $AB = a$

In $\Delta APB, \cos\theta = \dfrac{a}{4} \quad \Rightarrow \quad a = 4\cos\theta$

Also in ΔAPB, $\sin\theta = \dfrac{PB}{4} \Rightarrow PB = 4\sin\theta$

In ΔPCQ, $\cos\theta = \dfrac{PC}{3} \Rightarrow PC = 3\cos\theta$

Since $ABCD$ is a square

$AB = BC$

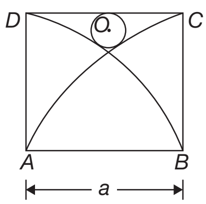
$AB = BP + PC$

$\therefore 4\cos\theta = 4\sin\theta + 3\cos\theta$

$\therefore \cos\theta = 4\sin\theta$

$\Rightarrow \quad \tan\theta = \dfrac{1}{4} \quad \Rightarrow \quad \cos\theta = \dfrac{4}{\sqrt{17}}$

$\therefore AB = a = 4\cos\theta = 4 \times \dfrac{4}{\sqrt{17}} = \dfrac{16\sqrt{17}}{17}$ cm.



Example 51 *In the figure ABCD is a square of side 'a' units. Find the radius 'r' of a smaller circle. Where arc DB and arc AC has centres at A and B respectively.*

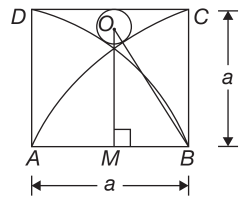
Solution:

Proof: Since if two circles are touching then the line segment joining their centres passes through their point of contact

$\therefore BO = a + r$

$\therefore MO = a - r$

And $BM = \dfrac{a}{2}$ by symmetry



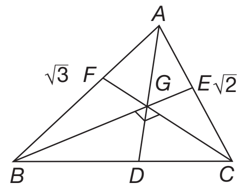
$\therefore$ In right angle ΔOMB

$BO^2 = MO^2 + BM^2$

$$\Rightarrow \quad (a+r)^2 = (a-r)^2 + \left(\frac{a}{2}\right)^2 \quad \Rightarrow \quad (a+r)^2 - (a-r)^2 = \frac{a^2}{4}$$

$$\Rightarrow \quad 4ar = \frac{a^2}{4} \quad \Rightarrow \quad r = \frac{a}{16}$$

Example 52 *Two sides of a triangle are $\sqrt{3}$ and $\sqrt{2}$ units. The medians to these two sides are mutually perpendicular. Prove that the third side has an integer measure.*



Solution: Let the medians BE and CF be perpendicular to each other.

Now $BG^2 + CG^2 = BC^2$ (1)

Also $4BE^2 = 2BC^2 + 2BA^2 - AC^2$ (2)

(From Apollonius theorem)

But $BG = \frac{2}{3}BE$ and so $BG^2 = \frac{4}{9}BE^2$ (3)

$\therefore BG^2 = \frac{1}{9}(2BC^2 + 2BA^2 - AC^2)$ (From Eqs. (2) and (3))

Similarly $CG^2 = \frac{1}{9}(2BC^2 + 2CA^2 - AB^2)$

Thus, $BG^2 + CG^2 = \frac{1}{9}(2BC^2 + 2BA^2 - AC^2 + 2BC^2 + 2CA^2 - AB^2)$

i.e., $BC^2 = \frac{1}{9}(4BC^2 + AB^2 + AC^2)$

$\therefore 9BC^2 = 4BC^2 + AB^2 + AC^2 \Rightarrow 5BC^2 = AB^2 + AC^2$

i.e., $5BC^2 = (\sqrt{3})^2 + (\sqrt{2})^2 = 5 \Rightarrow BC = 1$

which is an integer.

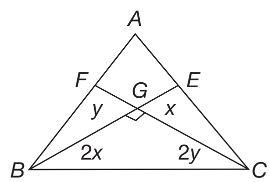
Aliter: Let FG be $y \Rightarrow GC = 2y$ and GE be $x \Rightarrow BG = 2x$

In ΔEGC, by Baudhayana (or Pythagoras) Theorem,

$$x^2 + 4y^2 = EC^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} \quad (1)$$

Similarly in ΔBGF,

$$4x^2 + y^2 = \frac{3}{4} \quad (2)$$



Now adding Eqs. (1) and (2), we get,

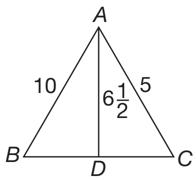
$$5x^2 + 5y^2 = \frac{5}{4} \Rightarrow (2x)^2 + (2y)^2 = 1$$

$$\Rightarrow BG^2 + GC^2 = 1$$

$$\Rightarrow BC^2 = 1$$

$$\Rightarrow BC = 1.$$

Example 53 *Two sides of a triangle are 10 cm and 5 cm in length and the length of the median to the third side is 6.5 cm. If the area of the triangle is $6\sqrt{p}$ cm^2, find the value of p.*



Solution: Let D be the mid-point of BC. By Apollonius theorem,

$$AB^2 + AC^2 = 2(BD^2 + AD^2) \qquad \text{(By Apollonius Theorem)}$$

or $\quad 4AD^2 = 2AB^2 + 2AC^2 - BC^2$

$\therefore \quad BC^2 = 2AB^2 + 2AC^2 - 4AD^2$

$$= 2(10)^2 + 2(5)^2 - 4\left(\frac{13}{2}\right)^2 = 81 \qquad \text{(on simplification)}$$

$$\therefore \quad BC = 9 \text{ cm} \Rightarrow s = \frac{9+10+5}{2} = 12$$

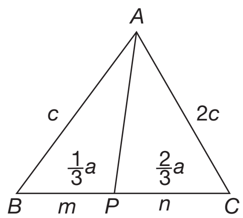
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} = 6\sqrt{p}$$

$$= \sqrt{12 \times 3 \times 7 \times 2} = 6\sqrt{p} \quad \Rightarrow p = 14.$$

Example 54 *The internal bisector of $\angle A$ of ΔABC meets BC at P and b = 2c in the usual notation. Prove that $(9AP^2 + 2a^2)$ is an integral multiple of c^2.*

Solution: As AP is bisector in the problem, we have

$$\frac{BP}{PC} = \frac{c}{2c} = \frac{1}{2} = \frac{\lambda}{\mu} \qquad \text{(Say)}$$



Also $BP : PC = 1 : 2 \Rightarrow BP = \frac{1}{3}a; PC = \frac{2}{3}a$

By applying Stewart's theorem in ΔABC, we get,

$$\mu AB^2 + \lambda AC^2 = (\lambda + \mu)AP^2 + \mu BP^2 + \lambda PC^2$$

$$\Rightarrow 2 \cdot c^2 + 1 \cdot 4c^2 = (2+1)AP^2 + 2 \cdot \frac{a^2}{9} + 1 \cdot \frac{4a^2}{9}$$

$$\Rightarrow 6c^2 = 3AP^2 + \frac{2}{3}a^2$$

$$\Rightarrow 9AP^2 + 2a^2 = 18c^2.$$

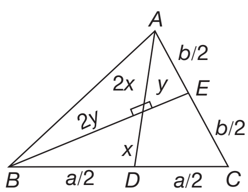
Example 55 *In triangle ABC, the medians from B and A to the opposite sides are mutually perpendicular to each other. If a, b, c are the measures of BC, CA, AB respectively, prove that,*

$$\frac{1}{2} < \frac{b}{a} < 2.$$

Solution: Let G be the centroid of ΔABC.

Since it trisect each median, let $AG = 2x$, $GD = x$, $BG = 2y$, $GE = y$.

Now from right triangles AGB and AGE and BGD, respectively we get,



$$4x^2 + y^2 = \frac{b^2}{4} \qquad (1)$$

$$4x^2 + 4y^2 = c^2 \qquad (2)$$

$$4y^2 + x^2 = \frac{a^2}{4} \qquad (3)$$

Adding Eqs. (1) and (3), we get

$$5x^2 + 5y^2 = \frac{a^2 + b^2}{4}$$

$$\Rightarrow x^2 + y^2 = \frac{a^2 + b^2}{20} \qquad (4)$$

From Eqs. (2) and (4) we get $c^2 = 4(x^2 + y^2) = \dfrac{a^2 + b^2}{5}$

Thus $a^2 + b^2 = 5c^2$ (5)

Also from Eqs. (2) and (3) we can infer that $c^2 < a^2$

And similarly from Eqs. (1) and (3) $c^2 < b^2$ so that 'c' is the smallest side.

$$\therefore \frac{a^2 + b^2}{5} < a^2 \quad \text{and} \quad \frac{a^2 + b^2}{5} < b^2 \qquad \text{(from Eq. (5))}$$

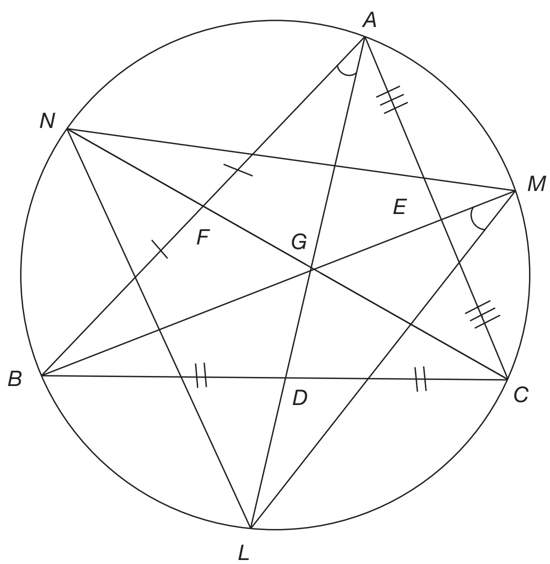
i.e., $b^2 < 4a^2$ and $a^2 < 4b^2$

i.e., $\dfrac{b}{a} < 2$ and $\dfrac{a}{b} < 2 \Rightarrow \dfrac{b}{a} > \dfrac{1}{2}$

Thus $\dfrac{1}{2} < \dfrac{b}{a} < 2$.

Example 56 *Let ABC be scalene triangle. The medians from A, B, C meet the circumcircle of ΔABC again at L, M and N respectively. If LM = LN, prove that, $AB^2 + AC^2 = 2BC^2$.*

Solution:



Let G be the centroid of ΔABC.

Now,

$\Delta LNG \sim \Delta CAG \quad \text{(AA)}$

$$\therefore \frac{LN}{AC} = \frac{GL}{GC} \qquad (1)$$

$\Delta LMG \sim \Delta BAG \quad \text{(AA)}$

$$\therefore \frac{LM}{AB} = \frac{GL}{GB} \qquad (2)$$

$$\text{Thus } \frac{AB}{AC} = \frac{GB}{GC} \qquad \text{(From Eq. (1)} \div \text{Eq. (2)and using } LN = LM\text{)}$$

$$\therefore \quad \frac{AB^2}{AC^2} = \frac{GB^2}{GC^2}$$

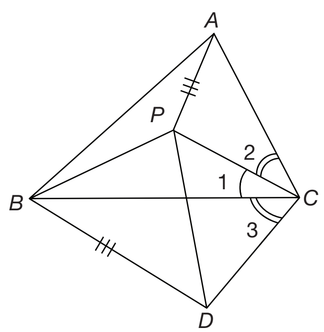
$$= \frac{\frac{1}{9}(2AB^2 + 2BC^2 - AC^2)}{\frac{1}{9}(2AC^2 + 2BC^2 - AB^2)} \qquad \text{(By Apollonius Theorem)}$$

$$\Rightarrow \frac{AB^2 - AC^2}{AC^2} = \frac{3(AB^2 - AC^2)}{2AC^2 + 2BC^2 - AB^2} \qquad \text{(Substracting 1 from both sides)}$$

$$\Rightarrow 2AC^2 + 2BC^2 - AB^2 = 3AC^2 \qquad \text{(As trinagle is scalene, } AB^2 \neq AC^2\text{)}$$

$$\Rightarrow 2BC^2 = AB^2 + AC^2.$$

Example 57 *In an equilateral ΔABC, a point P is taken in the interior of ΔABC such that $PA^2 = PB^2 + PC^2$ find $\angle BPC$.*



Solution: Construct $\angle BCD = \angle ACP$ and $CD = CP$

In ΔACP and ΔBCD

$AC = BC$ (Given)

$\angle ACP = \angle BCD$ (Construction)

$CP = CD$ (Construction)

$\therefore$ By SAS Congruency

$\Delta ACP \cong \Delta BCD$

$\therefore AP = BD$

Also $\angle 1 + \angle 3 = \angle 2 + \angle 3 = 60°$ (As $\angle 3 = \angle 2$)

And $PC = CD$

$\therefore \Delta PCD$ is an equilateral Δ with $PC = PD = CD$ and $\angle DPC = 60°$

Since

$PA^2 = PB^2 + PC^2$

$BD^2 = PB^2 + PD^2$ (As $PA = BD$, $PC = PD$)

$\therefore$ By converse of Baudhayna (or Pythagoras) theorem

$\angle BPD = 90°$

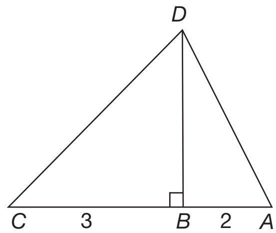
$\therefore \angle BPC = \angle BPD + \angle DPC$

$= 90° + 60°$

$\therefore \angle BPC = 150°.$

Build-up Your Understanding 6

1. Two sides of a triangle are 4 and 9. The median drawn to third side has length 6. Find the length of the third side.
2. ABC is an isosceles triangle with $AC = BC$. The medians AD and BE are perpendicular to each other and intersect at G. If $GD = a$ unit, find the area of the quadrilateral $CDGE$.
3. A right triangle has legs a and b and the hypotenuse c. Two segments from the right angle to the hypotenuse are drawn, dividing it into three equal parts of length $x = \frac{c}{3}$. If the segments have length p and q, prove that $p^2 + q^2 = 5x^2$.
4. Let ABC be a triangle and let D, E, F lie on the sides BC, CA, AB respectively, such that AD, BE and CF are concurrent at P. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the $[ABC]$.
5. In ΔABD, DB is perpendicular to AC at B so that $AB = 2$ and $BC = 3$ as shown in the figure. Furthermore, $\angle ADC = 45°$. Use this information to find the area of ΔADC.



6. On side AB of square $ABCD$ right ΔABF with hypotenuse AB is drawn externally to the square. If $AF = 6$ and $BF = 8$, find EF where E is the point of intersection of diagonals of the square. Also find EF when ΔABF is drawn internally to the square.
7. Point P on side AB of right ΔABC is such that $BP = PA = 2$. Point Q is on the hypotenuse AC so that PQ is perpendicular to AC. If $CB = 3$ find the length of BQ. Also find the area of the quadrilateral $CBPQ$.
8. (i) Let G be the centroid of triangle ABC and P is an arbitrary point. Prove that

   $$PA^2 + PB^2 + PC^2 = 3PG^2 + \frac{1}{3}(a^2 + b^2 + c^2).$$

   Note: This result is known as **Leibniz Theorem.**

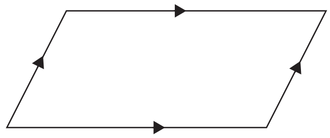
   (ii) Hence, or otherwise, find the formula of OG in terms of a, b, c, where O is the circumcentre.
9. Let ΔABC be right angle triangle with $\angle A = 90°$ and AL be its altitude. Let r, r_1, r_2 the inradii of ΔABC, ΔABL, ΔACL, respectively. Prove that $r_1^2 + r_2^2 = r^2$.
10. $ABCD$ and $A'B'C'D'$ are two non-congruent squares in a plane, placed by a displacement; (*i.e.*, $A'B' \parallel AB$, etc.) Prove that, $AA'^2 + CC'^2 = BB'^2 + DD'^2$.
11. Quadrilaterals $ABCP$ and $A'B'C'P'$ are inscribed in two concentric circles. If triangles ABC and $A'B'C'$ are equilateral, prove that $P'A^2 + P'B^2 + P'C^2 = PA'^2 + PB'^2 + PC'^2$
12. Let Q be the centre of the inscribed circle of a triangle ABC. Prove that for any point P, $a(PA^2) + b(PB^2) + c(PC^2) = a(QA)^2 + b(QB)^2 + c(QC)^2 + (a + b + c)QP^2$, where $a = BC$, $b = CA$ and $c = AB$.

Gottfried Wilhelm Leibniz

1 Jul 1646–14 Nov 1716
Nationality: German

8.9 QUADRILATERALS

A quadrilateral is a polygon with four edges (or sides) and four vertices or corners. They may be concave or convex. In our present discussion we are taking convex only. There are following important convex quadrilaterals:



8.9.1 Parallelogram

In a quadrilateral if both the pairs of opposite sides are parallel then it is called a parallelogram.

Some properties of a parallelogram:

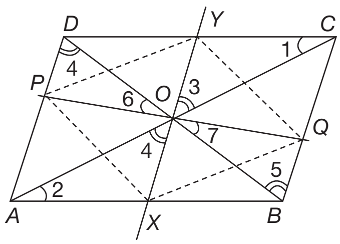
1. A diagonal of a parallelogram divides it into two congruent triangles.
2. In a parallelogram, opposite sides are equal.
3. Two opposite angles of a parallelogram are equal.
4. The diagonals of a parallelogram bisect each other.
5. In a parallelogram, the bisectors of any two consecutive angles intersects at right angle.
6. The angle bisectors of a parallelogram from a rectangle.
7. In a parallelogram sum of any two consecutive angles is 180°.
8. In a quadrilateral, if both opposite sides are equal then it is a parallelogram.
9. In a quadrilateral, if both opposite angles are equal then it is a parallelogram.
10. If the diagonals of a quadrilateral bisects each other then it is a parallelogram.
11. If one pair of opposite side of a quadrilateral is equal and parallel then it is a parallelogram.

Example 58 *The diagonals of a parallelogram ABCD intersects at O. A line through O intersects AB at X and DC at Y another line passing through O intersects AD at P and BC at Q. Prove that XQYP is a parallelogram.*

Solution:

Given: $ABCD$ is a parallelogram; AC, BD interests at O. XOY, POQ are two lines cutting AB at X, CD at Y also AD at P and BC at Q.

To prove: $XQYP$ is a parallelogram



Proof: In ΔCOY and ΔAOX

$\angle 1 = \angle 2$ (Alternate interior angles)

$CO = AO$

$\angle 3 = \angle 4$ (VOA)

$\therefore$ By ASA congruence

$\Delta COY \cong \Delta AOX$

$\Rightarrow \quad OY = OX$ (1)

Similarly in ΔPOD and ΔQOB

$\angle 4 = \angle 5$ (Alternate interior angles)

$OD = OB$ (Given)

$\angle 6 = \angle 7$ (VOA)

$\therefore$ By ASA congruence

$\Delta POD \cong \Delta QOB$

$\therefore OP = OQ$ (2)

$\therefore$ From (1) and (2), in quadrilateral $XQYP$ diagonals bisects each other and hence it is a parallelogram.

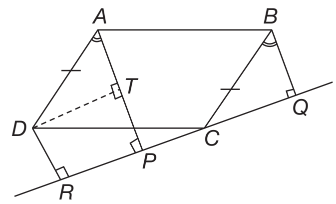
Examples 59 *ABCD is a parallelogram. Through C a straight line RQ is drawn outside the parallelogram and AP, BQ, DR are drawn perpendiculars to RQ. Show that* $DR + BQ = AP$

Solution:

Given: $ABCD$ is a parallelogram. DR, AP, BQ are perpendiculars on any line passes through C and out side the parallelogram.

To prove: $DR + BQ = AP$

Construction: Draw $DT \perp AP$



Proof: In quadrilateral $DRPT$, $\angle R = \angle P = \angle T = 90°$

$\therefore DRPT$ is a rectangle $\therefore DR = TP$ (1)

In ΔDAT and ΔCBQ

$DA = CB$

$\angle DAT = \angle CBQ$ (Angle between two parallel lines) $AO \parallel BC$ and $AT \parallel BQ$

$\angle DTA = \angle CQB = 90°$

$\therefore$ By AAS congruence

$\Delta DAT \cong \Delta CBQ$

$\therefore AT = BQ$ (2)

(1) + (2) $\therefore$ $DR + BQ = AT + TP$

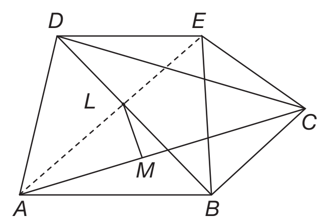
$\Rightarrow DR + BQ = AP$

Example 60 *L and M are the mid-points of the diagonals BD and AC respectively of the quadrilateral ABCD. Through D draw DE equal and parallel to AB. Show that EC is parallel to LM and is double of it.*

Solution:

Given quadrilateral $ABCD$, L and M are the mid-points of diagonals BD and AC respectively.

To prove: $LM \parallel EC$ and $LM = \frac{1}{2}EC$



Proof: Since $DE = AB$, $DE \parallel AB$ and in a quadrilateral if one pair of opposite side is equal and parallel then it is a parallelogram

$\therefore ABED$ is a parallelogram

Its diagonals bisects each other so L is also the midpoint of AE

In ΔAEC, L and M are the midpoint of AE and AC respectively

$\therefore$ By midpoint theorem $LM \parallel EC$

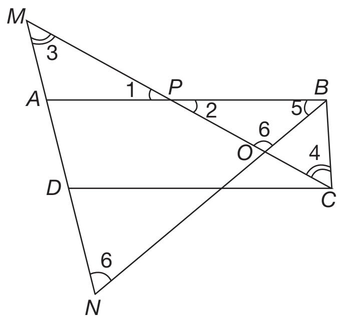
$\Rightarrow LM = \frac{1}{2}EC$ (Proved)

Example 61 *In a parallelogram ABCD, AB = 2BC · AD is produced both ways so that AM = AD = DN. Show that BN is perpendicular to CM*

Solution:

Given: $ABCD$ is a parallelogram with $AB = 2BC$, $AM = AD = DN$

To proof: $MC \perp BN$



Proof: In ΔPMA and ΔPCB

$\angle 1 = \angle 2$ VOA

$\angle 3 = \angle 4$ (Alternate interior angles)

$\therefore AM = BC$ (As $AM = AD = BC$)

$\therefore$ By AAS congruence

$\Delta PMA \cong \Delta PCB$

$AP = BP \Rightarrow P$ is the midpoint of AB

$\Rightarrow \quad BC = AP = PB = AM$ (As $AB = 2BC$)

$\therefore \angle 1 = \angle 3 = \angle 2 = \angle 4$

Also in $\Delta ABN, AB = 2BC,$

$AN = 2AD = 2BC$

$\therefore AB = AN$

$\therefore \angle 5 = \angle 6$

$\angle PAN = \angle 1 + \angle 3$ (Exterior angle property, in ΔAPM)

$= \angle 2 + \angle 2 = 2\angle 2 = \angle BAN$

In $\Delta ABN, \angle BAN + \angle BNA + \angle ABN = 180°$

$\Rightarrow 2\angle 2 + \angle 5 + \angle 6 = 180°$

$\Rightarrow 2\angle 2 + 2\angle 5 = 180°$

$\Rightarrow \angle 2 + \angle 5 = 90°$

In $\Delta POB, \quad \angle 2 + \angle 5 + \angle 7 = 180°$

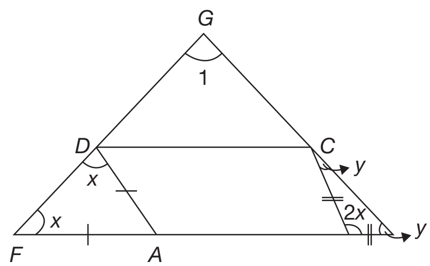
$\Rightarrow 90° + \angle 7 = 180°$

$\Rightarrow \angle 7 = 90°$

$\Rightarrow CM \perp BN$

Example 62 *The side AB of parallelogram is produced both ways to F and G, so that $AF = AD$ and $BG = BC$. Prove that FD and GC produced intersect at right angles.*

Solution:



Given: $ABCD$ is a parallelogram AB is produced both ways $AF = AD$ and $BG = BC$.

To prove: FD and GC produced cut at right angles

Proof: Since in $\Delta AFD, AF = AD \therefore \angle AFD = \angle ADF = x$ (Say)

$\angle DAB = 2x$ (Exterior angle theorem)

$AD \| CB \Rightarrow \angle CBG = \angle DAB = 2x$

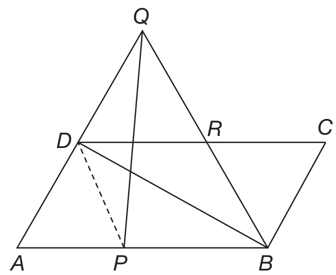
In $\Delta BCG, \quad BC = BG \Rightarrow \angle BCG = \angle BGC = y$ (Say)

And $2x + y + y = 180° \Rightarrow x + y = 90°$

In $\Delta FGH, x + y + \angle 1 = 180°$

$\Rightarrow \angle 1 = 90°.$ (As $x + y = 90°$)

Example 63 *In the sides AB, AQ of ΔAQB, the points P and D are so chosen that $[APQ] = [ABD]$. DC is drawn parallel to AB to cut BQ in R. BC drawn parallel to AD meets DR produced in C. Prove that $RC = AP$.*



Solution:

Construction: Join PD

Proof: Since [APQ] = [ABD], subtract area of ΔAPD to both sides.

$[APQ] - [APD] = [ABD] - [APD] \Rightarrow [PDQ] = [PDB]$

Triangles having same base and equal areas must lie between the same parallel

$\Rightarrow PD \,||\, BQ$ Or $PD \,||\, BR$ also $DR \,||\, PB$ (Given)

$\therefore$ $DPBR$ is a parallelogram $BP = DR$ (1)

Also $DC \,||\, AB$ and $AD \,||\, BC$

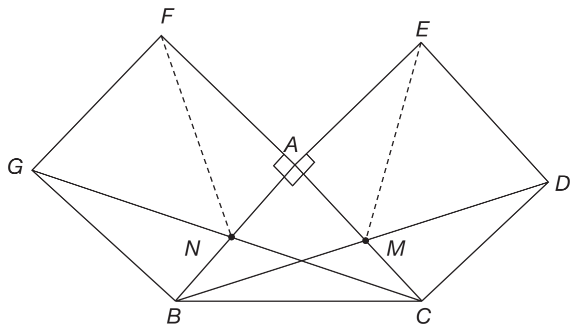
$\therefore$ $ABCD$ is also a parallelogram $\Rightarrow AB = DC$ (2)

From Eq. (2) – Eq. (1) we get $AB - BP = DC - DR$

$AP = RC$ proved.

Example 64 *In ΔABC, $\angle A$ is a right angle. Squares ACDE and ABGF are described on AC and AB externally to the triangle. BD cuts AC in M and CG cuts AB in N. Show that $AM = AN$.*

Solution:



Constructions: Join FN and ME.

Proof: Since if a triangle and a parallelogram having the same base and between the same parallel then area of triangle is half the area of parallelogram.

$$[GNF] = \frac{1}{2}[GBAF]$$

And $[GBN] + [AFN] = \frac{1}{2}[GBAF]$

Also $[GBC] = \frac{1}{2}[GBAF]$ (Between two parallels FC and GB with same base GB)

$\Rightarrow [GBN] = [AFN] = [GBC] = [GBN] + [NBC]$

$\Rightarrow [AFN] = [NBC]$

Adding $[ANC]$ to both sides, we get,

$[AFN] + [ANC] = [NBC] + [ANC]$

$\Rightarrow [FNC] = [ABC]$ (1)

Similarly $[DCM] + [EAM] = \frac{1}{2}[ACDE] = [DCB]$

$$\Rightarrow [DCM]+[EAM]=[DCM]+[BCM]$$

$$\Rightarrow [EAM]=[BCM]$$

$\Rightarrow [EAM] + [AMB] = [BCM] + [AMB]$ (Adding $[AMB]$ to both sides)

$\Rightarrow [EMB] = [ABC]$ (2)

$\therefore$ From Eqs. (1) and (2), we get, $[FNC] = [EMB]$

$$\Rightarrow \frac{1}{2} FC \times AN = \frac{1}{2} EB \times AM$$

$\Rightarrow AN = AM.$ ($FC = EB$ as $FC = FA + AC = AB + AE = EB$)

Aliter: $\Delta CAN \sim \Delta CFG$

$$\frac{AN}{FG} = \frac{CA}{CF}$$

$\Rightarrow \dfrac{AN}{AB} = \dfrac{CA}{AC + AB}$ (As $FG = AB$)

$\Rightarrow AN = \dfrac{AB \cdot CA}{AC + AB}$ (1)

Also $\Delta BAM \sim \Delta BED$

$$\frac{AM}{ED} = \frac{BA}{BE}$$

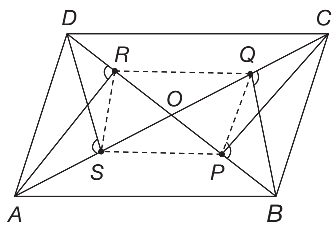
$\Rightarrow \dfrac{AM}{AC} = \dfrac{AB}{AB + AC}$ (As $ED = AC$)

$\Rightarrow AM = \dfrac{AB \cdot AC}{AB + AC}$ (2)

From Eqs. (1) and (2)
$AN = AM$

Example 65 *Prove that the feet of the perpendiculars drawn from the vertices of a parallelogram onto its diagonals are the vertices of another parallelogram.*

Solution:



Let the diagonals of the given parallelogram $ABCD$ intersects at O and P, Q, R, S are the feet of the perpendiculars from the vertices on the diagonals. In triangles OSD and OQB, we have $\angle OSD = \angle OQB = 90°$.

$\angle SOD = \angle QOB$ (VOA)

$OD = OB$ (Diagonals bisects each other)

By AAS congruence

$\Delta OSD \cong \Delta OQB$

$\Rightarrow OS = OQ$ (CPCT)

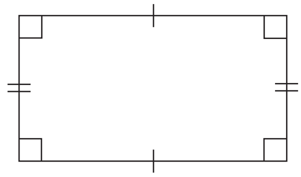
Similarly $\Delta ORA \cong \Delta OPC$

$\Rightarrow OR = OP$ (CPCT)

Thus in quadrilateral $PQRS$, diagonals bisects each other and consequently $PQRS$ is a parallelogram.

8.9.2 Rectangle

A parallelogram in which any one angle is right angle is called rectangle.

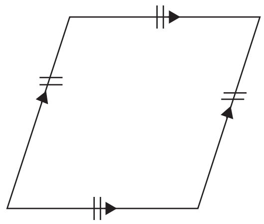


Properties:

1. Opposite sides are parallel and equal.
2. Opposite angles are equal and of 90°.
3. Diagonals are equal and bisects each other.
4. When a rectangle is inscribed in a circle the diameter of the circle is equal to the diagonal of the rectangle.
5. For the given perimeter of rectangle, a square has the maximum area.
6. The figure formed by joining the mid-points of the adjacent sides of a rectangle is a rhombus.
7. The quadrilateral formed by joining the intersection of the angle bisectors of a parallelogram is a rectangle.
8. If P is any point in the plane of the rectangle $ABCD$, then $PA^2 + PC^2 = PB^2 + PD^2$.

8.9.3 Rhombus

A parallelogram in which any two adjacent sides are equal is called rhombus.



Properties:

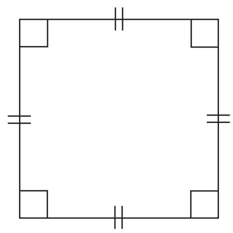
1. Opposite sides are parallel.
2. All sides are equal.
3. Diagonals are perpendicular bisectors to each other.
4. Diagonals bisects the opposite pair of angles.
5. Figure formed by joining the mid-points of the adjacent sides of a rhombus is a rectangle.
6. A parallelogram is a rhombus if its diagonals are perpendicular to each other.
7. Any parallelogram circumscribing a circle is a rhombus.
8. Area of rhombus $= \frac{1}{2} \times$ Product of diagonals

   $=$ Base $\times$ Height

   $=$ Product of adjacent sides $\times$ Sine of the included angle

8.9.4 Square

Square is a rectangle whose all sides are equal or a rhombus whose all angles are equal thus each square is a parallelogram, a rectangle and a rhombus.



Properties:

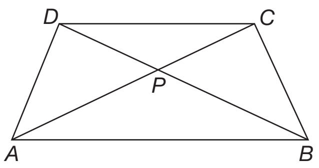
1. All sides are equal.
2. Opposite pair of sides are equal.
3. Diagonals are equal and are perpendicular bisector to each other.
4. Diagonal of an inscribed square is equal to the diameter of the circumscribing circle.
5. Side of a circumscribed square is equal to the diameter of the inscribed circle.
6. The figure formed by joining the mid-points of the adjacent side of a square is a square
7. Angles formed by the diagonals and a side of square is each equal to 45°.

8.9.5 Trapezium

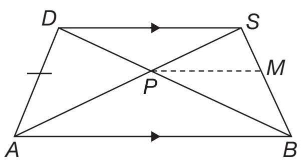
A quadrilateral whose one pair of side is parallel.

Properties:

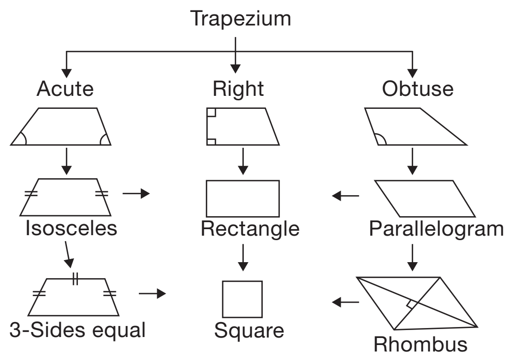
1. The line joining the mid-points of the oblique (non-parallel) sides is half the sum of the parallel sides.

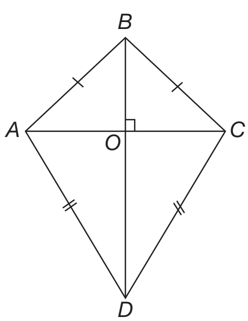


2. If the non-parallel sides are equal then the diagonals will also be equal to each other and converse is also true. The corresponding trapezium is called isosceles trapezium.
3. Diagonals intersects each other proportionally in the ratio of lengths of parallel sides, *i.e.*, $\frac{AP}{PC} = \frac{BP}{PD} = \frac{AB}{DC}$.
4. By joining the mid-points of adjacent sides of a trapezium four similar triangles are obtained.
5. If a trapezium is inscribed in a circle then it is an isosceles trapezium with equal oblique sides.
6. Area of trapezium $= \frac{1}{2} \times$ (sum of the parallel sides) $\times$ Height.
7. If $ABCD$ is a trapezium with $AB \parallel CD$ then $AC^2 + BD^2 = AD^2 + BC^2 + 2AB \cdot CD$
8. In an isosceles trapezium base angles are equal and other two angles are also equal.
9. If $ABCD$ is an isosceles trapezium with ΔORS and diagonals intersects at P then following results are true.



(i) $AD = BC$
(ii) $AC = BD$
(iii) $AP = PB$; $PD = PC$
(iv) $PA \times PC = PB \times PD$
(v) $\frac{PC}{PA} = \frac{PD}{PB}$
(vi) $\angle PAB = \angle PBA = \angle PDC = \angle PCD$
(vii) $\angle DAB = \angle CBA$; $\angle ADC = \angle BCD$
(viii) $\angle PAD = \angle PBC$; $\angle ADB = \angle ACB$
(ix) $AC^2 = AD^2 + AB \cdot CD$
(x) If $PM \parallel AB \parallel CD$ then $\frac{1}{PM} = \frac{1}{AB} + \frac{1}{CD}$ or $\frac{AB \cdot CD}{AB + CD}$
(xi) $ABCD$ is a cyclic quadrilateral and then all the properties of cyclic quadrilateral also apply. In this case it will be an isosceles trapezium.





8.9.6 Kite

In a kite two pairs of adjacent sides are equal

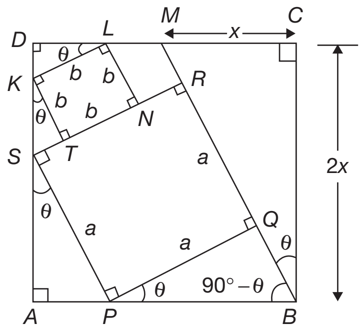
Properties:

1. $AB = BC$ and $AD = CD$.
2. Diagonals intersects at right angle.
3. Longer diagonal is the perpendicular bisector of shorter diagonal.
4. The quadrilateral formed by the mid-points of the adjacent the sides of a kite is a rectangle.
5. Area is $\frac{1}{2}$ product of diagonals

Note: Area of any quadrilateral $= \frac{1}{2} \times$ Product of diagonals $\times$ Sine of the included angle between diagonals.

Example 66 *ABCD is a square. M is a mid-point of CD. PQRS is a square of maximum possible area in trapezium ABMD. KLNT is another square as shown in diagram whose area is* $180\,cm^2$. *Find area of square PQRS and area of square ABCD.*

Solution:



Let $AB = BC = 2x \Rightarrow CM = x$, Let $PQ = a$, $KL = b$, Let $\angle CBM = \theta$

In ΔCBM, $\tan\theta = \frac{x}{2x} = \frac{1}{2} \quad \Rightarrow \quad \sin\theta = \frac{1}{\sqrt{5}}$ and $\cos\theta = \frac{2}{\sqrt{5}}$

In ΔBPQ, $\cos\theta = \frac{a}{BP} = \frac{2}{\sqrt{5}} \quad \Rightarrow \quad BP \frac{\sqrt{5}a}{2}$

In ΔAPS, $\sin\theta = \frac{AP}{a} = \frac{1}{\sqrt{5}} \quad \Rightarrow \quad AP = \frac{a}{\sqrt{5}}$

And $\cos\theta = \frac{AS}{a} = \frac{2}{\sqrt{5}} \quad \Rightarrow \quad AS = \frac{2a}{\sqrt{5}}$

In ΔKTS, $\cos\theta = \frac{b}{KS} = \frac{2}{\sqrt{5}} \Rightarrow KS = \frac{\sqrt{5}\,b}{2}$

In ΔKDL, $\sin\theta = \frac{KD}{b} = \frac{1}{\sqrt{5}} \quad \Rightarrow \quad KD = \frac{b}{\sqrt{5}}$

Since $AD = AB$

$KD + KS + SA = AP + PB$

$$\frac{b}{\sqrt{5}} + \frac{\sqrt{5}b}{2} + \frac{2a}{\sqrt{5}} = \frac{a}{\sqrt{5}} + \frac{\sqrt{5}a}{2}$$

$\Rightarrow 2b + 5b + 4a = 2a + 5a$ (Multiplying $2\sqrt{5}$ on both side)

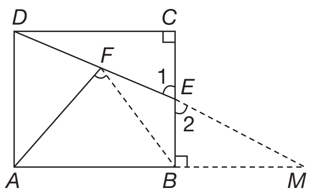
$\Rightarrow 7b = 3a$

$\therefore \frac{b}{a} = \frac{3}{7} \Rightarrow \frac{b^2}{a^2} = \frac{9}{49} \Rightarrow \frac{180}{a^2} = \frac{9}{49}$ (As $b^2 = [KLNT] = 180$)

$\Rightarrow [PQRS] = a^2 = 20 \times 49 = 980 \Rightarrow a = 14\sqrt{5}$

$\Rightarrow 2x = AB = AP + PB = \frac{a}{\sqrt{5}} + \frac{a\sqrt{5}}{2} = \frac{7a}{2\sqrt{5}} = \frac{7 \times 14\sqrt{5}}{2\sqrt{5}} = 49$

$\Rightarrow [ABCD] = (2x)^2 = 49^2 = 2401\,\text{cm}^2$.



Example 67 *In the figure ABCD is a square. E is the mid-point of CB. AF is drawn perpendicular to DE. If side of the square is 2017 cm find the length of FB in cm.*

Solution:

Construction: Produce DE to cut AB produce at M.

Proof: In ΔECD and ΔEBM

$\angle 1 = \angle 2$ (VOA)

$EC = EB$ (Given)

$\angle ECD = \angle EBM = 90°$

$\therefore$ By ASA congruences

$\Delta ECD \cong \Delta EBM$

$\therefore$ $CD = BM$

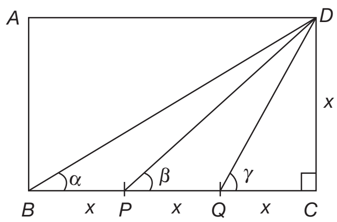
But $CD = AB$

$\therefore$ $AB = BM$

i.e., In right angled triangle AFM, B is the mid-point of the hypotenuse AM

$BF = BA = 2017$ cm.

Example 68 *Let ABCD be a rectangle such that BC = 3AB. P and Q are points on the side BC such that BP = PQ = CQ. Using geometrical or trigonometrical relations or otherwise show that ∠DBC + ∠DPC = ∠DQC.*



Solution: Let $CD = x$ then $AD = 3x = BC$

$\therefore$ $BP = PQ = QC = x$

In ΔDBC, $\tan\alpha = \dfrac{x}{3x} = \dfrac{1}{3}$

In ΔDPC, $\tan\beta = \dfrac{x}{2x} = \dfrac{1}{2}$

ΔDQC, $\tan\gamma = \dfrac{x}{x} = 1 = \tan 45° \Rightarrow \gamma = 45°$

Let us consider $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan a \tan\beta} = \dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}} = \dfrac{\frac{5}{6}}{\frac{5}{6}} = 1 = \tan 45°$

$\therefore \alpha + \beta = 45° = \gamma$

$\therefore \angle DQC = \angle DBC + \angle DPC.$

Aliter: Using Baudhayana theorem we get

$BD = \sqrt{10}x, DP = \sqrt{5}x$ and $DQ = \sqrt{2}x$

Since, $\dfrac{BD}{DP} = \dfrac{DQ}{PQ} = \dfrac{BQ}{DQ} = \dfrac{\sqrt{2}}{1}$

$\therefore$ By SSS similarly

$\therefore$ $\Delta BDQ \sim \Delta DPQ$

$\angle DBQ = \angle PDQ = \alpha$

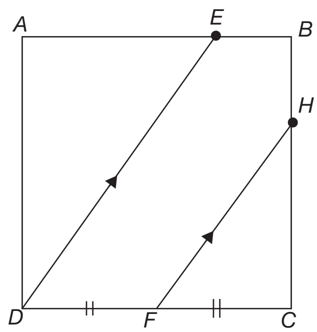
In ΔDPQ

$\angle DQC = \angle DPQ + \angle PDQ$ [By exterior angle property]

$\Rightarrow \gamma = \alpha + \beta$

Build-up Your Understanding 7

1. $ABCD$ is a parallelogram. The side CD is bisected at P and BP meets AC at X. Prove that $3AX = 2AC$.
2. $ABCD$ is a parallelogram. X divides AB in the ratio 3 : 2 and Y divides CD in the ratio 4 : 1. If XY cuts AC at Z, Prove that $7AZ = 3AC$.
3. $ABCD$ is a trapezium with $AB \parallel CD$ and $AB = 2CD$. If the diagonals meet at O, then prove that $3AO = 2AC$. If AD and BC meet at F, then prove that $AD = DF$.
4. $ABCD$ is a parallelogram. A straight line through A meets BD at X, BC at T and OC at Z. Prove that $AX : XZ = AY : AZ$.
5. $ABCD$ and $AECF$ are two parallelograms and side EF is parallel to AD. Suppose AF and DE meet at X and BF, CE meet at Y, then prove that $XY \parallel AB$.
6. In square $ABCD$, line segments are drawn from A to the mid-point of BC, from B to the mid-point of CD, from C to the mid-point of DA, and from D to the mid-point of AB. The four segments form a smaller square within square $ABCD$. If $AB = 1$, what is the area of the smaller square?
7. If area of a parallelogram is 6 units square and an octagon is formed by intersections of lines joining each vertex of the parallelogram to the mid-points of opposite sides of it, then find the area of the octagon.
8. Consider trapezium $ABCD$ such that $AB \parallel DC$, $AB = 4$, $DC = 10$, diagonals AC and BD are perpendiculars to each other. Sides DA and CB extended meets each other at E. $\angle DEC = 45°$. Find the area of the trapezium.
9. The distance between two parallel sides $\overline{AB}$ and $\overline{CD}$ of a trapezoid is 12 units. $AB = 24$ units; $CD = 15$ units. E is the mid-point of $\overline{AB}$. 'O' is the point of intersection of $\overline{DE}$ with $\overline{AC}$. Prove that the area of this quadrilateral $EBCO$ is 112 sq. units.
10. $ABCD$ is a quadrilateral and Q, P are mid-points of AB, CD respectively, AP and DQ meet at X; BP and CQ meet at Y; Prove, in the usual notation $[ADX] + [BCY] = [PXQY]$.
11. A trapezoid was formed by truncating an isosceles triangle ABC, through two points, taken as follows: D on AB and E on AC. BE and CD are connected. $\overline{BE} \cap \overline{CD} = \{F\}$. The area of the original triangle ABC is 60 cm^2 and that of the trapezoid is 45 cm^2. Find the area of ΔBFC.
12. Squares $ABDE$ and $ACFG$ are drawn outside $DABC$. Let P, Q be points on EG such that BP and CQ are perpendicular to BC. Prove that $BP + CQ \geq BC + EG$. When does equality hold?
13. Let $ABCD$ be a non-isosceles trapezium in which $AB \parallel CD$ and $AB > CD$. Further, $ABCD$ possesses in-centre I, which touches CD at E. Let, M be the mid-point of AB and MI meet CD at F. Show that $DE = FC$, if and only if, $AB = 2\ CD$.
14. Let, $ABCD$ be a square, F be the mid-point of DC, and E be any point on AB, such that $AE > EB$. H is a point on BC, such that FH is parallel to DE. Prove that EH is tangent to the inscribed circle of the square $ABCD$.

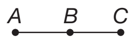


15. $ABCD$ is a parallelogram. E and F are the mid-points of AB and AD. Show that the area of the quadrilateral $AECF$ is half the area of the parallelogram $ABCD$.
16. $ABCD$ is a parallelogram and BF is drawn to intersect AC, DC and AD produced at E, G and F, respectively. Prove that EB is the geometric mean of EG and EF.

8.10 Concurrency and Collinearity

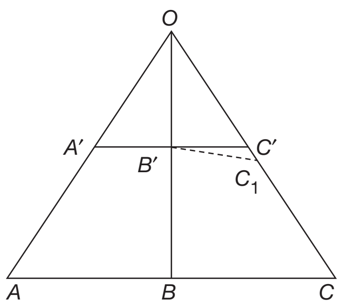
8.10.1 Definitions

1. A line segment joining a vertex of a triangle to any point on the opposite side (the point may be on the extension of the opposite side also) is called a cevian.
2. Three straight lines are said to be concurrent if all three passes through a common point.
3. Three points are said to be collinear, if they lie on a straight line.
4. **Directed length:** Given are any two distinct points A, B on a straight line. They determine a line segment of definite length. If we associate with this line segment, the direction from A to B, and denote it as AB. Then, the same line segment with the direction from B to A is denoted as BA, and we have $AB = -BA$ or $AB + BA = 0$. If M is a point on AB, such that M lies between A and B, then we may say that M divides the line segment AB in the ratio $AM : MB$, internally and, if the point N lies outside AB, then N divides AB in the ratio $AN : NB$, externally. Here, NB is negative, if we consider the direction from A to B as positive and hence, the ratio $AN : NB$ is negative on the other hand ratio $AM : MB$ is positive.

   If A, B, C are three points on a straight line in the order, we introduce a direction in the following manner
   AB, BC, AC are taken to be positive and BA, CB, CA are taken to be negative. Thus $AB + BC = AC$ and $AB + BC + CA = 0$

8.10.2 Theorem



If A, B, C and A', B', C' are points on two parallel lines such that $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$ then AA', BB', CC' are concurrent if they are not parallel.

Proof: Let AA' and BB' meet at O. where AA' and BB' are not parallel. Join OC and let it cut $A'B'$ at C_1.
By similarity

$$\frac{BC}{B'C_1} = \frac{OB}{OB'} = \frac{AB}{A'B'}$$

$$\therefore \quad \frac{BC}{B'C_1} = \frac{BC}{B'C'} \qquad \left(\text{As } \frac{AB}{A'B'} = \frac{BC}{B'C'}\right)$$

$$\Rightarrow \quad B'C_1 = B'C'$$

$\Rightarrow$ C_1 and C' coincide
Thus CC' passes through O.

Lazare Nicolas Marguerite, Count Carnot

13 May 1753–2 Aug 1823
Nationality: French

8.10.3 Carnot's Theorem

Let points D, E, and F be located on the sides BC, AC, and respectively AB of ΔABC. The perpendiculars to the sides of the triangle at points D, E, and F are concurrent if and only if

$$BD^2 - DC^2 + CE^2 - \mathrm{E}A^2 + AF^2 - FB^2 = 0$$

Proof: Let us first prove if perpendiculars are concurrent then results hold.
Let O be point of concurrency and OD, OE, OF are drawn perpendicular to the sides BC, CA, AB respectively of a triangle ABC

$$BD^2 = OB^2 - OD^2$$

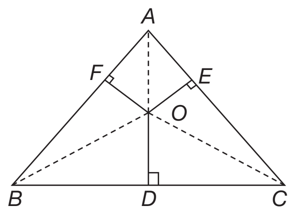
$$DC^2 = OC^2 - OD^2$$

$$\Rightarrow \quad BD^2 - DC^2 = OB^2 - OC^2 \qquad (1)$$

Similarly

$$CE^2 - EA^2 = OC^2 - OA^2 \qquad (2)$$

$$AF^2 - FB^2 = OA^2 - OB^2 \qquad (3)$$

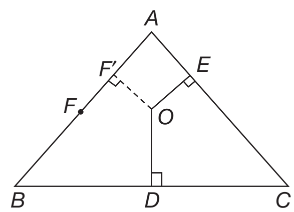


Adding Eqs. (1), (2) and (3), we get,

$$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = OB^2 - OC^2 + OC^2 - OA^2 + OA^2 - OB^2 = 0.$$

Proof of Converse: If D, E, F be points on the sides BC, CA, AB of a triangle ABC such that $BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0$, then the perpendiculars at D, E, F to the respective sides are concurrent.

Proof: Let the perpendiculars at D, E, to BC, CA respectively meet at O. Let OF' be the perpendicular from O to AB
Using previous result:



$$BD^2 - DC^2 + CE^2 - AE^2 + AF'^2 - F'B^2 = 0 \qquad (1)$$

But it is given that

$$BD^2 - DC^2 + CE^2 - AE^2 + AF^2 - FB^2 = 0 \qquad (2)$$

$\therefore$ From Eqs. (1) and (2)

$$AF'^2 - F'B^2 = AF^2 - FB^2$$

$$(AF' + F'B)(AF' - F'B) = (AF + FB)(AF - FB)$$

$$AB(AF' - F'B) = AB(AF - FB)$$

$$\Rightarrow \quad AF' - F'B = AF - FB \;\; (\text{As } AB \neq 0)$$

$$\therefore \quad AF - AF' = FB - F'B$$

$$\Rightarrow FF' = -F'F$$

$$\Rightarrow 2FF' = 0 \;\Rightarrow\; FF' = 0$$

That is, F and F' coincide

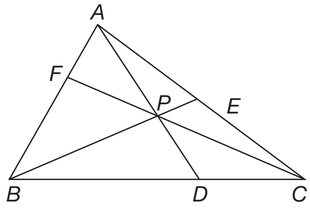
8.10.4 Ceva's Theorem

If points D, E, F are taken on the sides BC, CA, AB of ΔABC so that the lines AD, BE, CF are concurrent at a point P, then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1 \qquad (\text{OR}) \qquad BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$$

Giovanni Ceva

7 Dec 1647–15 Jun 1734
Nationality: Italian



Proof: By Ratio proportion theorem (or area lemma), we have $\dfrac{[ABD]}{[ADC]} = \dfrac{BD}{DC}$ (1)

And $$\frac{[BPD]}{[CPD]} = \frac{BD}{DC} \quad (2)$$

$\therefore$ From Eqs. (1) and (2)

$$\frac{BD}{DC} = \frac{[ABD]}{[ADC]} = \frac{[BPD]}{[CPD]} = \frac{[ABD]-[BPD]}{[ADC]-[CPD]} = \frac{[ABP]}{[ACP]}$$

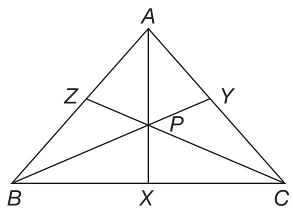
Let $[BPC] = \Delta_1$, $[ACP] = \Delta_2$ and $[ABP] = \Delta_3$

$$\therefore \quad \frac{BD}{DC} = \frac{\Delta_3}{\Delta_2}$$

Similarly $\dfrac{CE}{EA} = \dfrac{[BPC]}{[APB]} = \dfrac{\Delta_1}{\Delta_3}$

$$\frac{AF}{FB} = \frac{[APC]}{[BPC]} = \frac{\Delta_2}{\Delta_1}$$

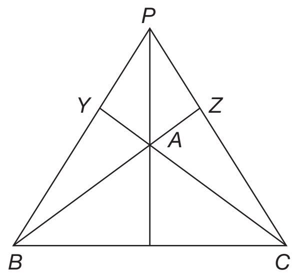
$$\therefore \quad \frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = \frac{\Delta_3}{\Delta_2}\cdot\frac{\Delta_1}{\Delta_3}\cdot\frac{\Delta_2}{\Delta_1} = 1.$$



Notes:

1. In the figure BX and XC are of same sign. CY and YA are of same sign and AZ and ZB are of same sign. Thus

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} \text{ is positive.}$$



2. In Ceva's theorem, if P lies outside as in the figure, then BX, XC are positive, CY is positive, YA is negative, AZ is positive ZB is negative. Thus $\dfrac{BX}{XC}$ is positive; $\dfrac{CY}{YA}$ is negative, $\dfrac{AZ}{ZB}$ is negative.

Hence $\dfrac{BX}{XC}\cdot\dfrac{CY}{YA}\cdot\dfrac{AZ}{ZB}$ is positive.

Thus $\dfrac{BX}{XC}\cdot\dfrac{CY}{YA}\cdot\dfrac{AZ}{ZB} = +1.$

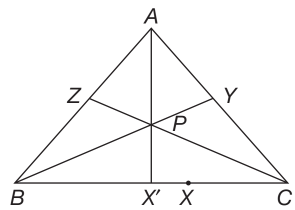
8.10.4.1 Trigonometric Form of Ceva's Theorem

Let X, Y, Z be the points taken respectively on the sides BC, CA, AB of ΔABC. Then the lines AX, BY, CZ are concurrent if only if

$$\frac{\sin\angle CAX}{\sin\angle XAB}\cdot\frac{\sin\angle ABY}{\sin\angle YBC}\cdot\frac{\sin\angle BCZ}{\sin\angle ZCA} = 1.$$

8.10.4.2 Converse of Ceva's Theorem

If three cevian AX, BY, CZ satisfy $\dfrac{BX}{XC}\cdot\dfrac{CY}{YA}\cdot\dfrac{AZ}{ZB} = +1,$ then they are concurrent



Proof: Let BY and CZ meet at P
Let AP meet BC at X'.

Then by Ceva's theorem

$$\frac{BX'}{X'C}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=+1 \tag{1}$$

But it is given that

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=+1 \tag{2}$$

$\therefore$ From Eqs. (1) and (2) we have

$$\frac{BX'}{X'C}=\frac{BX}{XC}$$

Adding 1 to both sides

$$\frac{BX'}{X'C}+1=\frac{BX}{XC}+1$$

$$\Rightarrow \quad \frac{BX'+X'C}{X'C}=\frac{BX+XC}{XC}$$

$$\Rightarrow \quad \frac{BC}{X'C}=\frac{BC}{XC}$$

$$\Rightarrow \quad X'C=XC$$

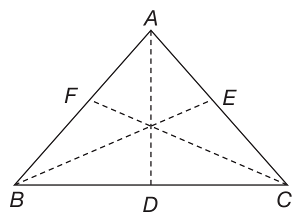
$$\text{or} \quad X'C-XC=0$$

$$\Rightarrow \quad X'X=0$$

Therefore X', X coincide. Thus the three cevians are concurrent.

Note: The converse of Ceva's theorem is more useful than the theorem in the sense that most of the elementary theorems regarding concurrency can be proved using the theorem

Example 69 *Proved that the medians of a triangle are concurrent.*



Solution: If D, E, F are the mid-points of BC, CA, AB respectively then $BD = DC$; $CE = EA$; $AF = AB$

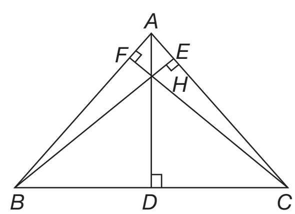
$$\therefore \quad \frac{BD}{DC}=1;\ \frac{CE}{EA}=1;\ \frac{AF}{FB}=1$$

$$\therefore \quad \frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$$

Thus the Cevians AD, BE, CF are concurrent.

For aliter please refer example 31 on page 8.27.

Example 70 *Prove that the altitudes of a triangle are concurrent.*



Solution: For acute angle triangle ABC

$$BE \perp CA,\ CF \perp AB$$

$\therefore$ In ΔAEB and ΔAFC

$\angle A = \angle A$ (Common)

$\angle AEB = \angle AFC = 90°$

$\therefore$ By AA similarity, $\Delta AEB \sim \Delta AFC$

$$\Rightarrow \frac{AF}{AE} = \frac{AC}{AB}$$

Similarly $AD \perp BC$ then $\Delta BFC \sim \Delta BDA$

$$\Rightarrow \frac{BD}{BF} = \frac{BA}{BC}$$

Also $\Delta CEB \sim \Delta CDA \Rightarrow \frac{CE}{CD} = \frac{CB}{CA}$

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \left(\frac{BD}{BF}\right) \cdot \left(\frac{CE}{CD}\right) \cdot \left(\frac{AF}{EA}\right)$$

$$= \frac{BA}{BC} \cdot \frac{BC}{CA} \cdot \frac{CA}{AB} = 1$$

Aliter 1:

$$\cot B = \frac{BD}{AD} \quad \Rightarrow \quad BD = AD \cot B$$

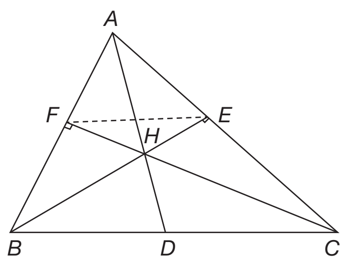
$$\cot C = \frac{DC}{AD} \quad \Rightarrow \quad DC = AD \cot C$$

$$\therefore \frac{BD}{DC} = \frac{\cot B}{\cot C}$$

Similarly, $\frac{CE}{EA} = \frac{\cot C}{\cot A}$ and $\frac{AF}{FB} = \frac{\cot A}{\cot B}$

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{\cot B}{\cot C} \cdot \frac{\cot C}{\cot A} \cdot \frac{\cot A}{\cot B} = +1$$

Hence AD, BE, CF are concurrent



Aliter 2: (Without Ceva's theorem)

Let $BE \perp AC$, $CF \perp AB$

Let BE, CF intersect at H.

Join AH and produce it to cut BC at D.

Now we have to prove $AD \perp BC$.

Since $\angle BFC = \angle BEC = 90°$

$\therefore B, F, E, C$ are concyclic

$$\therefore \angle BFE + \angle BCE = 180°$$

$$90° + \angle CFE + \angle BCE = 180°$$

$$\therefore \angle CFE + \angle BCE = 90° \text{ or } \angle CFE + \angle DCA = 90° \qquad (1)$$

Also $\angle HFA + \angle HEA = 90° + 90° = 180°$

$\therefore H, F, A, E$ are concyclic

$$\therefore \angle HFE = \angle HAE$$

$$\Rightarrow \quad \angle CFE = \angle HAE = \angle DAC \qquad (2)$$

$\therefore$ From Eqs. (1) and (2)

$$\angle DAC + \angle DCA = 90°$$

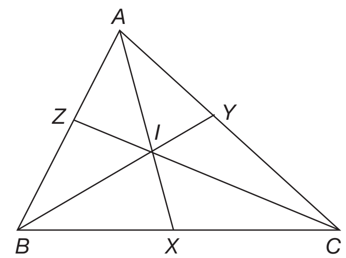
$\therefore$ By ASP of a triangle

In ΔADC

$$\angle ADC = 180° - (\angle DAC + \angle DCA) = 180° - 90° = 90°$$
$$\Rightarrow AD \perp BC.$$

Example 71 *Prove that internal bisectors of the angles of a triangle are concurrent.*

Solution: If AX, BY, CZ are the angle bisector then by internal angle bisector theorem we have following:



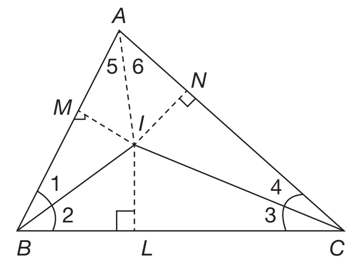
$$\frac{BX}{XC} = \frac{AB}{AC};\quad \frac{CY}{YA} = \frac{BC}{BA} \text{ and } \frac{AZ}{ZB} = \frac{AC}{BC}$$

$$\therefore \quad \frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = \frac{\cancel{AB}}{\cancel{AC}}\cdot\frac{\cancel{BC}}{\cancel{AB}}\cdot\frac{\cancel{AC}}{\cancel{BC}} = +1$$

Aliter: Let BI, CI, are the internal angle bisectors of $\angle B$ and $\angle C$ of ΔBAC. Join AI.

Now we have to prove that AI bisects $\angle A$.

Construction Draw $IL \perp$ BC, $IM \perp AB$ and $IN \perp AC$



Proof: In ΔIMB and ΔILM

$$\angle IBM = \angle ILM = 90°$$
$$\angle 1 = \angle 2 \quad \text{(Given)}$$
$$IB = IB \quad \text{(Common)}$$

$\therefore$ By AAS Congruences

$$\Delta IMB \cong \Delta ILB$$
$$\therefore IM = IL \quad \text{(CPCT)} \quad (1)$$

In ΔINC and ΔILC

$$\angle INC = \angle ILC = 90°$$
$$\angle 3 = \angle 4 \quad \text{(Given)}$$
$$IC = IC \quad \text{(Common)}$$

$\therefore$ By AAS congruences

$$\Delta INC \cong \Delta \text{ILC}$$
$$\therefore IN = IL \quad \text{(CPCT)} \quad (2)$$

$\therefore$ From Eqs. (1) and (2)

$$IL = IM = IN$$

Now in ΔIMA and ΔINA

$$\angle IMA = \angle INA = 90°$$
$$IM = IN \quad \text{(Proved above)}$$
$$IA = IA \quad \text{(Common)}$$

$\therefore$ By RHS congruences

$$\Delta IMA \cong \Delta INA$$
$$\Rightarrow \angle IAM = \angle IAN, \textit{ i.e.}, \angle 5 = \angle 6.$$

Thus AI bisects $\angle A$ and thus in a triangle all the angle bisectors are concurrent.

Notes:

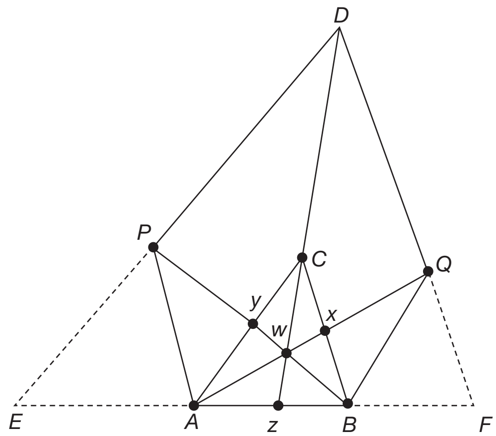
1. Taking I as a centre and IL as a radius draw a circle which passes through M and N and is called in-circle of a triangle, its radius IL is called in-radius and its centre I is called in-centre.
2. If the in-circle of ΔABC touches BC, CA, AB at L, N, M respectively then AL, BN, CM, are also concurrent (the point of concurrence is a called the ***Gergonne Point*** of ΔABC).

Joseph Diaz Gergonne

19 Jun 1771–4 May 1859
Nationality: France

With regard the ***Gergonne Point,*** it is interesting to note the following more general result which is know as ***Problem Of Joseph Diez Gergonne.***

Example 72 *If through the vertices of a ΔABC, two lines AP, BQ of arbitrary length are drawn in the direction of C such that AP parallel to BC and BQ parallel to AC, and if lines PD and QD are drawn respectively parallel to BQ and AP, meeting in D, then the lines AQ, BP and CD are concurrent.*



Proof: Let AQ cut BC in X, BP cut AC in Y and let AQ and BP intersect at W, let DC meet AB in Z. We will prove that DC passes through W.

In ΔQXB and ΔAXC

$$\angle QXB = \angle AXC \qquad \text{(VOA)}$$

$$\angle QBX = \angle ACX \qquad \text{(Alternate interior angles)}$$

$\therefore$ By AA similarity $\Delta QXB \sim \Delta AXC$

$$\therefore \quad \frac{BX}{XC} = \frac{QB}{AC} \qquad (1)$$

In ΔBYC and ΔPYA

$$\angle BYC = \angle PYA \quad \text{(VOA)}$$

$$\angle BCY = \angle PAY \qquad \text{(Alternate interior angles)}$$

$\therefore$ By AA similarly

$$\Delta BYC \sim \Delta PYA$$

$$\Rightarrow \quad \frac{CY}{YA} = \frac{BC}{AP} \qquad (2)$$

Let DP and BA meet in E and let DQ, AB meet in F.

In ΔEAP and ΔBFQ

$$\angle EAP = \angle ABC = \angle BFQ \qquad \text{(Corresponding angles)}$$
$$\angle AEP = \angle BAC = \angle FBQ \qquad \text{(Corresponding angles)}$$
$$\therefore \Delta EAP \sim \Delta ABC \sim \Delta BFQ$$

So from $\Delta EAP \sim \Delta ABC$, we get,

$$\therefore \quad \frac{EA}{AB} = \frac{AP}{BC} = \frac{PE}{CA} = \lambda \tag{3}$$

And from $\Delta BFQ \sim \Delta ABC$, we get,

$$\frac{BF}{AB} = \frac{FQ}{BC} = \frac{QB}{CA} = \mu \tag{4}$$

Hence

$$\frac{\frac{EA}{AB}}{\frac{BF}{AB}} = \frac{\lambda}{\mu}$$

$$\Rightarrow \quad \frac{EA}{BF} = \frac{\lambda}{\mu} \tag{5}$$

Since in ΔAZC and ΔEZD

$$\angle ZAC = \angle ZED \qquad \text{(Corresponding)}$$
$$\angle AZC = \angle EZD \qquad \text{(Common)}$$

$\therefore$ By AA similarity

$$\Delta AZC \sim \Delta EZD$$

$$\Rightarrow \quad \frac{AZ}{EZ} = \frac{ZC}{ZD} \tag{6}$$

Also, In ΔZBC and ΔZFD

$$\angle ZBC = \angle ZFD \qquad \text{(Corresponding angles)}$$
$$\angle BZC = \angle FZD \qquad \text{(Common)}$$

$\therefore$ By AA similarly

$$\Delta ZBC \sim \Delta ZFD$$

$$\therefore \quad \frac{ZB}{ZF} = \frac{ZC}{ZD} \tag{7}$$

From Eqs. (6) and (7)

$$\frac{AZ}{EZ} = \frac{ZB}{ZF}$$

$$\Rightarrow \quad \frac{AZ}{ZB} = \frac{EZ}{ZF} = \frac{EZ - AZ}{ZF - ZB} = \frac{EA}{BF} = \frac{\lambda}{\mu} \tag{8}$$

From Eqs. (1) and (4) $\dfrac{BX}{XC} = \dfrac{QB}{AC} = \mu$ (9)

From Eqs. (2) and (3) $\dfrac{CY}{YA} = \dfrac{BC}{AP} = \dfrac{1}{\lambda}$ (10)

$\therefore$ From Eqs. (8), (9) and (10), we get

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{\lambda}{\mu} \cdot \mu \cdot \frac{1}{\lambda} = 1$$

Hence by converse of Ceva's theorem the lines AQ, BP and CD are concurrent at W.

Notes:

1. If we take $AP = BQ = AB$, then

   $\dfrac{AZ}{ZB} = \dfrac{\lambda}{\mu} = \dfrac{AP / BC}{QB / CA} = \dfrac{CA}{CB}$ (from Eqs. (9) and (10))

   Also $\dfrac{BX}{XC} = \dfrac{QB}{AC} = \dfrac{AB}{AC}$ (from Eq. (1))

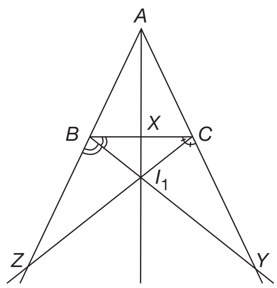
   Thus by converse of internal angle bisector theorem CZ and AX are angle bisector of $\angle C$ and $\angle A$ respectively and hence W is the in-centre of ΔABC.
2. If we take $AP = BC$ and $BQ = AC$ then from Eqs. (2) and (1), $\dfrac{CY}{YA} = 1$, *i.e.*, $CY = YA$ and $\dfrac{BX}{XC} = 1$, *i.e.*, $BX = XC$ then W is the centroid of ΔABC.
3. Finally if X and Y are the points of contact of the in-circle and P is taken as the point at which BY cuts the parallel through A and Q the point at which AX cuts the parallel through B, then W is the **Gergonne Point** of ΔABC.

Example 73 *Prove that the internal angle bisector of an angle of a triangle and the other two external bisectors are concurrent.*

Solution: Given In ΔABC

AX, is the internal angle bisector of $\angle BAC$. BY, CZ are the exterior angle bisector of $\angle B$ and $\angle C$ which cuts AC produced at Y and AB produced at Z respectively.

To prove: AX, BY, CZ are concurrent



Proof: In ΔABC by internal angle bisector theorem $\dfrac{AB}{AC} = \dfrac{BX}{XC}$

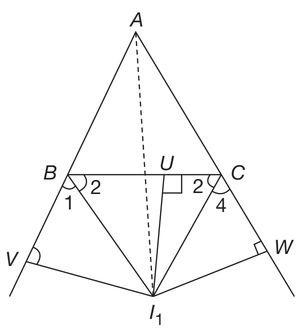
By exterior angle bisector theorem $\dfrac{BC}{BA} = \dfrac{CY}{YA}$ and $\dfrac{CA}{CB} = \dfrac{AZ}{ZB}$

$$\therefore \quad \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{AB}{AC} \times \frac{BC}{AB} \times \frac{AC}{BC} = +1$$

Note: Here CY is positive and YA is negative

$\therefore \quad \dfrac{CY}{YA}$ is negative and AZ is positive and ZB is negative

$\therefore \quad \dfrac{AZ}{ZB}$ is negative but their product is positive.



Aliter: Let BI_1, CI_1 are the exterior angle bisectors. Join AI_1
Now our aim is to prove AI_1 is the angle bisector of $\angle BAC$

Construction: Draw $I_1U \perp$ BC

$I_1V \perp AB$ produced

$I_1W \perp AC$ produced

Proof: In ΔBI_1U and ΔBI_1V

$$\angle BUI_1 = \angle BVI_1 = 90°$$

$$\angle 1 = \angle 2 \quad \text{(Given)}$$

$$BI_1 = BI_1 \quad \text{(Common)}$$

$\therefore$ By AAS congruencies

$$\Delta BUI_1 \cong \Delta BVI_1$$

$$\Rightarrow \quad I_1U = I_1V \quad (1)$$

In ΔCUI_1 and ΔCWI_1

$$\angle CUI_1 = \angle CWI_1 = 90°$$

$$\angle 3 = \angle 4$$

$$CI_1 = CI_1 \quad \text{(Common)}$$

$\therefore$ By AAS congruence

$$\Delta CUI_1 \cong \Delta CWI_1$$

$$I_1U = I_1W \quad (2)$$

$\therefore$ From Eqs. (1) and (2) $I_1U = I_1V = I_1W$ (3)

In ΔAVI_1 and ΔAWZ_1

$$\angle AVI_1 = \angle AWI_1$$

$$I_1V = I_1W \quad \text{(From Eq. (3))}$$

$$AI_1 = AI_1 \quad \text{(Common)}$$

$\therefore$ By RHS congruence

$$\Delta AVI_1 \cong \Delta AWI_1$$

$$\Rightarrow \quad \angle I_1AV = \angle I_1AW$$

Hence AI_1 bisects $\angle BAC$.

Example 74 *Let ABC be a triangle and let D, E, F be the points on its sides such that starting at A, D divides the perimeters of the triangle into two equal parts, starting at B, E divides the perimeter of the triangle into two equal parts and starting at C, F divides the perimeter of the triangle into two equal parts. Prove that D, E, F, lie on the sides BC, CA, AB respectively and the lines AD, BE, CF are concurrent.*

Solution: Let $2s = a + b + c$ be the perimeter of ΔABC.

Now $c < a + b$ and $b < c + a \Rightarrow c + c < a + b + c$ and $b + c + a < c + a + c + a$

$$\Rightarrow \quad 2c < a + b + c < 2(c + a)$$

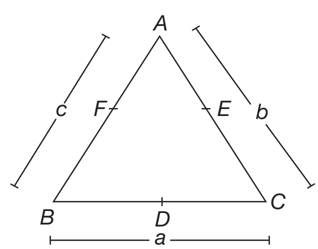
$$\Rightarrow \quad 2c < 2s < 2(c + a)$$

$$\Rightarrow \quad c < s < c + a$$

So, $AB = c < s = AB + BD < c + a = AB + BC$

i.e., $AB + BD < AB + BC$

$\therefore$ D lies on BC.



Similarly E lies on CA and F on AB

Also

$$c + BD = s = DC + b$$

$$\therefore \quad BD = s - c \text{ and } DC = s - b$$

and $a + CE = s = EA + C$

$$\Rightarrow \quad CE = s - a;\ EA = s - c$$

and $b + AF = s = FB + a$

$$\Rightarrow \quad AF = s - b \text{ and } FB = s - a$$

Hence

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = \left(\frac{s-c}{s-b}\right)\left(\frac{s-a}{s-c}\right)\left(\frac{s-b}{s-a}\right) = 1$$

Hence by converse of ceva's theorem AD, BE and CF are concurrent.

Note: D, E, F are the points where ex-circles are touching the sides of the triangle.

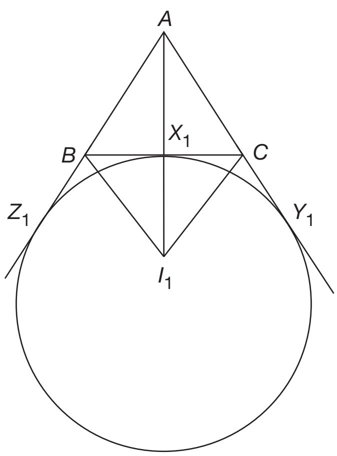
Christian Heinrich van Nagel

28 Feb 1803–27 Oct 1882
Nationality: German

Example. 75 *If the ex-circle of ΔABC, opposite the vertices A, B, C touch BC, CA, AB at X_1, Y_2, Z_3 respectively then prove that AX_1, BY_2, CZ_3 are concurrent (the point of concurrence is called the **Nagel Point** of ΔABC).*

Solution: Let the ex-circle opposite A touch AC produced at Y_1 and AB produced at Z_1 then $BX_1 = BZ_1$, and $CX_1 = CY_1$

And



$$AZ_1 = AY_1$$

Hence $AB + BZ_1 = AC + CY_1$

$$AB + BX_1 = AC + CX_1 = \frac{1}{2}(AB + BX_1 + X_1C + AC)$$

$$\therefore \quad AB + BX_1 = AC + CX_1 = \frac{1}{2}(AB + BC + CA) = s$$

Hence X_1 bisects the perimeter of ΔABC and lies on BC.

Similarly Y_2, Z_3 lies on AC and AB and bisects the perimeter of ΔABC.

Also $BX_1 = s - c$, $CX_1 = s - b$, $CY_2 = s - a$, $AY_2 = s - c$, $AZ_3 = s - b$ and $BZ_3 = s - a$

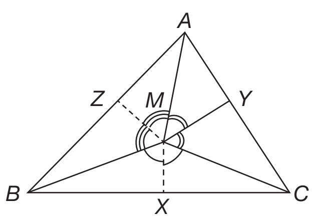
$$\therefore \quad \frac{BX_1}{X_1C}\cdot\frac{CY_2}{Y_2A}\cdot\frac{AZ_3}{Z_3B} = \left(\frac{s-c}{s-b}\right)\left(\frac{s-a}{s-c}\right)\left(\frac{s-b}{s-a}\right)$$

$$\Rightarrow \frac{BX_1}{X_1C}\cdot\frac{CY_2}{Y_2A}\cdot\frac{AZ_3}{Z_3B} = 1$$

Hence by converse of Ceva's Theorem, AX_1, BY_2, CZ_3 are concurrent.

Example 76 *M is an interior point of a triangle ABC. Bisectors of interior angles BMC, CMA, AMB intersect BC, CA, AB respectively at X, Y, Z. Prove that AX, BY, CZ are concurrent.*

If P is the point of concurrence and $\frac{PA}{PX}\cdot\frac{PB}{PY}\cdot\frac{PC}{PZ} = 8$, then show that M is the circumcentre and P is the centroid of ΔABC.



Solution: In ΔMBC

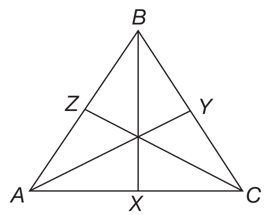
MX is the angle bisector of $\angle BMC$

So $\dfrac{BX}{XC} = \dfrac{MB}{MC}$

Similarly in ΔMCA, MY bisects $\angle AMC$ and by internal angle bisector theorem $\frac{MC}{MA} = \frac{CY}{YA}$ and in ΔAMB, $\frac{MA}{MB} = \frac{AZ}{ZB}$

$$\therefore \quad \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{\cancel{MB}}{\cancel{MC}} \cdot \frac{\cancel{MC}}{\cancel{MA}} \cdot \frac{\cancel{MA}}{\cancel{MB}} = 1$$

Hence by converse of Ceva's theorem AX, BY, CZ are concurrent.



If P is the point of concurrence of AX, BY, CZ. Let $[BPC] = \Delta_1$, $[APC] = \Delta_2$, $[APB] = \Delta_3$

$$\therefore \quad \frac{PA}{PX} = \frac{[APB]}{[BPX]} = \frac{[APC]}{[PCX]} = \frac{[APB]+[APC]}{[BPX]+[PCX]} = \frac{[APB]+[APC]}{[BPC]}$$

$$\therefore \quad \frac{PA}{PX} = \frac{\Delta_3 + \Delta_2}{\Delta_1}$$

Similarly $\frac{PB}{PY} = \frac{\Delta_1 + \Delta_3}{\Delta_2}$ and $\frac{PC}{PZ} = \frac{\Delta_2 + \Delta_1}{\Delta_3}$

$$\therefore \quad \frac{PA}{PX} \cdot \frac{PB}{PY} \cdot \frac{PC}{PZ} = \left(\frac{\Delta_3 + \Delta_2}{\Delta_1}\right)\left(\frac{\Delta_1 + \Delta_3}{\Delta_2}\right)\left(\frac{\Delta_2 + \Delta_1}{\Delta_3}\right) = 8 \quad \text{(Given)}$$

Since AM $\geq$ GM.

i.e., $\frac{a+b}{2} \geq \sqrt{ab}$ or $a + b \geq 2\sqrt{ab}$

$$\therefore \quad \Delta_1 + \Delta_2 \geq 2\sqrt{\Delta_1 \Delta_2}$$

Similarly $\Delta_2 + \Delta_3 \geq 2\sqrt{\Delta_2 \Delta_3}$ and $\Delta_3 + \Delta_1 \geq 2\sqrt{\Delta_3 \Delta_1}$

$\therefore$ Multiplying $(\Delta_1 + \Delta_2)(\Delta_2 + \Delta_3)(\Delta_3 + \Delta_1) \geq 8\Delta_1\Delta_2\Delta_3$

$$\therefore \quad \left(\frac{\Delta_1 + \Delta_2}{\Delta_3}\right)\left(\frac{\Delta_2 + \Delta_3}{\Delta_1}\right)\left(\frac{\Delta_3 + \Delta_1}{\Delta_2}\right) \geq 8$$

Thus equality holds if $\Delta_1 = \Delta_2 = \Delta_3$

i.e., $\frac{BX}{XC} = \frac{[ABX]}{[ACX]} = \frac{[PBX]}{[PCX]} = \frac{[ABX]-[PBX]}{[ACX]-[PCX]} = \frac{[ABP]}{[APC]} = \frac{\Delta_3}{\Delta_2}$

As $\Delta_3 = \Delta_2$

$$\therefore \quad \frac{BX}{XC} = 1 \quad \Rightarrow \quad BX = XC$$

But MX is the bisector of $\angle BMC$, and $BX = XC$

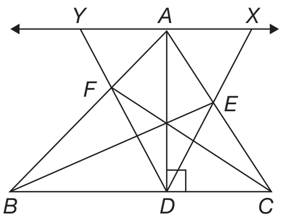
$\therefore$ ΔMBC of an isosceles triangle $MB = MC$. Similarly $MC = MA$.

Thus M is the circumcentre of ΔABC.

Since $BX = XC$ $\therefore$ AX is a median similarly BY, CZ are also medians.

$\therefore$ Their point of intersection P is the centroid of the triangle.

Example 77 *AD, BE, CF are three concurrent lines drawn from the vertices of the triangle ABC to points D, E, F on the opposite sides. If AD is an altitude of the triangle ABC, show that AD bisects the angle $\angle FDE$.*



Solution: Through A, draw a line parallel to BC. DE meets this line at X and DF meets this line at Y. Consider ΔAXE and ΔCDE

$$\angle AEX = \angle CED \qquad \text{(VOA)}$$

$$\angle EAX = \angle ECD \qquad \text{(Alternate interior angles)}$$

$\therefore$ By AA similarly

$$\Delta AXE \sim \Delta CDE$$

$$\therefore \quad \frac{CE}{EA} = \frac{CD}{AX} \qquad (1)$$

Similarly $\Delta AFY \sim \Delta BFD$

$$\therefore \quad \frac{AF}{FB} = \frac{AY}{BD} \qquad (2)$$

Since the lines AD, BE, CF are concurrent

$\therefore$ By Ceva's theorem

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \qquad (3)$$

$\therefore$ From Eqs. (1), (2) and (3) we get

$$\frac{BD}{DC} \cdot \frac{CD}{AX} \cdot \frac{AY}{BD} = 1$$

$$\Rightarrow \quad \frac{AY}{AX} = 1 \qquad \Rightarrow \quad AY = AX$$

Since $AD \perp BC$ and $\angle DAX = \angle ADB = 90°$ (Alternate interior angles)

$$\therefore \quad \angle DAX = \angle DAY = 90° \text{ and } AX = AY$$

$\therefore$ DA is the perpendiculars bisector of XY. $DXDY$ is an isosceles triangle

$$\therefore \angle XDA = \angle YDA$$

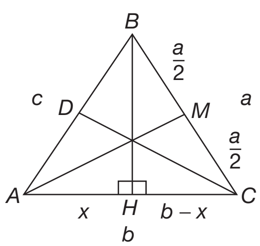
$$\Rightarrow \quad \angle EDA = \angle FDA$$

$\therefore$ AD bisects $\angle EDF$.

Note: If AD, BE, CF are the altitudes and their point of intersection is H, (orthocentre) then DH bisects $\angle EDF$ and EH bisects $\angle DEF$ and hence H is the in-centre of orthic triangle DEF.

Example 78 *In any triangle ABC, the median AM, the altitude BH and the angle bisector CD are concurrent. Prove, in the usual notation for the triangle,*

$$\frac{b^2 + c^2 - a^2}{b^2 + a^2 - c^2} = \frac{b}{a}.$$



Solution:

Assume $AH = x$;

$$\therefore HC = b - x.$$

By bisector theorem, $\dfrac{BD}{DA} = \dfrac{BC}{AC} = \dfrac{a}{b}$

Also $BM = CM$.

By Ceva's theorem, $\dfrac{AH}{HC} \cdot \dfrac{CM}{MB} \cdot \dfrac{BD}{DA} = 1 \Rightarrow \dfrac{AH}{HC} = \dfrac{b}{a}$ (1)

Consider right angled triangle ABH where $BH^2 = c^2 - x^2$
And from right triangle BHC, $BH^2 = a^2 - (b - x)^2$.
Thus,

$$c^2 - x^2 = a^2 - b^2 - x^2 + 2bx$$

$$\Rightarrow x = \frac{b^2 + c^2 - a^2}{2b}$$

$$\therefore b - x = b - \frac{b^2 + c^2 - a^2}{2b} = \frac{b^2 + a^2 - c^2}{2b}$$

Thus, $\dfrac{AH}{HC} = \dfrac{x}{b-x} = \dfrac{b^2 + c^2 - a^2}{2b} \times \dfrac{2b}{b^2 + a^2 - c^2}$

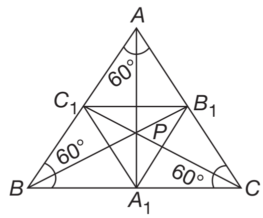
i.e., $\dfrac{b}{a} = \dfrac{b^2 + c^2 - a^2}{b^2 + a^2 - c^2}$. (From Eq.(1))

Example 79 *Let ABC be an equilateral triangle and let P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB at A_1, B_1, C_1 respectively.*

(i) *Prove the inequality: $A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A$.*

(ii) *When does the equality hold?*

Solution:



Now, $\cos C = \cos 60° = \dfrac{1}{2}$.

Apply cosine formula for A_1B_1 in ΔA_1B_1C.

$$A_1B_1^2 = CA_1^2 + CB_1^2 - 2CA_1 \cdot CB_1 \cdot \cos C$$

i.e., $A_1B_1^2 = CA_1^2 + CB_1^2 - CA_1 \cdot CB_1$ $\left(\text{As } \cos C = \dfrac{1}{2}\right)$

This is like $x^2 + y^2 - xy$

Apply Sophie inequality: $x^2 + y^2 \geq xy$ $(\text{where } x, y \in \mathbb{R})$

$$\therefore A_1B_1^2 \geq CA_1 \cdot CB_1 \quad (1)$$

Similarly $B_1C_1^2 \geq AB_1 \cdot AC_1$ and $C_1A_1^2 \geq BA_1 \cdot BC_1$

Thus $A_1B_1^2 \cdot B_1C_1^2 \cdot C_1A_1^2 \geq (CA_1 \cdot CB_1 \cdot AB_1 \cdot AC_1 \cdot BA_1 \cdot BC_1)$ (2)

But by Ceva's theorem, as the lines AA_1, BB_1, CC_1 are concurrent, we have,

$$BA_1 \cdot CB_1 \cdot AC_1 = CA_1 \cdot BC_1 \cdot AB_1.$$

Thus Eq. (2) becomes

$$(A_1B_1 \cdot B_1C_1 \cdot C_1A_1)^2 \geq (BA_1 \cdot CB_1 \cdot AB_1)^2 \quad (3)$$

$$\therefore A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A$$

Equality holds if $CA_1 = CB_1$; $AB_1 = AC_1$; $BA_1 = BC_1$.

This happens when P is the centre of the incircle of ΔABC.

Build-up Your Understanding 8

1. Prove that the necessary and sufficient condition that lines from the vertices A, B, C of ΔABC to points X, Y, Z on the opposite sides are concurrent is that
$$\frac{\sin\angle BAX}{\sin\angle CAX}\cdot\frac{\sin\angle CBY}{\sin\angle ABY}\cdot\frac{\sin\angle ACZ}{\sin\angle BCZ}=+1$$
2. Three squares are drawn on the sides of ΔABC (*i.e.*, the square on AB has AB as one of its sides and lies outside ΔABC. Show that the lines drawn from the vertices A, B, C to the centres of the opposite squares are concurrent.
3. Let ABC be a triangle, and let D, E, F be the feet of the altitudes from A, B, C. Construct the incircles of triangles AEF, BDF, and CDE; let the points of tangency with DE, EF, and FD be C'', A'', and B'', respectively. Prove that AA'', BB'', CC'' concurrent.
4. Three circles (whose centres form the vertices of a triangle) touch two by two. Prove that the three common tangents at the points of contact are concurrent.
5. In an acute triangle ABC with $AB \neq AC$, let V be the intersection of the angle bisector of A with BC, and let D be the foot of the perpendicular from A to BC. If E and F are the intersections of the circumcircle of AVD with CA and AB, respectively, show that the lines AD, BE, CF concurrent. **[Korea MO, 1997]**
6. Let $ABCDEF$ be a convex cyclic hexagon. Prove that AD, BE, CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.
7. If a given straight line AB is divided internally at P and externally at Q in the same ratio, then AB is said to be divided harmonically at P and Q. P and Q are called the harmonic conjugates of A and B. D, E, F are points on the sides BC, CA, AB of a triangle such that AD, BE, CF are concurrent, If EF cuts BC produced at D', prove D and D' are the harmonic conjugates of B and C.
8. The circles k_l and k_2 with respective centres O_1 and O_2 are externally tangent at the point C, while the circle k with centre O is externally tangent to k_1 and k_2. Let l be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k perpendicular to l. Assume that O and A lie on the same side of l. Show that the lines AO_1, BO_2, l have a common point. **[Bulgaria MO, 1996]**
9. Let ABC be a triangle. Construct rectangles $ACDE$, $AFGB$, and $BHIC$, one on each side of ABC. Prove that the perpendicular bisectors to the segments EF, GH, and ID are concurrent.
10. A line from vertex C of ΔABC bisects the median from A. Prove that it divides the side AB in the ratio $1:2$.
11. Triangle ABC is inscribed in ΔXYZ and circumscribed about ΔPQR. If AP, BQ, CR are concurrent and AX, BY, CZ are concurrent, prove that PX, QY, RZ are concurrent.
12. The in-circle of ΔABC touches the sides BC, CA, AB at D, E, F, respectively. The centres of ex-circles opposite A, B, C are P, Q, R. Show that PD, QE and RF concurrent.
13. Triangle ABC has in-centre I. The in-circle touches BC, CA at P and Q, respectively. A', C' are mid-points of sides BC, AB. Prove that the lines AF, PQ, $A'C'$ are concurrent.
14. In ΔABC, the in-circle Σ touches the sides BC, CA, AB at D, E, F. Let, P be any point within the circle and, let the segments AP, BP, CP meet Σ at X, Y, Z. Prove that DX, EY, FZ are concurrent.

8.10.5 Menelaus Theorem

If a transversal cuts the sides BC, CA, AB of a triangle ABC at X, Y, Z respectively then

$$\frac{BX}{XC}\frac{CY}{YA}\cdot\frac{AZ}{ZB} = -1$$

Proof: Let h_1, h_2, h_3 be the lengths of perpendiculars AP, BQ, CR respectively from A, B, C on the transversal.

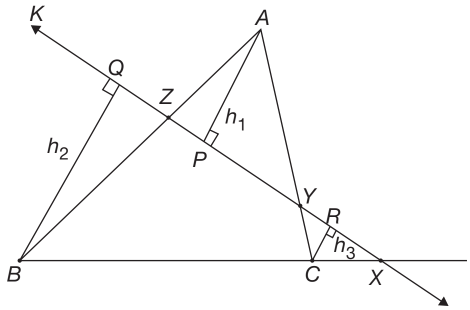


Figure (i)

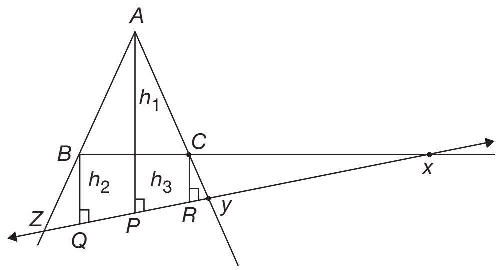


Figure (ii)

In ΔBQX and ΔCRX

$$\angle BQX = \angle CRX = 90°$$

$$\angle BXQ = \angle CXR \qquad \text{(Common)}$$

∴ By AA similarly

$$\Delta BQX \sim \Delta CRX$$

$$\therefore \quad \frac{BX}{XC} = \frac{BQ}{CR} = \frac{h_2}{h_3}$$

Similarly $\Delta CRY \sim \Delta APY$

$$\therefore \quad \frac{CY}{YA} = \frac{CR}{AP} = \frac{h_3}{h_1}$$

and $\Delta APZ \sim \Delta BQZ$

$$\therefore \quad \frac{AZ}{ZB} = \frac{AP}{BQ} = \frac{h_1}{h_2}$$

$$\therefore \quad \frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = \frac{h_2}{h_3}\cdot\frac{h_3}{h_1}\cdot\frac{h_1}{h_2} = 1$$

Menelaus of Alexandria

c. 70 CE–c. 140 CE

Nationality: Greek

In Figure (i) as per the directed line segments we have BX is positive and XC is negative. Therefore $\frac{BX}{XC}$ is negative and the other two ratios are positive.

$$\therefore \quad \frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = -1.$$

Remark: You may take $\frac{BZ}{ZA}\cdot\frac{AY}{YC}\cdot\frac{CX}{XB} = -1$

In second figure (when transversal cutting all the sides externally)
As per directed line segments we have BX is positive but XC is negative.

$\therefore \quad \frac{BX}{XC}$ is negative and also CY is positive but YA is negative.

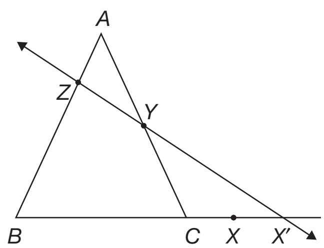
$\therefore \quad \frac{CY}{YA}$ is also negative and AZ is positive but ZB is negative.

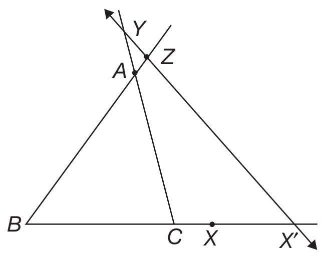
$\therefore \quad \frac{AZ}{ZB}$ is negative.

Thus $\frac{BX}{XC}, \frac{CY}{YA}, \frac{AZ}{ZB}$ all are negative

$$\therefore \quad \frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = -1$$

8.10.5.1 Converse of Menelaus Theorem





If X, Y, Z are three points on each of the sides BC, CA, AB, of ΔABC or on their extensions such that $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = -1$, then X, Y, Z are collinear.

Proof: Since it is given that X, Y, Z are on BC, CA, AB, or on their extensions such that

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = -1 \qquad (1)$$

Let if possible ZY produced meets BC produced at X'.

$\therefore$ By Menelaus theorem

$$\frac{BX'}{X'C}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = -1 \qquad (2)$$

$\therefore$ From Eqs. (1) and (2) $\frac{BX}{XC} = \frac{BX'}{X'C}$

Subtract 1 from both sides $\frac{BX}{XC} - 1 = \frac{BX'}{X'C} - 1$

$$\Rightarrow \quad \frac{BX - XC}{XC} = \frac{BX' - X'C}{X'C}$$

$$\frac{BC}{XC} = \frac{BC}{X'C}$$

$$\Rightarrow \quad \frac{1}{XC} = \frac{1}{X'C}$$

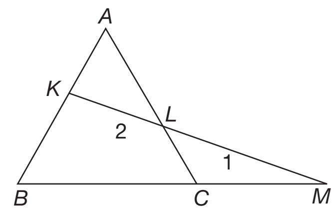
$$XC = X'C$$
$$X'C - XC = 0$$
$$XX' = 0$$

That is, X and X' coincides and thus X, Y, Z are collinear.

Note: The reader may have a doubt that whether X lies on the right or left of X'. In the proof given above X has been taken on the left of X'. If X lies on the right of X' then also you can prove.

Example 80 *In a triangle ABC, AB = AC. A transversal intersects AB and AC internally at K and L respectively. It intersects BC produced at M. If KL = 2LM, find KB/LC.*

Solution: In ΔAKL, consider BCM as the transversal which intersects AK, AL, KL at B, C and M respectively.



$\therefore$ By Menelaus theorem

$$\frac{KB}{\cancel{BA}} \cdot \frac{\cancel{AC}}{CL} \cdot \frac{LM}{MK} = -1 \text{ (As } AB = AC \text{)}$$

$$\frac{KB}{CL} \times \frac{1}{3} = -1 \text{ (As } KL = 2LM \Rightarrow KM = 3LM \text{)}$$

$$\frac{KB}{CL} = -3 \quad \Rightarrow \quad \frac{KB}{LC} = \frac{3}{1}.$$

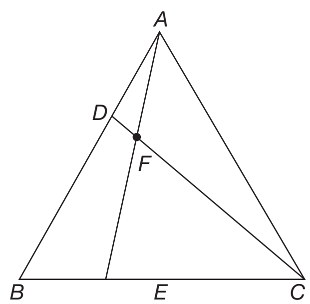
Example 81 *ABC is a triangle and D and E are interior points of the sides AB and BC respectively such that* $\frac{AD}{DB} = \frac{1}{3}$ *and* $\frac{CE}{EB} = 3$. *If AE and CD intersect at F, find* $\frac{CF}{FD}$.

Solution:

In ΔBCD, consider EFA as a transversal.

It cuts BC, CD, DB at E, F and A respectively.

Then by Menelaus theorem



$$\frac{BE}{EC} \cdot \frac{CF}{FD} \cdot \frac{DA}{AB} = -1 \qquad (1)$$

Since $\frac{AD}{DB} = \frac{1}{3} \quad \Rightarrow \quad \frac{AD}{AB} = \frac{1}{4}$

Also $\frac{CE}{EB} = 3 \Rightarrow \quad \frac{BE}{EC} = \frac{1}{3}$

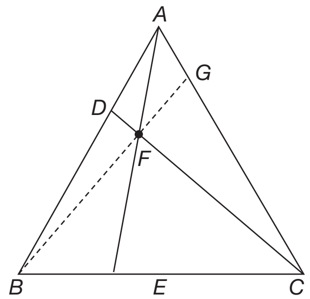
$\therefore$ Eq. (1) becomes $\frac{1}{3} \cdot \frac{CF}{FD} \cdot \frac{1}{4} = -1$

$$\frac{CF}{FD} = \frac{-12}{1} \text{ or } \frac{CF}{FD} = \frac{12}{1}$$

Aliter:

Construction: Join BF and produce it to cut AC at G

Since by Ceva's theorem



$$\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CG}{GA}=1$$

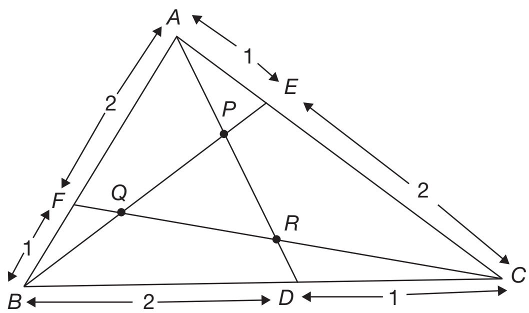
$$\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{CG}{GA}=1$$

$$\therefore \quad \frac{CG}{GA}=\frac{9}{1}$$

Also by **van Aubel's** theorem, we get $\dfrac{CF}{FD}=\dfrac{CG}{GA}+\dfrac{CE}{EB}=\dfrac{9}{1}+\dfrac{3}{1}=\dfrac{12}{1}$

Example 82 *On the sides BC, CA, AB of ΔABC, points D, E, F are taken in such a way that $\dfrac{BD}{DC}=\dfrac{CE}{EA}=\dfrac{AF}{FB}=\dfrac{2}{1}$. Show that the area of the triangle determined by the lines AD, BE, CF is $\dfrac{1}{7}$ of area of ΔABC.*

Solution:



Using Menelaus theorem in ΔABD with transversal CF, we get $\dfrac{AR}{RD}\times\dfrac{DC}{CB}\times\dfrac{BF}{FA}=1$

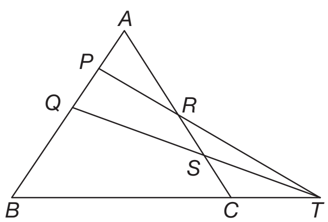
$$\Rightarrow \frac{AR}{RD}\times\frac{1}{3}\times\frac{1}{2}=1\Rightarrow\frac{RD}{AR}=\frac{1}{6}.\text{ Also } [ADC]=\frac{1}{3}[ABC]=\frac{1}{3}\Delta$$

Now, $[ARC]=\dfrac{6}{7}[ADC]=\dfrac{2}{7}\Delta$. Similarly, $[BQC]=\dfrac{2}{7}\times\Delta$ and $[APB]=\dfrac{2}{7}\times\Delta$

$$\text{Now } [PQR]=[ABC]-[ARC]-[BQC]-[APB]=\Delta-\frac{2}{7}\Delta-\frac{2}{7}\Delta-\frac{2}{7}\Delta$$

$$\Rightarrow [PQR]=\frac{1}{7}\times\Delta.$$

Example 83 *ABC is a triangle; P, Q are points on AB, so that, $6PQ=3AP=2QB$; R, S are points on AC, such that, $6RS=3SC=2AR$. Prove that, PR, QS and BC (produced) are concurrent.*



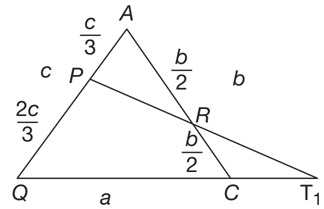
Solution:

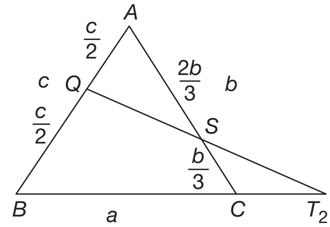
Let, $6PQ=3AP=2QB=c$

$$\therefore PQ=\frac{c}{6};\ AP=\frac{c}{3};\ QB=\frac{c}{2}.$$

Also let, $6RS = 3SC = 2AR = b$

$$\therefore RS = \frac{b}{6};\ SC = \frac{b}{3};\ AR = \frac{b}{2}.$$





Applying Menelaus' theorem to first and second figures

$$\left(\frac{BT_1}{T_1C}\right)\left(\frac{CR}{RA}\right)\left(\frac{AP}{PB}\right) = -1 \quad \text{and} \quad \left(\frac{BT_2}{T_2C}\right)\left(\frac{CS}{SA}\right)\left(\frac{AQ}{QB}\right) = -1 \qquad (1)$$

(Where PRT_1 and QST_2 are transversals to first and second figure respectively)

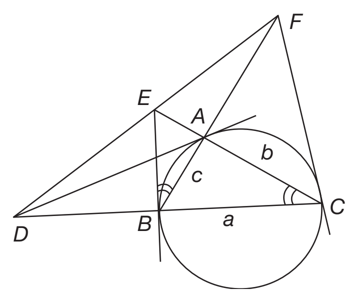
$$\Rightarrow \left(\frac{BT_1}{T_1C}\right)\left(\frac{\frac{b}{2}}{\frac{b}{2}}\right)\left(\frac{\frac{c}{3}}{\frac{2c}{3}}\right) = -1 \quad \Rightarrow \quad \frac{BT_1}{T_1C} = -2$$

$$\text{And } \left(\frac{BT_2}{T_2C}\right)\left(\frac{\frac{b}{3}}{\frac{2b}{3}}\right)\left(\frac{\frac{c}{2}}{\frac{c}{2}}\right) = -1 \quad \Rightarrow \quad \frac{BT_2}{T_2C} = -2$$

This means $\frac{BT_1}{T_1C} = \frac{BT_2}{T_2C}$ implying BC is divided externally in the *same* ratio at two *distinct* points T_1 and T_2.

This is not possible implying $T_1 = T_2 = T$ (T_1, T_2 must coincide). Thus PR, QS, BC (produced) are concurrent at T.

Example 84 *Prove that the tangents at the vertices of a triangle to its circumcircle meets the opposite sides in three collinear points.*



Given In ΔABC, tangent at A to the circumcircles meets CB produced at D. Tangent at B to the circumcircle meets CA produced at E and tangent at C to the circumcircle meets BA produced at F.

To prove D, E, F are collinear points

Proof: In ΔDAB and ΔDCA

$$\angle ADB = \angle CDA \qquad \text{(Common)}$$

$$\angle DAB = \angle DCA \qquad \text{(Alternate segment theorem)}$$

$\therefore$ By AA similarly $\Delta BAD \sim \Delta ACD$

$$\therefore \frac{BD}{AD} = \frac{AD}{CD} = \frac{BA}{AC} = \frac{c}{b}$$

Now $\frac{BD}{AD} \cdot \frac{AD}{CD} = \left(\frac{AB}{AC}\right)^2$

$$\Rightarrow \frac{BD}{CD} = \frac{AB^2}{AC^2} = \frac{c^2}{b^2}$$

$$\Rightarrow \frac{BD}{DC} = \frac{-c^2}{b^2} \qquad (\because BD \text{ and } DC \text{ are in opposite directions})$$

Similarly, $\dfrac{CE}{EA} = \dfrac{-a^2}{c^2}$ and $\dfrac{AF}{FB} = \dfrac{-b^2}{a^2}$

$$\therefore \quad \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \left(\frac{-c^2}{b^2}\right)\left(\frac{-a^2}{c^2}\right)\left(\frac{-b^2}{a^2}\right) = -1$$

$\therefore$ By converse of Menelaus theorem D, E, F are collinear.

Example 85 *Three points X, Y, Z are taken on the sides BC, CA, AB respectively of a ΔABC such that AX, BY, CZ are concurrent. YZ meets BC in X'. ZX meets AC in Y', XY meets BA in Z'. Prove that*

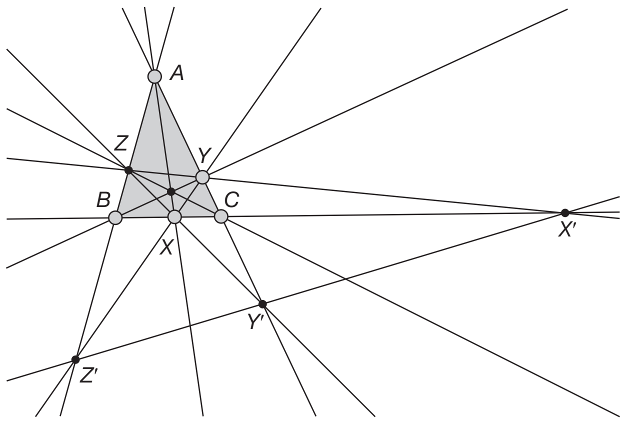
(i) *X', Y', Z' are collinear*
(ii) *AX, BY', CZ' are concurrent*
(iii) *AX', BY, CZ' are concurrent*
(iv) *AX', BY', CZ are concurrent*

Solution: Since AX, BY, CZ are concurrent
$\therefore$ By Ceva's theorem

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1 \tag{1}$$

The transversal $X'YZ$ cuts the sides of ΔABC, by Menelaus theorem

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1 \tag{2}$$



Similarly the transversal ZXY' and XYZ' with respect to ΔABC

$$\frac{BX}{XC} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ}{ZB} = -1 \tag{3}$$

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ'}{Z'B} = -1 \tag{4}$$

(i) To prove X', Y', Z' are collinear take those equations which include X', Y', Z'. So multiplying Eqs. (2), (3) and (4) we get

$$\left(\frac{BX'}{X'C}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}\right)\left(\frac{BX}{XC}\cdot\frac{CY'}{Y'A}\cdot\frac{AZ}{ZB}\right)\left(\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ'}{Z'B}\right)=(-1)\times(-1)\times(-1)$$

$$\Rightarrow \frac{BX'}{X'C}\left(\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{AB}\right)\frac{CY'}{Y'A}\left(\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}\right)\frac{AZ'}{Z'B}=-1$$

$\therefore$ By using Eq. (1), we get, $\dfrac{BX'}{X'C}\cdot\dfrac{CY'}{Y'A}\cdot\dfrac{AZ'}{Z'B}=-1$

$\therefore$ By converse of Menelaus theorem X', Y', Z' are collinear

(ii) Now to prove AX, BY', CZ' are concurrent take those equations which includes X, Y', Z'

Thus multiplying Eqs. (3) and (4), we get

$$\left(\frac{BX}{XC}\cdot\frac{CY'}{Y'A}\cdot\frac{AZ}{ZB}\right)\left(\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ'}{Z'B}\right)=(-1)(-1)$$

$$\Rightarrow\left(\frac{BX}{XC}\cdot\frac{CY'}{Y'A}\cdot\frac{AZ'}{Z'B}\right)\left(\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}\right)=+1$$

$$\Rightarrow\frac{BX}{XC}\cdot\frac{CY'}{Y'A}\cdot\frac{AZ'}{Z'B}=1$$

Thus by converse of Ceva's theorem AX, BY', CZ' are concurrent

(iii) Multiplying Eqs. (2) and (4), we get

$$\left(\frac{BX'}{X'C}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}\right)\left(\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ'}{Z'B}\right)=(-1)(-1)$$

$$\Rightarrow\left(\frac{BX'}{X'C}\cdot\frac{CY}{YA}\cdot\frac{AZ'}{Z'B}\right)\left(\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}\right)=+1$$

$$\Rightarrow\frac{BX'}{X'C}\cdot\frac{CY}{YA}\cdot\frac{AZ'}{Z'B}=1$$

$\therefore$ By converse of Ceva's theorem AX', BY, CZ' are concurrent

(iv) Multiplying Eqs. (2) and (3) we get

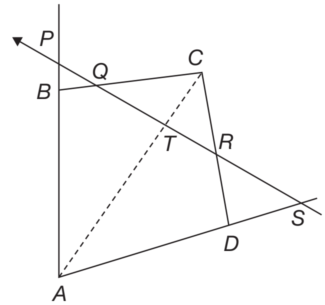
$$\left(\frac{BX'}{X'C}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}\right)\left(\frac{BX}{XC}\cdot\frac{CY'}{Y'A}\cdot\frac{AZ}{ZB}\right)=(-1)(-1)$$

$$\Rightarrow\left(\frac{BX'}{X'C}\cdot\frac{CY'}{Y'A}\cdot\frac{AZ}{ZB}\right)\left(\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}\right)=1$$

$$\Rightarrow\frac{BX'}{X'C}\cdot\frac{CY'}{Y'A}\cdot\frac{AZ}{ZB}=1$$

$\therefore$ By converse of Ceva's theorem AX', BY', CZ are concurrent

Example 86 *A transversal cuts the sides AB, BC, CD, DA of a quadrilateral at P, Q, R, S respectively prove that* $\dfrac{AP}{PB}\cdot\dfrac{BQ}{QC}\cdot\dfrac{CR}{RD}\cdot\dfrac{DS}{SA}=+1$



Construction: Join AC which cuts the line at T

Solution: In ΔABC and ΔADC apply Menelaus theorem on the given transversal, we get

$$\frac{AP}{PB}\cdot\frac{BQ}{QC}\cdot\frac{CT}{TA}=-1 \qquad (1)$$

and $$\frac{CT}{TA}\cdot\frac{AS}{SD}\cdot\frac{DR}{RC}=-1 \tag{2}$$

$\therefore$ From Eq. (1) ÷ Eq. (2) we get $\dfrac{AP}{PB}\cdot\dfrac{BQ}{QC}\cdot\dfrac{CR}{RD}\cdot\dfrac{DS}{SA}=+1$

8.10.6 Pappus Theorem

Pappus of Alexandria

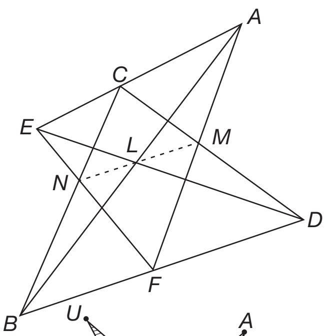
c. 290–c. 350 AD
Nationality: Greek

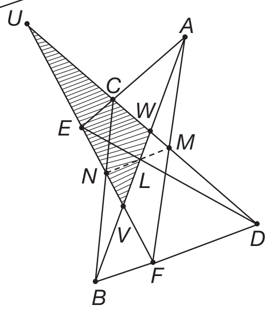
If A, C, E are three points on one straight line. B, D, F on another and if the three lines AB, CD, EF meet respectively DE, FA and BC at L, M, N, then these three points L, M, N are collinear.

Given: ACE and BDF are any two lines. AB, CD, EF intersects DE, FA, and BC at L, M, N respectively

To prove: L, M, N are collinear

Construction: Produce FE and DC to intersect at U
Let FU cuts BA at V and DU cuts BA at W.





Proof: Now to prove L, M, N are collinear in ΔUVW, we have to prove

$$\frac{UN}{NV}\cdot\frac{VL}{LW}\cdot\frac{WM}{MU}=-1$$

In ΔUVW, consider LDE as transversal and then by Menelaus theorem

$$\frac{UE}{EV}\cdot\frac{VL}{LW}\cdot\frac{WD}{DU}=-1 \tag{1}$$

In ΔUVW consider AMF as transversal and then by Menelaus theorem

$$\frac{UF}{FV}\cdot\frac{VA}{AW}\cdot\frac{WM}{MU}=-1 \tag{2}$$

In ΔUVW, by considering BCN as transversal and then by Menelaus theorem

$$\frac{UN}{NV}\cdot\frac{VB}{BW}\cdot\frac{WC}{CU}=-1 \tag{3}$$

In ΔUVW, consider ACE as transversal and then by Menelaus theorem

$$\frac{UE}{EV}\cdot\frac{VA}{AW}\cdot\frac{WC}{CU}=-1 \tag{4}$$

In ΔUVW consider BFD as transversal and by Menelaus theorem

$$\frac{UF}{FV}\cdot\frac{VB}{BW}\cdot\frac{WD}{DU}=-1 \tag{5}$$

Multiply Eqs. (1), (2) and (3)

$$\left(\frac{UE}{EV}\cdot\boxed{\frac{VL}{LW}}\cdot\frac{WD}{DU}\right)\left(\frac{UF}{FV}\cdot\frac{VA}{AW}\cdot\boxed{\frac{WM}{MU}}\right)\left(\boxed{\frac{UN}{NU}}\cdot\frac{VB}{BW}\cdot\frac{WC}{CU}\right)=(-1)(-1)(-1)$$

$$\left(\frac{UN}{NV}\cdot\frac{VL}{LW}\cdot\frac{WM}{MU}\right)\left(\frac{UE}{EV}\cdot\frac{VA}{AW}\cdot\frac{WC}{CU}\right)\left(\frac{UF}{FV}\cdot\frac{VB}{BW}\cdot\frac{WD}{DU}\right)=-1$$

By Eq. (4) and (5)

$$\left(\frac{UN}{NV}\cdot\frac{VL}{LW}\cdot\frac{WM}{MV}\right)(-1)(-1) = -1$$

$$\Rightarrow \frac{UN}{NV}\cdot\frac{VL}{LW}\cdot\frac{WM}{MU} = -1$$

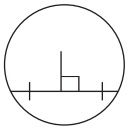
$\therefore$ By converse of Menelaus theorem, L, M, N are collinear.

Build-up Your Understanding 9

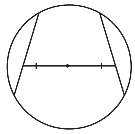
1. Prove that the external bisectors of the three angle of a scalene triangle meet their respective opposite sides at three collinear points.
2. In ΔABC points D and E respectively divide the sides BC and CA in the ratios $\dfrac{BD}{DC} = m$, $\dfrac{AE}{EC} = n$. The segment AD and BE intersect in a point X. Find the ratio $\dfrac{AX}{XD}$.
3. The external bisector of angle A of triangle ABC meets BC produced at L, and the internal bisector of angle B meets CA at M. If LM meets AB at R, prove that CR bisects the angle C.
4. In a parallelogram $ABCD$ with $\angle A < 90°$, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, E are collinear. **[Turkey MO, 1996]**
5. Let M be an interior point of triangle ABC. AM meets BC at D, BM meets CA at E, CM meets AB at F. Prove that $[DEF] \le 1/4\,[ABC]$. **[The 26th and 31st IMO Shortlisted Problem]**
6. Suppose PA, PB, PC be three rays for which $\angle APC = \angle APB + \angle BPC < 180°$. Prove that A, B, C are collinear if and only if $\dfrac{\sin\angle APC}{PB} = \dfrac{\sin\angle APB}{PC} + \dfrac{\sin\angle BPC}{PA}$.
7. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that $AM/AC = CN/CE = r$. Determine r if B, M, N are collinear **[IMO, 1982]**
8. Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let G and O denote the centroid and circumcentre of the ΔABC. Prove that G, O, D are collinear. **[Bulgaria MO, 1997]**
9. The semicircle with side BC of ΔABC as diameter intersects sides AB, AC at points D, E, respectively. Let F, G be the feet of the perpendiculars from D, E to side BC respectively. Let M the intersection of DG and EF. Prove that $AM \perp BC$.
10. Consider a triangle ABC and a point P within the triangle. Lines AP, BP, CP intersects the opposite sides in points D, E, F respectively. Prove that out of the numbers $\dfrac{AP}{PD}, \dfrac{BP}{PE}, \dfrac{CP}{PF}$ at least one is ≤ 2 and at least one is ≥ 2.
11. Consider a triangle ABC with its inscribed circle whose centre I, touching BC at D. Let the mid-points of AD, BC be M, N. Prove that M, I, N are collinear.
12. **Construction of Harmonic Mean by Pappus:** O, A and B are collinear points. On the perpendicular to OB at B, mark-off $BD = BE$. Let the perpendicular to OB at A meet OD at F. Draw FE to cut OB at C. Prove that OC is the Harmonic Mean between OA and OB.

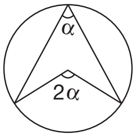
8.11 Circles

The following elementary theorems about circles are worth remembering:

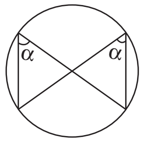


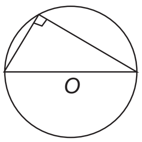
1. One and only one circle can be drawn so as to pass through three non-collinear points.
2. The perpendicular drawn from the centre of a circle to a chord of the circle bisects the chord. Conversely, the straight line joining the mid-point of a chord of a circle to the centre is perpendicular to the chord.
3. Equal chords of a circle are equidistant from the centre. Conversely if two chords of a circle are equidistant from the centre then they are equal.



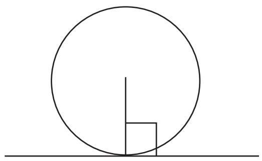


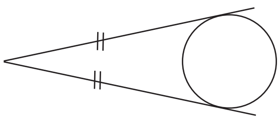
4. In the same circle or in equal circles, equal chords cut off equal arcs and conversely.
5. Angle subtended by an arc of a circle at the centre of the circle is twice the angle subtended by the same are at any point on the remaining part of the circle.
6. Angles in the same segment of a circle are equal and conversely.





7. Angle in a semi-circle is a right angle. Angle in a segment smaller than (resp bigger than) a semi-circle is an obtuse (resp. acute) angle.
8. Radius drawn at point of contact of a tangent to the circle is perpendicular to the tangent.

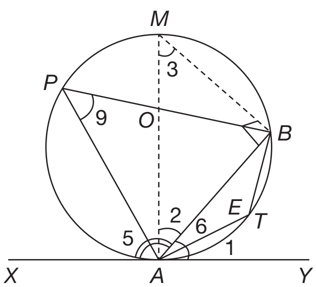




9. From an external point we can draw two tangents to the circle. Both tangents are equal in length.

Proofs of above theorems are left as an exercise. It is highly recommended before going further please do the proofs of above.

8.11.1 Alternate Segment Theorem



If through a point on a circle, a tangent and a chord be drawn the angle which the tangent makes with the chord is equal to the angle in the alternate segment.

Given: In the figure XAY is a tangent at A to the circle. AB is a chord and P is any point in its alternate segment.

To prove: (i) $\angle BAY = \angle APB$ (ii) $\angle BAX = \angle ATB$

Proof: Let O be the centre of the circle. Join AO and produce it to cut the circle at M. Join BM

$\therefore \quad \angle 1 + \angle 2 = 90°$

Also $\angle 2 + \angle 3 = 90°$

$\therefore \quad \angle 1 + \angle 2 = \angle 2 + \angle 3$

$\Rightarrow \quad \angle 1 = \angle 3$

Also $\angle 3 = \angle 4$ (Angles in a same segment)

$\therefore \quad \angle 1 = \angle 4$

Now $\angle 1 + \angle 5 = 180°$ (Linear pair)

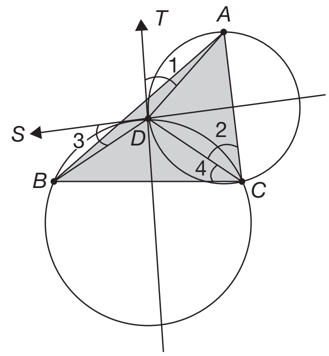
Also $PATB$ is a cyclic quadrilateral

$\therefore \quad \angle 4 + \angle 6 = 180°$

$\Rightarrow \quad \angle 1 + \angle 5 = \angle 4 + \angle 6$

$\Rightarrow \quad \angle 5 = \angle 6$ $(\because \quad \angle 1 = \angle 4)$

Example 87 *Let D be a point in the interior of an acute angled triangle ABC, such that* $\angle ADB = \angle ACB + \dfrac{\pi}{2}$. *Prove that the circumcircles of the triangles ACD and BCD cut each other orthogonally.*



Solution: Draw tangents DT and DS to the circles ADC and BDC at D.

Then $\angle ADT = \angle ACD$ [*i.e.*, $\angle(1) = \angle(2)$] (Alternate segment theorem)

$\angle BDS = \angle BCD$ [*i.e.*, $\angle(3) = \angle(4)$] (Alternate segment theorem)

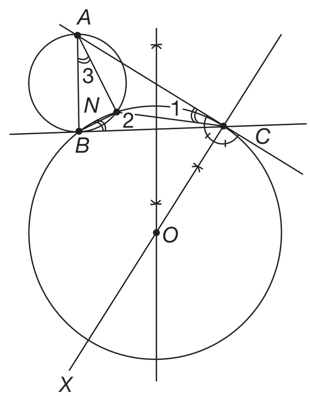
This implies that $\angle SDT = \angle BDA - (\angle BDS + \angle ADT)$

$\Rightarrow \angle SDT = 90° + \angle C - \angle C = 90°$

Thus the tangents to the two circles ADC and BDC are perpendicular, *i.e.*, the circles cut each other orthogonally.

Note: Angle between the tangents (or normals) to the two circles at their point of intersection is called angle between the circles and if this angle is 90° then circles are said to be orthogonal.

Example 88 *Given a right angle ABC, construct a point N in the interior of the triangle, such that the angles ∠NBC, ∠NCA, ∠NAB are all equal. Justify your construction.*



Solution:

Draw a semicircle on AB as diameter.

Draw $CX \perp AC$.

Draw the perpendicular bisector of BC and extend it to meet CX at O. With 'O' as centre and OC as radius draw arc of a circle to intersect the semicircle on AB as diameter at N.

N is the required point.

Proof: Join AN, BN, CN. Then $\angle NAB = \angle NBC = \angle NCA$.

CA is a tangent and CN is a chord of the circle with centre 'O'.

$\therefore \angle ACN = \angle CBN$ (angle in the alternate segment) (1)

BC is a tangent and BN is a chord to circle on AB as diameter.

$\therefore \angle CBN = \angle NAB$ (angle in the alternate segment) (2)

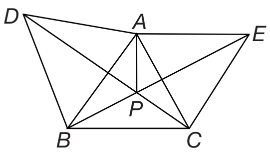
Thus from Eqs. (1) and (2), we get, $\angle NAB = \angle NBC = \angle NCA$.

Note: This point N is called ***Brocard Point***.

Pierre Rene Jean Baptiste Henri Brocard

12 May 1845–16 Jan 1922

Nationality: French



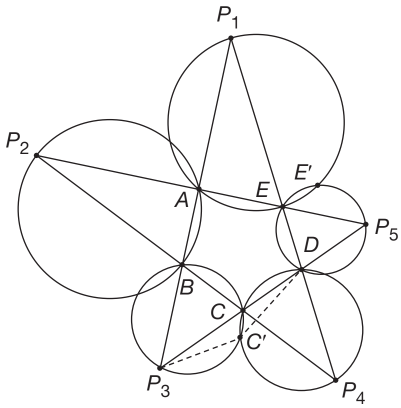
Archimedes

c. 287 BCE–212 BCE
or 211 BCE
Nationality: Greek

Build-up Your Understanding 10

1. Prove that if two chords of a circle bisect each other, they are diameters.
2. If three chords of a circle are such that each pair of chords bisects the third; all the three chords are at the same distance from the centre of the circle.
3. D is a point in the base BC of a ΔABC and through B, D, C lines are drawn perpendicular to AB, AD, AC respectively meeting one another in E, F, G. Prove that A, E, G, F are concyclic.
4. Let A, B be two given points and $k \neq 1$ a positive real number. Prove that the locus of points P satisfying $PA/PB = k$ is a circle whose centre lies on AB.
   Note: The circle obtained in the above problem is called '**Circle of Apollonius**'
5. A triangle inscribed in a circle of radius 5 has 2 sides measuring 5 and 6 respectively. Find the measure of the third side of the triangle.
6. We begin with ΔABC and construct equilateral triangle ABD and ACE with their vertices D and E in the exterior of ΔABC. Segments DC and EB intersect at point P as shown in the figure. Find $\angle APD$.
7. H is the orthocentre of an acute –angled triangle ABC with circumcentre 'O'. Let P be a point on the arc, not containing A of the circumcircle, different from B and C. Let D be a point, such that $AD = PC$ and $AD \parallel PC$. Let K be the orthocentre of ΔACD. Prove that K lies on the circumcircle of ΔABC.
8. Point D is the mid-point of arc AC of a circle; point B is on minor arc CD; and E is the point on AB such that DE is perpendicular to AB. Prove that $AE = BE + BC$.
   Note: This problem is known as ***'Archimedes broken-chord theorem'***
9. Two circles C_1 and C_2 intersect at two distinct points P and Q in a plane. Let a line passing through 'P' meet the circles C_1 and C_2 in A and B respectively. Let Y be the mid-point of AB. Let QY meet the circles C_1 and C_2 in X and Z respectively. Prove that Y is the mid-point of XZ also.
10. Two circles intersect at points A and B. An arbitrary line through B intersects the first circle again at C and the second circle again at D. The tangents to the first circle at C and to the second circle at D intersect at M. The line parallel to CM which passes through the point of intersection of AM and CD intersects AC at K. Prove that BK is tangent to the second circle.
11. Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E and M intersects the lines BC and AC at F and G, respectively. If $\dfrac{AM}{AB} = t$, find $\dfrac{EG}{EF}$ in terms of t. **[IMO, 1990]**
12. Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR. **[APMO, 1999]**
13. ABC is an isosceles triangle with $AB = AC$. Suppose that
    (i) M is the mid-point of BC and O is the point on the line AM such that OB is perpendicular to AB;
    (ii) Q is an arbitrary point on the segment BC different from B and C;
    (iii) E lies on the line AB and F lies on the line AC such that E, Q and F are all distinct and collinear.
    Prove that OQ is perpendicular to EF if and only if $QE = QF$. **[IMO, 1994]**
14. $ABCDE$ is a convex pentagon. The sides of the pentagon intersect at P_1, P_2, P_3, P_4, and P_5 as shown in the Figure. Construct the circumcircles of the triangles P_1AE, P_2BA, P_3CB, P_4DC and P_5ED. These circumcircles meet at five points A',

B', C', D', E' which are different from A, B, C, D, E. Prove that the points A', B', C', D$'$, E$'$ are concyclic.

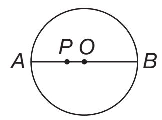


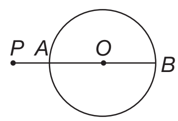
8.11.2 The Power of a Point

Let ω be a circle with centre O and radius r, and let P be a point. The power of P with respect to ω is defined to be the difference of squared length $PO^2 - r^2$.

This is positive, zero, or negative according as P is outside, on, or inside the circle ω.

Explanation:





Let line PO meet the circle ω at points A and B, so that AB is a diameter. Here we will be using directed lengths which is as follows:

For three collinear points P, A, B,

If PA and PB point in the same direction, then we will take PA and PB of same sign

$\Rightarrow PA \cdot PB$ is positive.

If PA and PB point in the opposite direction, then we will take PA and PB of opposite sign

$\Rightarrow PA \cdot PB$ is negative.

Now,

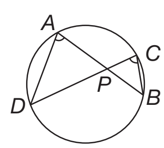
$$PA \cdot PB = (PO + OA)(PO + OB) = (PO - r)(PO + r) = PO^2 - r^2,$$

$$\Rightarrow PA \cdot PB = PO^2 - r^2 \tag{1}$$

Which is the power of the point P. Observe the right hand side of the Eq. (1),

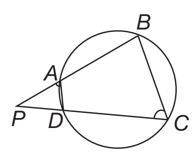
If P lies inside the circle, then $PO < r$, which forces $PO^2 - r^2$ to be negative and If P lies outside the circle, then $PO > r$, which forces $PO^2 - r^2$ to be positive.

8.11.3 Intersecting Chords Theorem



If a line L through P intersects a circle ω at two points A and B, the product $PA \cdot PB$ (of signed lengths) is equal to the power of P with respect to the circle.

More over if there are two lines through P one meets circle ω at points A and B, and let another line meets circle ω at points C and D. Then



$$PA \cdot PB = PC \cdot PD.$$

Proof: Let us consider two cases separately

Case 1: P lies inside the circle ω,
In ΔPAD and ΔPCB, we have

$$\angle PAD = \angle PCB$$

and $$\angle APD = \angle CPB,$$

So by AA similarity, ΔPAD and ΔPCB are similar.

$$\Rightarrow \frac{PA}{PD} = \frac{PC}{PB} \Rightarrow PA \cdot PB = PC \cdot PD.$$

Case 2: P lies outside the circle ω,
In ΔPAD and ΔPCB, we have

$$\angle PAD = \angle PCB$$

and $$\angle APD = \angle CPB,$$

So by AA similarity, ΔPAD and ΔPCB are similar.

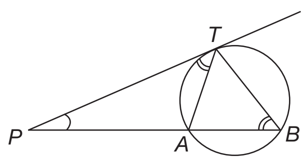
$$\Rightarrow \frac{PA}{PD} = \frac{PC}{PB} \Rightarrow PA \cdot PB = PC \cdot PD.$$

8.11.4 Tangent Secant Theorem

If through a point outside a circle a tangent and a chord be drawn. The square of the length of the tangent is equal to the rectangle contained by the segments of the chord.

Proof: In ΔPTA and ΔPBT

$$\angle TPA = \angle BPT \qquad \text{(Common)}$$
$$\angle PTA = \angle PBT \qquad \text{(Alternate segment theorem)}$$



$\therefore$ By AA similarly

$$\Delta PTA \sim \Delta PBT$$

$$\therefore \quad \frac{PT}{PB} = \frac{PA}{PT}$$

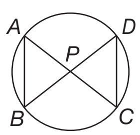
$$\Rightarrow \quad PT^2 = PA \times PB.$$

Note: Using the power of a point theorem and intersecting chord theorem, we infer that 'for any line passing through point P and meeting the circle ω at X and Y, $PX \cdot PY$ is always constant (independent of line passing through point P!) And it is equal to the power of point P with respect to the circle ω'.

By convention, this is positive, zero, or negative according as P is outside, on, or inside the circle ω. Also when P is outside the circle, the power equals to the square of the length of the tangent from P to the circle.

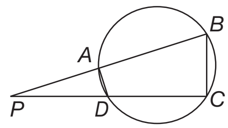
8.11.5 Theorem (Converse of Intersecting Chords Theorem)

Let A, B, C, D be four distinct points. Let lines AB and CD intersect at P. Then A, B, C, D are concyclic if and only if $PA \cdot PB = PC \cdot PD$.



Proof: In ΔAPD and ΔCPB.

$$\frac{PA}{PD} = \frac{PC}{PB} \qquad \text{(Using } PA \cdot PB = PC \cdot PD\text{)}$$



And $\angle APD = \angle CPB$ (Whether P inside the circle or outside the circle)

$$\Rightarrow \Delta APD \text{ and } \Delta CPB \qquad \text{(By SAS similarity)}$$

Thus,

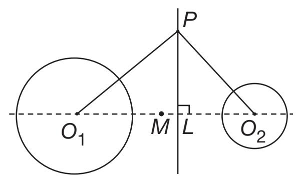
$$\angle PAD = \angle PCB.$$

$$\Rightarrow A, B, C, D \text{ are concyclic.}$$

Note: The above theorem is very useful for proving that four points are concyclic. It is one of the most commonly used criteria for proving concyclic points.

8.11.6 Radical Axis

Let ω_1 and ω_2 be two nonconcentric circles, then the locus of point with equal power with respect to both ω_1 and ω_2, is a line, called their radical axis. It is perpendicular to line joining centres of the circles.



Proof: Let ω_1 and ω_2 be two circles with different centres O_1 and O_2, and radii r_1 and r_2 respectively. Let $r_1 \geq r_2$. Let P be a point on the locus, then

$$\text{PO}_1^2 - r_1^2 = \text{PO}_2^2 - r_2^2 \qquad \text{(Given)} \qquad (1)$$

Now join PO_1, PO_2 and O_1O_2 and draw perpendicular from P to O_1O_2. Let L be the foot of the perpendicular. Also assume M be the mid-point of O_1O_2.

Now,

$$PO_1^2 - r_1^2 = O_1L^2 + PL^2 - r_1^2 \qquad \text{(Using Baudhayna theorem)} \qquad (2)$$

Also

$$PO_2^2 - r_2^2 = O_2L^2 + PL^2 - r_2^2 \qquad \text{(Using Baudhayna theorem)} \qquad (3)$$

From Eqs. (1), (2) and (3), we get,

$$O_1L^2 + PL^2 - r_1^2 = O_2L^2 + PL^2 - r_2^2$$

$$\Rightarrow O_1L^2 - O_2L^2 = r_1^2 - r_2^2$$

$$\Rightarrow (O_1L - O_2L)(O_1L + O_2L) = r_1^2 - r_2^2$$

$$\Rightarrow ((O_1M + ML) - (O_2M - ML))(O_1O_2) = r_1^2 - r_2^2 \quad \text{(As } M \text{ is the mid-point of } O_1O_2)$$

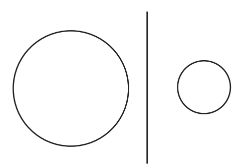
$$\Rightarrow 2ML \cdot O_1O_2 = r_1^2 - r_2^2$$

$\Rightarrow$ L is a fixed point

$\Rightarrow$ For any point P on the locus foot of perpendicular on O_1O_2 is always fix point L.

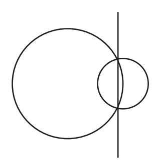
$\Rightarrow$ Locus is a straight line perpendicular to O_1O_2.

Note: It is always closer to the circumference of the larger circle.



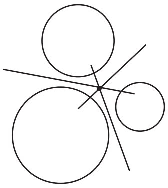
Corollary: Let ω_1 and ω_2 be two circles intersecting at the points A and B. Then their radical axis is precisely the common secant AB.

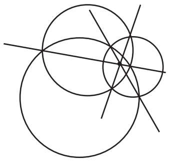
Proof: Clearly, points A and B have equal power (both zero) with respect to the circles. So A and B must lie on the radical axis. From radical axis theorem, we know that locus is a line, and two points determine that line



Note: If ω_1 and ω_2 be two circles intersecting at one point (*i.e.*, tangent to each other), then their radical axis is the common tangent at the point of contact.

8.11.7 Radical Centre





Let ω_1, ω_2 and ω_3 be three circles such that their centres are not collinear and no two concentric. Then their three pairwise radical axes are concurrent and point of concurrency is called radical centre.

Proof: Denote the three circles by ω_1, ω_2, and ω_3, and denote the radical axes of ω_i and ω_j by l_{ij}. As centres are non collinear, no two radical axis is parallel.
Let l_{12} and l_{13} meet at P.

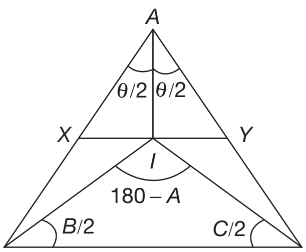
Since P lies on l_{12}, it has equal powers with respect to ω_1 and ω_2.

Similarly since P lies on l_{13}, it has equal powers with respect to ω_1 and ω_3.

Therefore, P has equal powers with respect to all three circles, and hence it must lie on l_{23} as well.

Note: If centres are collinear then their three pairwise radical axes are parallel.

Example 89 *ΔABC has incentre I. Let points X, Y be located on the line segments AB, AC respectively, so that, $BX \cdot AB = IB^2$ and $CY \cdot CA = IC^2$. Given that the points X, I, Y are collinear, find the possible values of $\angle A$.*



Solution: Let ABC be the triangle with incentre I. Let X, Y be points on AB, AC respectively such that, $BX \cdot BA = BI^2$ and $CY \cdot CA = CI^2$.
Hence by secant tangent theorem we can conclude that there are circles passing through AIX and AIY respectively, so that, BI is a tangent and BXA is secant in the first circle and CI is a tangent and CYA is a secant to the second circle.

Thus $\angle BIX = \angle BAI$ and $\angle CIY = \angle CAI$ (Alternate segment theorem)

i.e.,

$$\angle BIX = \frac{A}{2} \text{ and } \angle CIY = \frac{A}{2} \quad \text{(Alternate segment theorem)}$$

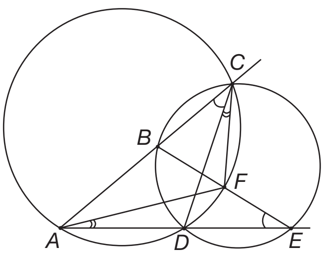
$$\Rightarrow \angle BIC = 180° - A \quad \text{(as } X, I, Y \text{ are collinear given)}$$

$$\text{Thus } \frac{B}{2} + (180° - A) + \frac{C}{2} = 180° \quad \text{(from } \Delta BIC\text{)}$$

$$\Rightarrow 90° - \frac{A}{2} - A = 0°$$

$$\Rightarrow \frac{3A}{2} = 90° \Rightarrow A = \frac{2 \times 90°}{3} = 60°.$$

Example 90 *From a point 'A', outside a circle, two straight lines ABC and ADE are drawn, intersecting the circle in B, C, D, and E respectively. A circle is described passing through A, C, D and cutting BE at F. Prove that $AD \cdot AE = AF^2$.*



Solution: Join CF.

Now $\angle BFA = \angle AEF + \angle EAF$ (Exterior angle property)

$= \angle BCD + \angle DCF$

(As BD subtends $\angle DEF$ and $\angle BCD$ on the same side of the circle and similarly DF subtends $\angle DAF$ and $\angle DCF$ on the same side of the second circle)

$\angle BFA = \angle BCF$

But these are angles on the alternate segment BF.

$\therefore$ AF is tangent and ABC is secant to circle BCF.

i.e., $AF^2 = AB \cdot AC$ (Tangent secant theorem)

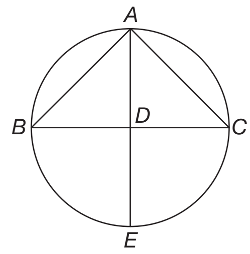
$AF^2 = AD \cdot AE$ (as $AB \cdot AC = AD \cdot AE$, Power of the point A).

Example 91 *If the internal bisector of $\angle A$ of a triangle meets the base BC at D, show that*

$$AD^2 + BD \cdot DC = AB \cdot AC.$$

Solution:

We have, as given,



$$\begin{aligned} AB \cdot AC &= AD \cdot AE \\ &= AD(AD + DE) = AD^2 + AD \cdot DE \end{aligned} \tag{1}$$

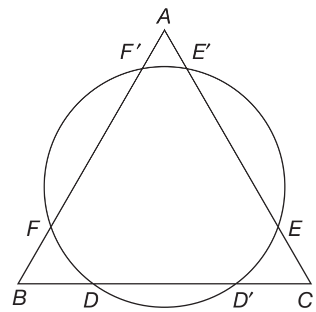
Since AE and BC are two chords of a circle which intersect at D, therefore by applying power of the point D, we get, $AD \cdot DE = BD \cdot DC$. (2)

Thus, from Eqs. (1) and (2) we get,

$$AB \cdot AC = AD^2 + BD \cdot DC.$$

Example 92 *A circle cuts the sides of ΔABC internally as follows; BC, at D, D'; CA at E, E' and AB at F', F. If AD, BE, CF are concurrent, prove that AD', BE', CF' are concurrent*

Solution: Let AD, BE, CF are concurrent, then by Ceva's theorem, we have



$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Also

$BD \cdot BD' = BF \cdot BF'$ (Power of the point B with respect to the circle)

$$\therefore \quad \frac{BD}{BF} = \frac{BF'}{BD'}. \tag{1}$$

Also $CD' \cdot CD = CE \cdot CE'$ (Power of the point C with respect to the circle)

$$\therefore \quad \frac{CE}{CD} = \frac{CD'}{CE'} \tag{2}$$

Also $AE' \cdot AE = AF' \cdot AF$ (Power of the point A with respect to the circle)

$$\frac{AF}{AE} = \frac{AE'}{AF'} \tag{3}$$

From Eqs. (1), (2) and (3) we get $\dfrac{BF'}{BD'} \cdot \dfrac{CD'}{CE'} \cdot \dfrac{AE'}{AF'} = \left(\dfrac{BD}{FB}\right) \cdot \left(\dfrac{CE}{CD}\right) \cdot \left(\dfrac{AF}{EA}\right) = 1$

$$\Rightarrow \left(\frac{BF'}{F'A}\right) \cdot \left(\frac{AE'}{E'C}\right) \cdot \left(\frac{CD'}{D'B}\right) = 1$$

$\therefore$ By converse of Ceva's theorem AD', BE', CF' are concurrent.

Example 93 *Given circles ω_1 and ω_2 intersecting at points X and Y, let l_1 be a line through the centre of ω_1 intersecting ω_2 at points P and Q and let l_2 be a line through the centre of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the centre of this circle lies on line XY.* **[USA MO, 2009]**

Solution: Let the circumcircle of $PQRS$ be ω_3.

Let the centre and the radius of the circle ω_k be O_k and r_k respectively, $k = 1, 2, 3$.

As O_1 lies on the line PQ, which is common chord (or radical axis) of ω_2, ω_3,

$\Rightarrow$ The power of O_1 with respect to ω_2, ω_3 are the same,

$$O_1O_2^2 - r_2^2 = O_1O_3^2 - r_3^2 \tag{1}$$

And similarly O_2 lies on the radical axis of ω_1, ω_3.

$\Rightarrow$ The power of O_2 with respect to ω_1, ω_3 are the same,

$$O_2O_1^2 - r_1^2 = O_2O_3^2 - r_3^2. \quad (2)$$

From Eq. (1) – Eq. (2), we get, $O_1O_3^2 - r_1^2 = O_2O_3^2 - r_2^2$

$\Rightarrow O_3$ lies on the radical axis of ω_1, ω_2. But radical axis of ω_1, ω_2 is line XY.

Hence X, Y, O_3 are collinear.

Example 94 *Let H be the orthocentre of acute angle triangle ABC. The tangents from A to the circle with diameter BC touch the circle at P and Q. Prove that P, Q, H are collinear.* **[China MO, 1996]**

Solution: Let A_1 and C_1 be the foot of the altitude from A and C respectively.
Let ω be the circle with diameter BC. Let D be the mid-point of BC.

Draw the circle ω_1 with diameter AD.

These two circles meets each other at P, Q (As $\angle APD = 90° = \angle AQD$)

$\Rightarrow PQ$ is the radical axis of the two circles.

As we need to prove H is collinear with P and Q, we need to prove that H is on the radical axis PQ of the two circles which is equivalent to prove that the H has equal power with respect to the two circles .

Power of H w.r.t. ω is $CH \cdot HC_1$ (1)

Since $\angle AA_1D = 90° \Rightarrow A_1 \in \omega_1$

$\Rightarrow$ Power of H w.r.t. ω_1 is $AH \cdot HA_1$. (2)

Now we can see that ACA_1C_1 is cyclic, writing power of H for this circle we get

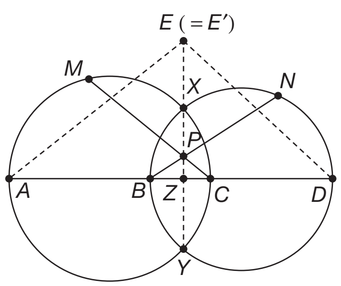
$AH \cdot HA_1 = CH \cdot HC_1$ (3)

From Eqs. (1), (2) and (3), we get

H has equal power with respect to the two circles ω, ω_1. Hence H must lie on their radical axis.

$\Rightarrow H$, P, Q are collinear.

Example 95 *Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.* **[IMO, 1995]**



Solution:

Draw DE parallel to CM meets XY at E, and draw AE_1 parallel to BN meets XY at E_1.

Claim: $E = E_1$.

Proof of claim: As Z is on the radical axis of the two circles, Equating power of the points with respect to two circles we get, $ZA \times ZC = ZB \times ZD$ (1)

$$\Delta AZE_1 \sim \Delta BZP \quad \text{(By construction)}$$

$$\Rightarrow \frac{ZE_1}{ZP} = \frac{ZA}{ZB} \quad (2)$$

Similarly ΔDZE ~ ΔCZP (By construction)

$$\Rightarrow \frac{ZP}{ZE} = \frac{ZC}{ZD} \quad (3)$$

From Eq. (2) × Eq. (3) we get,

$$\frac{ZE_1}{ZP} \times \frac{ZP}{ZE} = \frac{ZA}{ZB} \times \frac{ZC}{ZD}$$

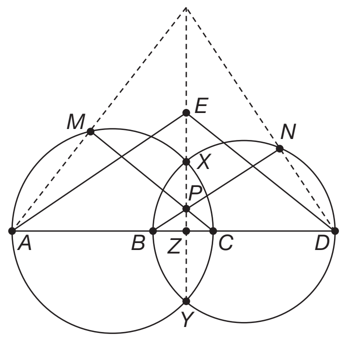
$$\Rightarrow \frac{ZE_1}{ZE} = 1 \quad \text{(From Eq. (1))}$$

$$\Rightarrow ZE = ZE_1.$$

Now, for ΔADE,

AM, DN and XY are the altitudes.

Hence they are concurrent.



Aliter:

$$\angle AMC = 90° \quad \text{(Angle in semicircle)}$$

$$\Rightarrow \quad \angle MCA = 90° - A$$

Also $\angle BND = 90°$. (Angle in semicircle)

As P is on XY (The radical axis of the two circles with diameters AC and BD), we get, $PN \cdot PB = PM \cdot PC$

$\Rightarrow$ Quadrilateral $MNBC$ is cyclic (by the converse of intersecting chord theorem)

Now in Cyclic Quadrilateral $MNBC$,

$\angle MCB = \angle MNB$ (Angle in same segment)

$\Rightarrow \quad \angle MND = \angle MNB + \angle BND$

Also $\angle MND = \angle MCB + 90° = 90° - A + 90° = 180° - \angle MAD$

$\Rightarrow$ Quadrilateral $AMND$ is cyclic.

Let the circumcircle of $AMND$ be circle ω.

Then,

AM is the radical axis of ω and the circle with diameter AC.

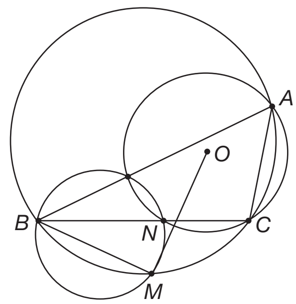
DN is the radical axis of ω and the circle with diameter BD.

Also we know XY is the radical axis circles with diameters AC and BD.

So from radical centre theorem, all three radical axis are concurrent.

Thus, AM, DN, XY are concurrent.

Example 96 *A circle with centre O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.* **[IMO, 1985]**



Solution: From the figure we can infer that the lines AC, KN, BM concur at the radical centre say P of the three circles involved.

Now from lemma 8.8.6, we have $OM \perp BP \Leftrightarrow OB^2 - OP^2 = MB^2 - MP^2$.

The quadrilateral PCNM is cyclic since $\angle PCN = \angle AKN = \angle BMN$.

$\Rightarrow PM \times PB = PC \times PA = OP^2 - r^2$ (By intersecting chords theorem) (1)

Where r is the circumradius of triangle AKC. Similarly,

Similarly $BM \times BP = BN \times BC = OB^2 - r^2$. (2)

From Eq. (2) – Eq. (1), we get

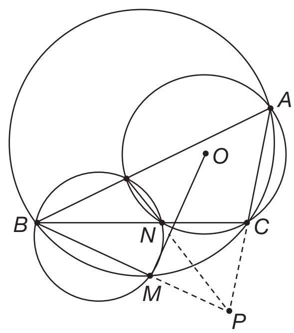
$OB^2 - OP^2 = BM \times BP - PM \times PB$

$= BP \times (BM - PM)$

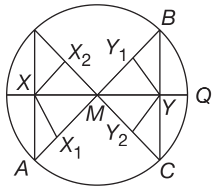
$= (BM + PM) \times (BM - PM)$

$= BM^2 - PM^2$

Hence, $OM \perp BP$.



Example 97 *Let PQ be a chord of a circle and M be the mid-point of PQ. Through M two chords AB and CD of the circle are drawn. Chords AD and BC intersect PQ at points X and Y respectively. Prove that M is the mid-point of the segment XY.*



Solution:

Construction: From X we draw perpendicular lines to AB and CD, with feet X_1 and X_2 respectively. From Y draw perpendicular lines to AB and CD, with feet Y_1 and Y_2 respectively. let $MX = x$, $MY = y$ and $PM = QM = a$.

Using similar triangles we get

$$\frac{x}{y} = \frac{XX_1}{YY_1} = \frac{XX_2}{YY_2}, \frac{XX_1}{YY_2} = \frac{AX}{CY} \text{ and } \frac{XX_2}{YY_1} = \frac{DX}{BY}$$

$$\Rightarrow \quad \frac{x^2}{y^2} = \frac{XX_1}{YY_1} \times \frac{XX_2}{YY_2}$$

$$= \frac{XX_1}{YY_2} \times \frac{XX_2}{YY_1}$$

$$= \frac{AX}{CY} \times \frac{DX}{BY}$$

$$= \frac{PX \times QX}{PY \times QY} \qquad \text{(By intersecting chords theorem)}$$

$$= \frac{(a+x)(a-x)}{(a+y)(a-y)} = \frac{a^2 - x^2}{a^2 - y^2}$$

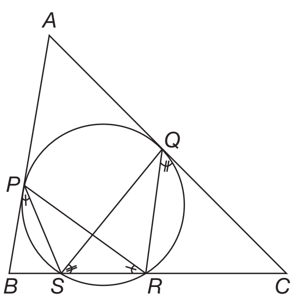
$$\Rightarrow \quad \frac{x^2}{y^2} = 1,$$

$$\Rightarrow \quad x = y.$$

Note: This problem is known as ***Butterfly theorem***.

Example 98 *Given a triangle ABC, let P and Q be points on segments AB and AC respectively, such that $AP = AQ$. Let S and R be distinct points on segment BC such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).* **[USA JMO, 2012]**

Solution: Since $\angle BPS = \angle PRS$, the circumcircle of triangle PRS is tangent to AB at P. Similarly, since $\angle CQR = \angle QSR$, the circumcircle of triangle QRS is tangent to AC at Q.



Now in order to prove P, Q, R, S concyclic, we will prove that circumcircles of triangles PRS and QRS are same.

If possible, let the circumcircles of triangles PRS and QRS are not the same circle.

Now $AP = AQ \Rightarrow A$ lies on the radical axis of both circles.

But radical axis of the circles is SR (As both circles pass through R and S)

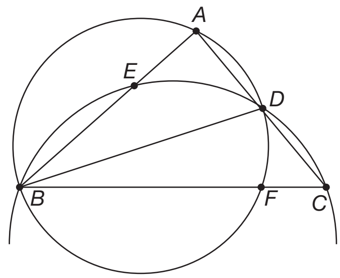
$\Rightarrow$ A lies on RS

$\Rightarrow$ A lies on BC, which is a contradiction.

$\Rightarrow$ The two circumcircles are the same circle

$\Rightarrow$ P, Q, R, and S are concyclic.

Example 99 *Let BD be the internal angle bisector of angle B in triangle ABC with D on side AC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that AE = CF.*



Solution:
Let the circumcircle of triangle BDC be ω_1 and the circumcircle of triangle ABD ω_2.

By angle bisector theorem, we get $\dfrac{AD}{CD} = \dfrac{AB}{CB}$ (1)

By applying intersecting chords theorem for point A with respect to circle ω_1, we get,

$$AE \times AB = AD \times AC \Rightarrow \quad AE = \frac{AD \times AC}{AB} \tag{2}$$

Also by Applying intersecting chords theorem for point C with respect to circle ω_2, we get,

$$CF \times CB = CD \times CA \quad \Rightarrow \quad CF = \frac{CD \times CA}{CB} \tag{3}$$

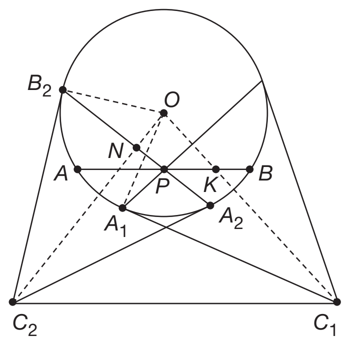
Dividing Eq. (1) by Eq. (2) we get,

$$\frac{AE}{CF} = \frac{AD \times CB}{AB \times CD} \tag{4}$$

From Eqs. (1) and (4), we get,

$$\frac{AE}{CF} = 1 \Rightarrow AE = CF.$$

Example 100 *AB is a chord of a circle, which is not a diameter. Chords A_1 B_1 and A_2 B_2 intersect at the mid-point P of AB. Let the tangents to the circle at A_1 and B_1 intersect at C_1. Similarly, let the tangents to the circle at A_2 and B_2 intersect at C_2. Prove that C_1C_2 is parallel to AB.*



Solution: Let O be the centre of the circle, let OC_1 intersects A_1B_1 at M, let OC_2 intersects $A_2\,B_2$ at N, and let also OC_1 intersects AB at K.

Clearly, OM and ON are respectively the perpendicular bisectors of $A_1\,B_1$ and $A_2\,B_2$.

So, $\angle OMP = \angle ONP = 90°$, saying that O, M, P, N are concyclic.

$\Rightarrow \angle ONM = \angle OPM = 90° - \angle MOP = \angle OKA.$ (1)

Claim: M, C_1, C_2, N are concyclic.

Proof of claim: As ΔOA_1C_1 and ΔOB_2C_2 are right-angled triangles,

$$OM \times OC_1 = OA_1^{\,2} = OB_2^{\,2} = ON \times OC_2.$$

Or $$OM \times OC_1 = ON \times OC_2.$$

$\Rightarrow$ M, C_1, C_2, N are concyclic (by the converse of intersecting chords theorem)

Now,

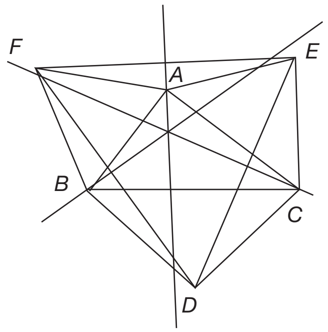
$$\angle OC_1C_2 = \angle ONM \qquad \text{(As } M, C_1, C_2, N \text{ are concyclic)}$$

$$\Rightarrow \angle OC_1C_2 = \angle OKA \text{ (from Eq. (1))}$$

$$\Rightarrow C_1C_2 \parallel AB$$

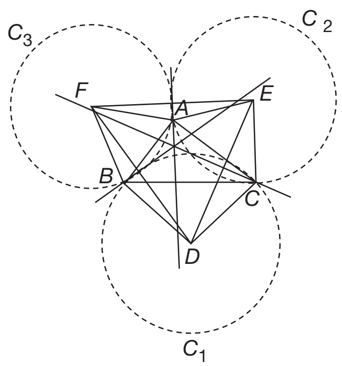
Example 101 *Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, and AB as their respective bases. Prove the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent.*

[USA MO, 1997]



Solution: Let ω_1 be the circle with centre D and radius DB, ω_2 be the circle with centre E and radius EC, ω_3 be the circle with centre F and radius FA.

Claim: The lines which need to prove concurrent are basically radical axes of the three pairs of circles (ω_1, ω_2), (ω_2, ω_3) and (ω_3, ω_1).



Proof of claim: As A is a point of the intersections of circles ω_2 and ω_3, A lies on the radical axis.

Also the radical axis of ω_2 and ω_3 is the line perpendicular to the line joining the centres E and F.

Hence, line through A and perpendicular to EF is radical axis of circles ω_2 and ω_3.

Similarly, the radical axis of ω_1 and ω_2 is the line through C perpendicular to DE, and the radical axis of ω_3 and ω_1 is the line through B perpendicular to FD.

From radical centre theorem, we conclude that these three radical axes concurrent.

Example 102 *Let quadrilateral ABCD be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the two tangents QE and QF to the circle where E and F are the points of tangency. Prove that the three points P, E, F are collinear.* **[CMO, 1997]**

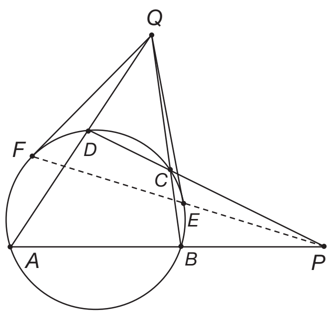
Solution:

Let ω_1 be the circumcircle and r_1 be circumradius of triangle ABC and O_1 be its centre.

Suppose the circumcircle ω_2 of QCD intersects the line PQ at Q and R.

Now $\angle PRC = \angle QDC = \angle ABC$

$\Rightarrow$ The points P, R, C, B are concyclic.

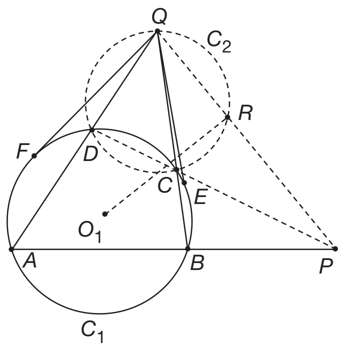


Let us first prove $O_1R \perp PQ$.

$O_1P^2 - r_1^{\,2} = PC \times PD = PR \times PQ$ (By intersecting chords theorem) (1)

Similarly,

$$O_1Q^2 - r_1^{\,2} = QC \times QB = QR \times QP, \tag{2}$$



From Eq. (1) – Eq. (2), we get,

$$\begin{aligned} O_1P^2 - O_1Q^2 &= PR \times PQ - QR \times QP \\ &= PQ \times (PR - QR) \\ &= (PR + QR) \times (PR - QR) \\ &= PR^2 - QR^2 \end{aligned}$$

$\Rightarrow O_1R \perp PQ$ (By lemma 8.8.6)

i.e., the points Q, F, O_1, E, R are also concyclic.

Let ω_3 be the circle passes through these five points. Now, we have three circles ω_1, ω_2, ω_3.

The radical axis of ω_1 and ω_2 is the line CD.

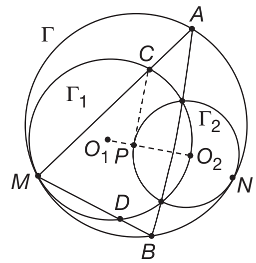
And the radical axis of ω_2 and ω_3 is the line QR.

These two radical axes intersect at P.

Hence, P lies on the radical axis of ω_3 and ω_1, namely EF.

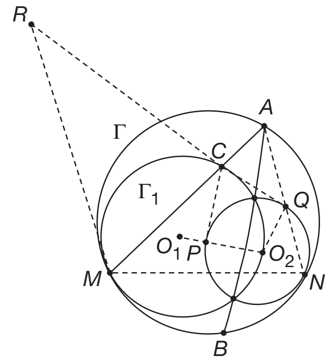
Example 103 *Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the centre of Γ_2. The* *line passing through the two points of intersection of* Γ_1 *and* Γ_2 *meets* Γ *at A and B. The lines MA and MB meet* Γ_1 *at C and D, respectively. Prove that CD is tangent to* Γ_2.

[IMO, 1999]



Solution: Let O_1 and O_2 be the centres of Γ_1 and Γ_2, respectively. The line O_1O_2 intersects Γ_2 at point P (see the adjacent figure). In order to prove CD tangent to Γ_2, we will prove $\angle CPO_2 = 90°$, by similar arguments $\angle DPO_2 = 90°$.

Let us join AN, which meets Γ_2 at point Q, let R be the intersection of the line CQ with the common tangent at M to Γ and Γ_1.



Claim: CQ is a common tangent of Γ_1 and Γ_2.

Proof of claim: As A is on the radical axis of Γ_1 and Γ_2, $AC \times AM = AQ \times AN$.

$\Rightarrow$ $CMNQ$ is cyclic (By the converse of intersecting chords theorem)

$\Rightarrow$ $\angle RCM = \angle MNQ$ (As $CMNQ$ is cyclic)

$\Rightarrow$ $\angle RCM = \angle RMC$ (Angle in alternate segment)

$\Rightarrow$ RC to tangent of Γ_1 from R. (Converse of angle in alternate segment)

$\Rightarrow$ CQ is a tangent of Γ_1

Similarly CQ is also a tangent of Γ_2.

Now in ΔCPO_2 and ΔCQO_2, we have,

$O_2P = O_2Q$,

$\angle PO_2C = 90° - 1/2\angle CO_1O_2 = 90° - \angle QCO_2$ (As CQ is a tangent of $\Gamma_1 \Rightarrow 1/2\angle CO_1O_2 = \angle QCO_2$)

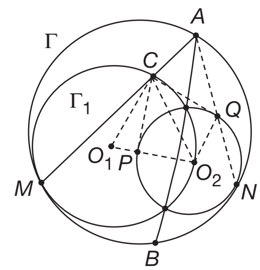
$\Rightarrow$ $\angle PO_2C = \angle QO_2C$ (As CQ is a tangent of Γ_2)

$\Rightarrow$ $\Delta CPO_2 \cong \Delta CQO_2$

$\Rightarrow$ $\angle CPO_2 = 90°$

By similar arguments we will get $\angle DPO_2 = 90°$

$\Rightarrow$ CPD are collinear and CD tangents to Γ_2 at P.



Build-up Your Understanding 11

1. Let ω_1 and ω_2 be two intersecting circles. Let a common tangent to ω_1 and ω_2 touch ω_1 at A and ω_2 at B. Show that the common chord of ω_1 and ω_2, when extended, bisects segment AB.
2. Given triangle ABC, let D, E be any points on BC. A circle through A cuts the lines AB, AC, AD, AE at the points P, Q, R, S, respectively. Prove that

$$\frac{AP \times AB - AR \times AD}{AS \times AE - AQ \times AC} = \frac{BD}{CE}.$$

3. Let ω_1 and ω_2 be concentric circles, with ω_1 in the interior of ω_1. From a point A on ω_1 one draws the tangent AB to ω_2 ($B \in \omega_2$). Let C be the second point of intersection of AB and ω_1, and let D be the mid-point of AB. A line passing through A intersects ω_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

[USA MO, 1998]

4. Let A, B, C be three points on a circle Γ with $AB = BC$. Let the tangents at A and B meet at D. Let DC meet Γ again at E. Prove that the line AE bisects segment BD.
5. AB is a chord of a circle, which is not a diameter. Chords A_1B_l and A_2B_2 intersect at the mid-point P of AB. Let the tangents to the circle at A_l and B_1 intersect at C_1. Similarly, let the tangents to the circle at A_2 and B_2 intersect at C_2. Prove that C_1C_2 is parallel to AB.

6. Let BB', CC' be altitudes of triangles ABC, and assume $AB \neq AC$. Let me be the mid-point of BC, H the orthocentre of ABC, and D the intersection of BC and $B'C'$. Show that DH is perpendicular to AM.

7. Let G be centroid of an ΔABC and circumcircle of ΔAGC touches the side AB at A. Given $BC = 6$, $AC = 8$ find AB.

8. Let C be a point on a semicircle of diameter AB and let D be the mid-point of arc AC. Let M be the projection of D onto the line BC and F the intersection of line AE with the semicircle. Prove that BF bisects the line segment DE.

9. A circle ω is tangent to two parallel lines l_1 and l_2. A second circle ω_1 is tangent to l_1 at A and to ω externally at C. A third circle ω_2 is tangent to l_2 at B, to ω externally at D and to ω_1 externally at E. Let Q be the intersection of AD and BC. Prove that $QC = QD = QE$. **[IMO Proposal, 1994]**

10. The circles S_1 and S_2 intersect at M and N. Show that if vertices A and C of a rectangle $ABCD$ lie on S_1 while vertices B and D lie on S_2, then the intersection of the diagonals of the rectangle lies on the line MN. **[Russia MO, 1997]**

11. Let ABC be an acute triangle. Let the line through B perpendicular to AC meet the circle with diameter AC at points P and Q, and let the line through C perpendicular to AB meet the circle with diameter AB at points R and S. Prove that P, Q, R, S are concyclic.

12. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, and AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines EF, FD, DE respectively, are concurrent. **[USA MO, 1997]**

13. Let D and E be the mid-point of sides AB and AC respectively and G be the centroid of the triangle. If A, D, G, E are concyclic, then prove that $b^2 + c^2 = 2a^2$.

14. Two circles P and Q with radii 1 and 2, respectively, intersect at X and Y. Circle P is to the left of circle Q. Prove that point A is to the left of XY if and only if $AQ^2 - AP^2 > 3$.

15. Let ABC be a triangle and let D and E be points on the sides AB and AC, respectively, such that DE is parallel to BC. Let P be any point interior to triangle ADE, and let F and G be the intersections of DE with the lines BP and CP, respectively. Let Q be the second intersection point of the circumcircles of triangles PDG and PFE. Prove that the points A, P, and Q are collinear.

16. Two circles Γ_1 and Γ_2 intersect at M and N. Let l be the common tangent to Γ_1 and Γ_2 so that M is closer to l than N is. Let l touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to l meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$. **[IMO, 2000]**

17. Let ABC be a triangle. A line parallel to BC meets sides AB and AC at D and E, respectively. Let P be a point inside triangle ADE, and let F and G be the intersection of DE with BP and CP, respectively. Show that A lies on the radical axis of the circumcircles of PDG and PFE. **[INMO, 1995]**

18. In an acute–angled triangle ABC, points D, E, F are located on the sides BC, CA, AB respectively, such that $\dfrac{CD}{CE} = \dfrac{CA}{CB} = \dfrac{AE}{AF} = \dfrac{AB}{AC}; \dfrac{BF}{BD} = \dfrac{BC}{BA};$

(i) Prove that AD, BE, CF are the altitudes of ΔABC.

(ii) Hence or otherwise, prove that, AD, BE, CF are concurrent.

19. O is the centre of a circle; OA is its radius. From a point C, in the exterior of the circle, CB is drawn perpendicular to OA. If CA cuts the circle at D, Prove that

(a) $CA \cdot AD = 2OA \cdot AB$

(b) Examine if the proposition is true when C is in the interior of the circle.

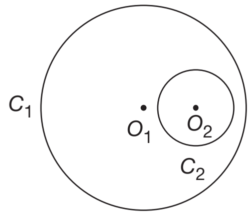
20. A circle with centre O is internally tangent to two circles inside it at points S and T. Suppose the two circles inside intersect at M and N with N closer to ST. Show that $OM \perp MN$ if and only if S, N, T are collinear.

21. $PQRS$ is a square. T is the mid-point of PQ. ST is produced to M, such that, $ST = 5TM$. Prove that, M lies on the circle circumscribing the square.

22. PT and PS are tangents from P to the circle with centre O. The line through P and O meets the circle at A and B. The chord of contact ST meets AB at C. Prove that PC is the Harmonic Mean between PB and PA.

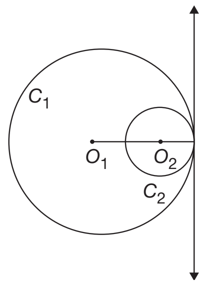
8.11.8 Common Tangents to Two Circles

Given two circles C_1 and C_2 with centres O_1 and O_2 with radii R and r respectively with $R > r$ and distance between their centres is 'd', then the number of common tangents that can be drawn to them varies from zero to four in the same plane of the circle depending upon the relative positions of the circles. Five different cases arises:



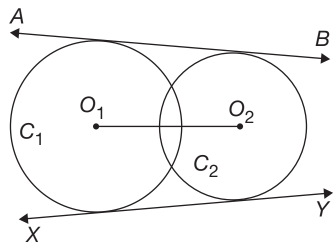
Case 1: The circle C_2 lies wholly within C_1 and the two circles do not touch each other (fig. 1). Here $d < R - r$

In this case the circle do not have any common tangent.



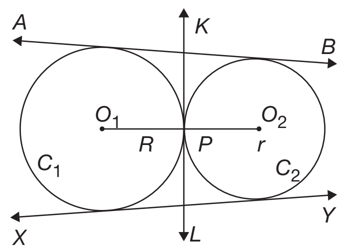
Case 2: The circle C_2 lies wholly within the circle C_1 and touches it internally at a point P (second figure). Here $d = R - r$.

In this case the circles have one common tangent at P. The line joining their centres also passes through the point of contact, *i.e.*, P of the circles.



Case 3: The circles C_1 and C_2 intersect each other (in two distinct points) as in fig. 3. Here $R - r < d < R + r$.

In this case the circles have two common tangents. Namely AB and XY. These tangents are called **Direct Common Tangents**.



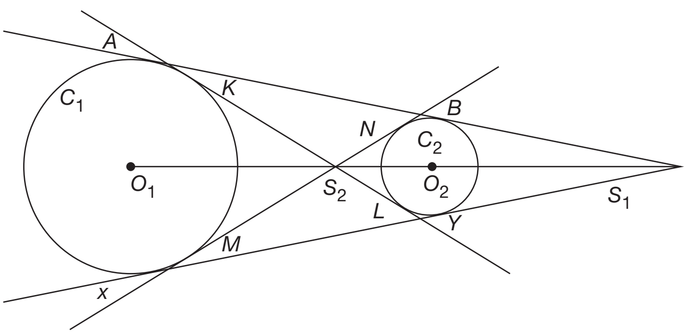
Case 4: The circle C_1 and C_2 touch each other externally as in the following figure. Here $d = R + r$.

In this case there are three common tangents. The two direct common tangents AB and XY and one common tangent KL at the point P where the circles touch each other.

In this case also the line segment joining the centres of the two circles passes through the point of contact.

Case 5: The circles C_1 and C_2 do not intersect and are placed as shown in fig. 5. Here $d > R + r$.

In this case there are four common tangents the two direct common tangents AB and XY and two transverse common tangents KL and MN.



8.11.8.1 Centres of Similitude of Two Circles

It can be easily seen that the direct common tangents to two circles intersect each other at a point on the line joining the centres. This point is called a centre of similitude of the circles. It divides the line joining the centres externally in the ratio of radii. That is in the figure of Case 5, we find S_1 is a centre of similitude of the circles C_1 and C_2. It divides O_1O_2 externally in the ratio $R : r$ so that $O_1S_1 : S_1O_2 = R : r$.

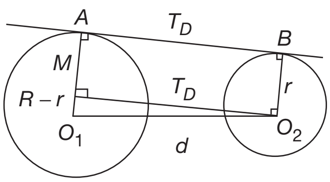
The transverse common tangents to two circles also intersect each other at a point on the line joining the centres. This point is also called a centre of similitude.

It divides the line joining the centres internally in the ratio of the radii. In the figure of Case 5, we find that S_2 is a centre of similitude of the circles. It divides O_1O_2 internally in the ratio $R : r$ so that

$$O_1S_2 : S_2O_2 = R : r$$

Thus there are two centres of similitude of two circles (lying outside each other and not intersecting at all). They divide the line joining the centres of the circles in the ratio of the radii, one internally and the other externally.

8.11.8.2 Length of the Direct Common Tangents



Let $O_1A = R$ and $O_2B = r$, $AB = T_D$ length of direct common tangent

Draw $O_2M \perp O_1A$

So quadrilateral $MABO_2$ will be a rectangle

$$\therefore \quad MO_2 = AB = T_D$$

$$MA = O_2B = r$$

$$\therefore \quad O_1M = O_1A - MA = R - r$$

$O_1O_2 = d$ (Distance between the centres)

In ΔO_1MO_2, by using Baudhayana (or Pythagoras) theorem

$$O_1O_2^2 = O_1M^2 + MO_2^2$$

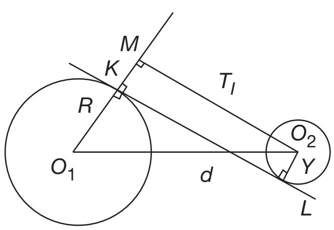
$$d^2 = (R-r)^2 + T_D{}^2$$

$$\Rightarrow \quad T_D{}^2 = d^2 - (R-r)^2 \Rightarrow T_D = \sqrt{d^2 - (R-r)^2}$$

Note: If two circles touch each other externally then $d = R + r$ and

$$T_D = \sqrt{(R+r)^2 - (R-r)^2} = \sqrt{4R \cdot r} = 2\sqrt{R \cdot r}.$$

8.11.8.3 Length of Transverse Common



Draw $O_2M \perp O_1K$ produced then O_2MKL is a rectangle

$$\therefore \quad KM = O_2L = r$$

$MO_2 = KL = T_I$ (Length of indirect tangent or transverse common tangent)

In right ΔO_1MO_2, $O_1O_2{}^2 = O_1M^2 + MO_2^2$

$$\Rightarrow d^2 = (R+r)^2 + T_I^2 \Rightarrow T_I^2 = d^2 - (R+r)^2$$

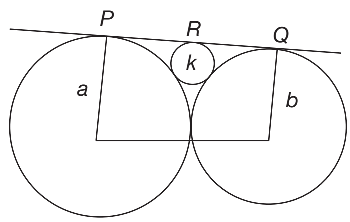
$$\Rightarrow T_I = \sqrt{d^2 - (R+r)^2}$$

Note: If two circles touch externally then $d = R + r$

$\therefore \quad T_I = \sqrt{(R+r)^2 - (R+r)^2} = 0$, *i.e.*, Length of transverse common tangent is zero.

Example 104: *Two circles with radii a and b respectively touch each other externally. Let c be the radius of a circle that touches these two circles as well as a common tangent to the two circles prove that*

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$



Solution: As when two circles touch externally then the length of their direct common tangent $= 2\sqrt{R \cdot r}$

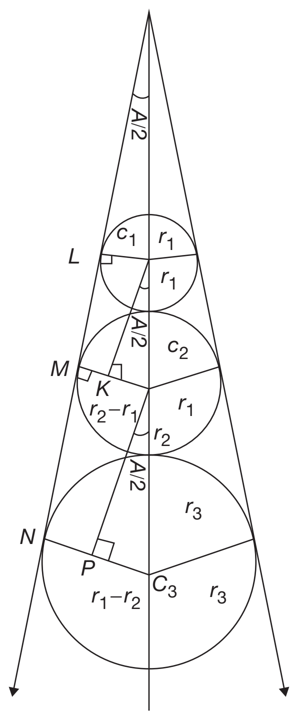
$$\therefore \quad PR = 2\sqrt{ac}; \quad RQ = 2\sqrt{bc}; \quad PQ = 2\sqrt{ab}$$

Now $PQ = PR + RQ$

$$\Rightarrow \quad 2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bc}$$

Divide both sides by $2\sqrt{abc}$, we get, $\dfrac{1}{\sqrt{c}} = \dfrac{1}{\sqrt{b}} + \dfrac{1}{\sqrt{a}}$.

Example 105 *Three circles C_1, C_2, C_3 with radii r_1, r_2, r_3 ($r_1 < r_2 < r_3$) respectively are given. They are placed such that C_2 lies to the right of C_1 and touches it externally. C_3 lies to the right of C_2 and touches it externally. Further there exists two straight lines each of which is a direct common tangent simultaneously to all the three circle. Find r_2 in terms of r_1 and r_3.*



Solution: $C_1L = r_1$; $C_2M = r_2$; $C_3N = r_3$

Draw $C_1K \perp C_2M$

$$\therefore \quad C_2K = C_2M - KM = C_2M - C_1L \Rightarrow C_2K = r_2 - r_1$$

Draw $C_2P \perp C_3N$

$$\therefore \quad C_3P = C_3N - PN = C_3N - C_2M \Rightarrow C_3P = r_3 - r_2$$

Since $C_1K \parallel C_2P$ and $C_{1L} \parallel C_{2M} \parallel C_{3N}$

$$\therefore \quad \angle C_1AL = \angle C_2C_1K = \angle C_3C_2P = \frac{A}{2} \quad \text{(Corresponding angles)}$$

$$\Rightarrow \Delta C_1C_2K \sim \Delta C_2C_3P \Rightarrow \frac{C_2K}{C_3P} = \frac{C_1C_2}{C_2C_3}$$

$$\Rightarrow \frac{r_2 - r_1}{r_3 - r_2} = \frac{r_1 + r_2}{r_2 + r_3} \quad \text{or} \quad \frac{r_2 - r_1}{r_2 + r_1} = \frac{r_3 - r_2}{r_3 + r_2}$$

Using Componendo and Dividendo, we get,

$$\frac{r_2 - r_1 + r_2 + r_1}{r_2 - r_1 - r_2 - r_1} = \frac{r_3 - r_2 + r_3 + r_2}{r_3 - r_2 - r_3 - r_2}$$

$$\Rightarrow \frac{2r_2}{-2r_1} = \frac{2r_3}{-2r_2}$$

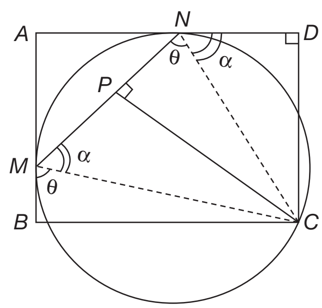
$$\Rightarrow \frac{r_2}{r_1} = \frac{r_3}{r_2} \Rightarrow r_2^2 = r_1 r_3$$

$$\Rightarrow r_2 = \sqrt{r_1 r_3}$$

Example 106 *A circle passes through the vertex C of a rectangle ABCD and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units. Find the area of the rectangle ABCD.*

Solution: Let $CP \perp MN$, $CP = 5$ units

By alternate segment theorem, $\angle CMB = \angle CNM = \theta$ also $\angle CND = \angle CMN = \alpha$



Consider quadrilaterals $DNPC$ and $PMBC$

$$\angle DNP = \alpha + \theta = \angle PMB$$

$$\angle NPC = 90° = \angle MBC$$

$$\text{And } \angle NDC = 90° = \angle MPC$$

By AAA similarity quadrilaterals are similar, hence

$$\frac{DC}{PC} = \frac{PC}{BC}$$

$$\Rightarrow BC \cdot DC = PC^2 = 5^2 = 25$$

$$\Rightarrow [ABCD] = 25 \text{ square units}$$

Aliter:

In ΔBMC, $\sin\theta = \dfrac{BC}{CM}$

In ΔNPC, $\sin\theta = \dfrac{CP}{CN}$

$$\therefore \frac{BC}{CM} = \frac{CP}{CN} \Rightarrow \frac{BC}{CP} = \frac{CM}{CN} \quad (1)$$

In ΔCND, $\sin\alpha = \dfrac{CD}{CN}$

In ΔCPM, $\sin\alpha = \dfrac{CP}{CM}$

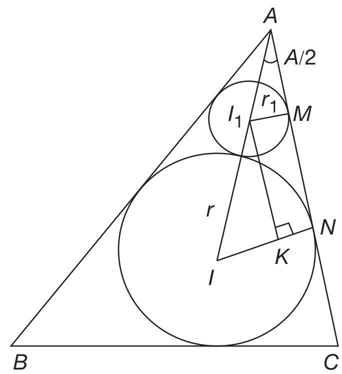
$$\Rightarrow \frac{CD}{CN} = \frac{CP}{CM} \Rightarrow \frac{CD}{CP} = \frac{CN}{CM} \quad (2)$$

Multiplying Eqs. (1) and (2), we get $\dfrac{BC}{CP} \cdot \dfrac{CD}{CP} = \dfrac{CM}{CN} \cdot \dfrac{CN}{CM} = 1$

$$\therefore BC \cdot CD = CP^2 = (5)^2 \Rightarrow BC \cdot CD = 25 \text{ sq. units}$$

$\therefore$ Area of rectangle = 25 sq. units.

Example 107 *Let ABC be a triangle and a circle C_1 be drawn lying inside the triangle, touching its in-circle C externally and also touching the two sides AB and AC. Show that the ratio of the radii of the circles C_1 and C is equal to* $\tan^2\left(\frac{\pi - A}{4}\right)$.



Solution: Draw $I_1K \perp IN$, $\therefore$ I_1KNM is a rectangle. $I_1K \parallel MN$

$$\angle II_1K = \angle I_1AM = \frac{A}{2}$$

In ΔII_1K, $\sin\frac{A}{2} = \frac{IK}{II_1} = \frac{r - r_1}{r + r_1}$

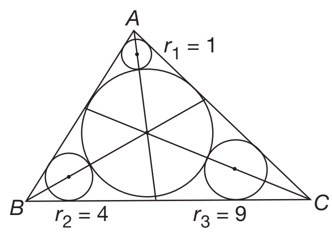
Applying componendo and dividendo, we get $\dfrac{1 - \sin\frac{A}{2}}{1 + \sin\frac{A}{2}} = \dfrac{r + r_1 - r + r_1}{r + r_1 + r - r_1}$

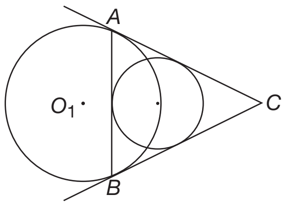
$$\Rightarrow \frac{1 - \cos\left(\frac{\pi}{2} - \frac{A}{2}\right)}{1 + \cos\left(\frac{\pi}{2} - \frac{A}{2}\right)} = \frac{2r_1}{2r} \Rightarrow \frac{2\sin^2\left(\frac{\pi - A}{4}\right)}{2\cos^2\left(\frac{\pi - A}{4}\right)} = \frac{r_1}{r}$$

$$\Rightarrow \tan^2\left(\frac{\pi - A}{4}\right) = \frac{r_1}{r}.$$

Build-up Your Understanding 12

1. Prove that of all straight lines drawn through a point of intersection of two circles and terminated by them, the one which is parallel to the line joining the centres is the greatest.
2. Two circles of equal radii cut each other at P and Q, so that the centre of one lies on the other. A straight line through P cuts the circle again at A and B. Prove that ΔQAB is equilateral.
3. A circle AOB, passing through the centre 'O' of another circle cuts the latter circle at A and B. A straight line APQ is drawn meeting the circle AOB in P and the other circle in Q. Prove that $PB = PQ$.
4. The altitude AD of ΔABC is produced to cut the circumcircle in K. Prove that $HD = DK$ where H is the orthocentre.
5. The chords AC, BD of a circle cut at right angles at O. Prove that the median of ΔDCO through O is perpendicular to AB. Also prove that the perpendicular from O on AB produced bisects CD.
6. BE, CF are the altitudes from B and on the opposite sides of a ΔABC. If P be the mid-point of BC. Show that $PE = PF$.
7. A triangle ABC is inscribed in a circle and $\angle A$ is bisected by AE meeting the circumference in F. Also $\angle C$ is bisected by CI meeting AE in I. Prove that EB, EC, FI are all equal.
8. Two circles touch internally at A and a chord APQ is drawn cutting them in P and Q. If the tangent at P meets the other circle in H and K. Prove that $HQ = KQ$.
9. ABC is a triangle. Circles with radii as shown are drawn inside the triangle each touching two sides and the incircle. Find the radius of the incircle of the ΔABC.

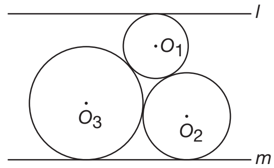


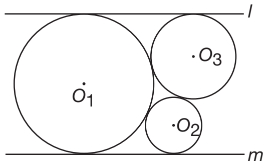


10. Let D be an arbitrary point on the side AB of a given triangle ABC and let E be the intersection point where CD intersects the external common tangent to the incircles of the triangles ACD and BCD. As D assumes all positions between A and B, prove that, the point E traces the arc of a circle.

11. The tangents at A and B on a given circle $O_1(r)$ intersect at C. Show that the in-centre of the triangle lies on the given circle.

12. Three circles $O_1(r_1)$, $O_2(r_2)$ and $O_3(r_3)$ touch each other externally. The line l is tangent to $O_1(r_2)$ and parallel to the exterior common tangent m to $O_2(r_2)$ and $O_3(r_3)$ which does not intersect $O_1(r_1)$. Find the distance between the lines l and m.





13. Two circles $O_1(r_1)$ and $O_2(r_2)$ $r_1 > r_2$, touch each other externally, and the line l is a common tangent. The line m is parallel and touches $O_1(r_1)$ and the circle $O_3(r_3)$ touches m and the two given circles externally. Show that $r_1^2 = 4r_2r_3$.

8.12 Quadrilaterals (Cyclic and Tangential)

8.12.1 Cyclic Quadrilateral

A quadrilateral which has a circle passing through all its four vertices is called a cyclic quadrilateral (or Inscribed quadrilateral). This circle is called circumcircle of the quadrilateral, its centre is the circumcentre and its radius is called the circumradius.

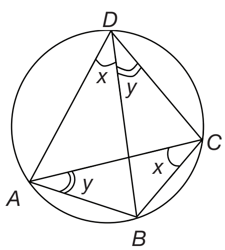
8.12.1.1 Theorem

If a quadrilateral is cyclic, then the sum of each pair of opposite angles is 180°

Proof: $\angle BCA = \angle BDA = x$ (Say)

And $\angle BAC = \angle BDC = y$ (Say)

In ΔBAC,



$$\angle BAC + \angle BCA + \angle B = 180°$$

$$x + y + \angle B = 180$$

but $$x + y = \angle D$$

$\therefore$ $$\angle D + \angle B = 180° \quad (1)$$

Also in quadrilateral $ABCD$,

$$\angle A + \angle B + \angle C + \angle D = 360°$$

$\therefore$ $$(\angle A + \angle C) + (\angle B + \angle D) = 360°$$

$$\angle A + \angle C = 180° \quad \text{(From Eq. (1))}$$

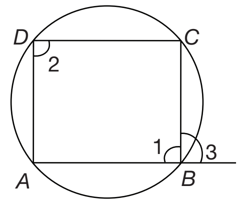
8.12.1.2 Corollary

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Since

$$\angle 1 + \angle 2 = 180°$$

Also



$$\angle 1 + \angle 3 = 180° \text{ (Linear pair)}$$
$$\therefore \quad \angle 1 + \angle 2 = \angle 1 + \angle 3$$
$$\Rightarrow \quad \angle 2 = \angle 3$$

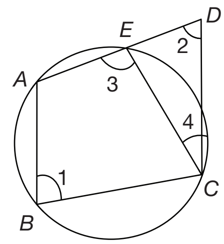
8.12.1.3 Theorem

If in a quadrilateral, the sum of a pair of opposite angles is 180°, then it is cyclic.

Proof: Let in quadrilateral $ABCD$, $\angle B + \angle D = 180°$

Consider a circle passing through A, B and C if possible let D be not on this circle.

Then two cases may arise either D lies outside the circle or inside the circle.



Case 1: If possible let D be outside the circle

Join AD which cuts the circle at E. Join CE. Since $ABCE$ is a cyclic quadrilateral

$\angle 1 + \angle 3 = 180°$

Also it is given that

$\angle 1 + \angle 2 = 180°$

$\therefore\ \angle 1 + \angle 3 = \angle 1 + \angle 2$

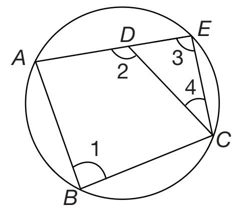
$\Rightarrow \quad \angle 3 = \angle 2$ (1)

but $\angle 3 = \angle 2 + \angle 4$ (Exterior angle property)

$\therefore\ \angle 3 > \angle 2$ (2)

$\therefore$ Eqs. (1) and (2) contradict each other.

Thus D cannot lie outside the circle.



Case 2: If possible let D be inside the circle.

Produce AD to cut the circumcircle at E

Join CE

Since $ABCE$ is a cyclic quadrilateral,

$\angle 1 + \angle 3 = 180°$

Also $\angle 1 + \angle 2 = 180°$ (given)

$\Rightarrow \quad \angle 1 + \angle 2 = \angle 1 + \angle 3$

$\Rightarrow \quad \angle 2 = \angle 3$ (3)

But $\angle 2 = \angle 3 + \angle 4$ (Exterior angle property)

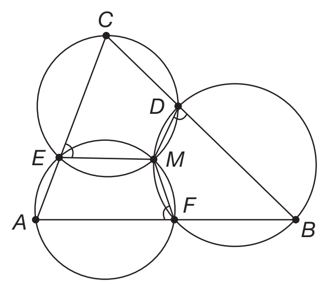
$\Rightarrow \quad \angle 2 > \angle 3$ (4)

$\therefore$ Eqs. (3) and (4) contradict with other

Thus D cannot lie inside the circle.

Thus D must lie on the circle

Example 108 *Let ABC be a triangle, with arbitrary points D, E and F on sides BC, AC, and AB respectively (or their extensions). Draw three circumcircles to triangles AEF, DBF, and DEC. Then prove that these circles intersect in a single point M.*



Solution: Let circumcircles of triangles AEF, DBF intersect each other at F and M.

From cyclic quadrilateral $AEMF$,

$\angle CEM = \angle MFA$ (Exterior angle property of Cyclic quadrilateral) (1)

From cyclic quadrilateral $FBDM$,

$\angle MFA = \angle MDB$ (Exterior angle property of Cyclic quadrilateral) (2)

From Eqs. (1) and (2), we get,

$\angle CEM = \angle MDB$

$\Rightarrow$ $CEMD$ is a cyclic quadrilateral

$\Rightarrow$ Circumcircle of ΔCED passes through M.

Auguste Miquel

1816–1851

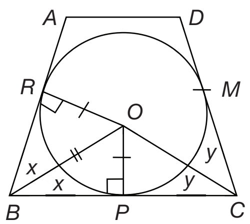
Nationality: French

Note: The problem statement is known as ***Miquel's theorem*** and point M is called the ***Miquel point***. Special case of this theorem in which DEF are not collinear called ***Pivot Theorem***. In case of DEF are collinear Miquel point lies on circumcircle of ΔABC.

Example 109 *Let $ABCD$ be a convex quadrilateral. Consider four circles C_1, C_2, C_3, and C_4 each of which touches 3 sides of this quadrilateral. C_1 touches AB, BC, CD, C_2 touches BC, CD, DA, C_3 touches CD, DA, AB and C_4 touches DA, AB, BC. Prove that the centres O_1, O_2, O_3, O_4 of the four circles form a cyclic quadrilateral.*

Solution: First we will prove some basic results.

Let C_1, touches, AB, BC, and CD at R, P and M respectively with centre O_1.



Now $O_1P \perp BC$ and $O_1R \perp AB$

In ΔO_1PB and ΔO_1RB

$\angle O_1PB = \angle O_1RB = 90°$

$O_1P = O_1R$ (Radius)

$O_1B = O_1B$ (Common)

$\therefore$ By RHS, congruency

$\Delta O_1PB \cong \Delta O_1RB$

$\angle O_1BP = \angle O_1BR$

i.e., O_1B bisects $\angle B$

Similarly O_1C bisects $\angle C$

So let $\angle O_1BP = \angle O_1BR = x$

And $\angle O_1CP = \angle O_1CM = y$

Now in quadrilateral $ABCD$

$\angle A + \angle B + \angle C + \angle D = 360°$

$\angle A + 2x + 2y + \angle D = 360°$

$2x + 2y = 360°\ (\angle A + \angle D)$

$x + y = 180° - \frac{1}{2}(\angle A + \angle D)$

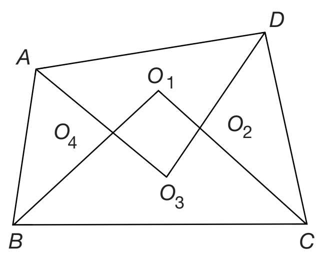
In ΔBO_1C,

$$x + y + \angle BO_1C = 180°$$

$$\Rightarrow \quad 180° - \frac{1}{2}(\angle A + \angle D) + \angle BO_1C = 180°$$

$$\Rightarrow \quad \angle BO_1C = \frac{1}{2}(\angle A + \angle D)$$

Similarly In ΔAO_3D,



$$\angle AO_3D = \frac{1}{2}(\angle C + \angle B)$$

Now in quadrilateral $O_1O_2O_3O_4$,

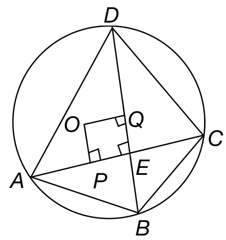
$$\angle O_4O_1O_2 + \angle O_2O_3O_4 = \frac{1}{2}(\angle A + \angle D) + \frac{1}{2}(\angle B + \angle C)$$

$$= \frac{1}{2}(\angle A + \angle B + \angle C + \angle D)$$

$$= \frac{1}{2} \times 360 = 180°$$

Hence quadrilateral $O_1O_2O_3O_4$ is a cyclic quadrilateral.

Example 110 *The diagonals AC and BD of a cyclic quadrilateral ABCD meet at right angle in E. Prove that* $EA^2 + EB^2 + EC^2 + ED^2 = 4R^2$, *where R is the radius of the circumscribing circles.*



Solution: Let O be the centre of the circle and P, Q be the feet's of the perpendicular from O to AC and BD.

Clearly $OPEQ$ is a rectangle

Now $EA^2 + EC^2 = (EP + PA)^2 + (PC - PE)^2$

$= EP^2 + PA^2 + 2PA \cdot PE + PC^2 + PE^2 - 2PC\,PE$ (As $PA = PC$)

$= 2(PA^2 + PE^2)$

Similarly

$EB^2 + ED^2 = 2(QD^2 + QE^2)$

$\therefore\ EA^2 + EB^2 + EC^2 + ED^2$

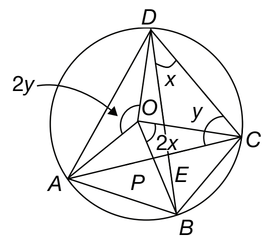
$= 2[PA^2 + PE^2 + QD^2 + QE^2]$

$= 2[PA^2 + OQ^2 + QD^2 + OP^2]$ (As $PE = OQ$, $QE = OP$)

$= 2[PA^2 + OP^2 + QD^2 + OQ^2]$

$= 2[OA^2 + OD^2] = 2[R^2 + R^2]$

$EA^2 + EB^2 + EC^2 + ED^2 = 4R^2$



Aliter: Let $\angle BDC = x$

then $\angle BOC = 2x$

Also let $\angle ACD = y$

Then $\angle AOD = 2y$

Also in ΔEDC $x + y = 90°$

$\therefore\ 2x + 2y = 180°$

$\therefore\ 2y = 180° - 2x$

Since, $EA^2 + ED^2 = AD^2$ and $EB^2 + EC^2 = BC^2$

$\therefore\ EA^2 + EB^2 + EC^2 + ED^2 = AD^2 + BC^2$

$= OA^2 + OD^2 - 2OA \cdot OD \cos 2y + OB^2 + OC^2 - 2OB \cdot OC \cos 2x$

$= R^2 + R^2 - 2R^2 \cos 2y + R^2 + R^2 - 2R^2 \cos 2x$

$= 4R^2 - 2R^2 [\cos 2y + \cos 2x]$

$= 4R^2 - 2R^2 [\cos(180° - 2x) + \cos 2x]$

$= 4R^2 - 2R^2 [-\cos 2x + \cos 2x]$

$= 4R^2$

8.12.2 Simson–Wallace Line

The feet's L, M, N of the perpendiculars on the sides BC, CA, AB of any ΔABC from any point X on the circumcircle of the triangles are collinear. The line LMN is called the Simson–Wallace line.

Robert Simson

14 Oct 1687–1 Oct 1768
Nationality: French

This concept was first published by William Wallace.

14 Oct 1687–1 Oct 1768
Nationality: Scottish

Proof: Join AX, XC. Join NM and ML.

Now to prove L, M, N collinear, we will prove $\angle LMX + \angle NMX = 180°$.

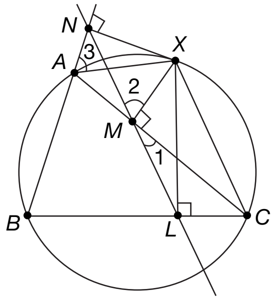
Since $\angle XMC = \angle XLC = 90°$

$\therefore$ $XMLC$ is a cyclic quadrilateral.

$\therefore\ \angle XML + \angle XCL = 180°$

$90° + \angle 1 + \angle C = 180°$

$\therefore\ \angle 1 + \angle C = 90°$ (1)



Since $\angle ANX + \angle AMX = 90° + 90° = 180°$

A, M, X, N are concyclic

$\angle XAN = \angle XMN$

i.e., $\angle 3 = \angle 2$

Also $AXCB$ is a cyclic quadrilateral

$\therefore\ \angle 3 = \angle C$

$\Rightarrow \angle 2 = \angle 3 = \angle C$ (2)

From Eqs. (1) and (2)

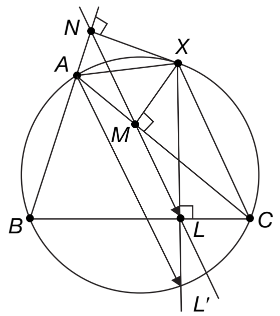
$\angle 1 + \angle 2 = 90°$ (3)

Now, $\angle LMX + \angle XMN = \angle 1 + 90° + \angle 2 = 180°$ (From Eq. (3))

$\Rightarrow\quad L, M, N$ are collinear.

Note: Converse is also true. That is, L, M, N are collinear then X lies on the circumcircle of the triangle.

Example 111 *If the perpendicular XL on side BC of ΔABC meets the circumcircle again at L′ then prove that AL′ is parallel to the Simson line of X.*



Proof: Since XL produced meets the circumcircle at L'

$$\therefore\ \angle XCA = \angle XL'A \qquad (1)$$

Also $\angle XMC = \angle XLC = 90°$

So $XMLC$ is a Cyclic quadrilateral

$\angle XCA = \angle XLM$ (2)

From Eqs. (1) and (2)

$\angle XL'A = \angle XLM$

$\therefore$ By converse of corresponding angle postulate $AL' \parallel LN$.

8.12.3 Ptolemy's Theorem

Claudius Ptolemy

c. AD 100–c. 170
Nationality: Greek

In a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the pairs of opposite sides.

Proof: Given $ABCD$ is a cyclic quadrilateral

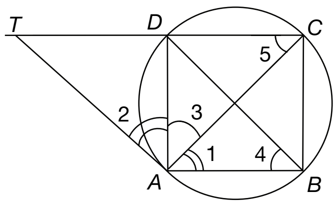
To prove

$AB \cdot CD + AD \cdot BC = AC \cdot BD$

Construction:

Draw $\angle DAT = \angle CAB$

which cuts CD produced at T



Proof: In ΔCAT and ΔBAD

$\angle ACT = \angle ABD$ (Angles in a same segment)

$\angle CAT = \angle 2 + \angle 3 = \angle 1 + \angle 3 = \angle BAD$ (As $\angle 1 = \angle 2$)

$\therefore$ By AA similarity

$\Delta CAT \sim \Delta BAD$

$$\therefore\ \frac{AC}{AB} = \frac{TC}{DB}$$

$$\Rightarrow\quad TC = \frac{AC}{AB} \cdot BD \qquad (1)$$

In ΔDAT and ΔBAC

$\angle DAT = \angle BAC$ (Construction)

$\angle TDA = \angle CBA$ (Exterior angle of a cyclic quadrilateral)

$\therefore$ By AA similarity

$\Delta DAT \sim \Delta BAC$

$$\therefore \frac{AD}{AB} = \frac{TD}{BC}$$

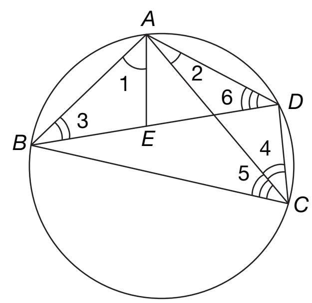
$$\Rightarrow \quad TD = \frac{AD}{AB} \cdot BC \qquad (2)$$

$\therefore TC = TD + DC$

$$\Rightarrow \quad \frac{AC}{AB} \cdot BD = \frac{AD}{AB} \cdot BC + DC \quad \text{(From Eqs. (1) and (2))}$$

$\Rightarrow \quad AC \cdot BD = AD \cdot BC + AB \cdot CD.$ Hence proved.

Aliter: Choose a point E in BD, so that $\angle BAE = \angle DAC$.



In Δs ABE and ACD, we have

$\angle BAE = \angle CAD$ (Construction)

$\angle ABE = \angle ACD$ (Angles in the same segment of a circle).

$\therefore$ Δs are equiangular and hence similar.

$$\therefore \frac{BE}{DC} = \frac{AB}{AC} \quad \text{or} \quad AB \cdot CD = AC \cdot BE \qquad (1)$$

Let us now consider triangles BAC and EAD,

$\angle BAC = \angle EDA$ (Add $\angle EAC$ to both $\angle 1$ and $\angle 2$)

$\angle BCA = \angle EDA$ (Angles in the same segment of a circle)

$\therefore$ The triangles are equiangular and hence similar.

$$\therefore \frac{BC}{ED} = \frac{AC}{AD} \quad \text{or} \quad BC \cdot AD = AC \cdot ED \qquad (2)$$

Adding corresponding sides of Eqs. (1) and (2),

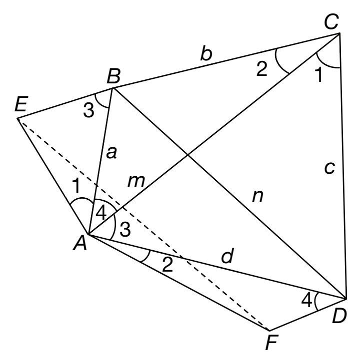
$$AB \cdot CD + AD \cdot BC = AC \cdot BE + AC \cdot ED \qquad (3)$$

i.e., $AB \cdot CD + AD \cdot BC = AC.(BE + ED)$, *i.e.*, $AC \cdot BD$.

8.12.4 Generalization of Ptolemy's Theorem (for All Convex Quadrilateral)

In any quadrilateral, product of the diagonals is less than or equal to the sum of the products of the pairs of opposite sides. Equality holds for cyclic quadrilateral only.

Proof:



Claim: $ABCD$ is a quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = m$ and $BD = n$

Then $m^2n^2 = a^2c^2 + b^2d^2 - 2abcd\cos(A + C)$

Proof of claim: Construct a $\Delta ABE \sim \Delta CAD$ on the side AB so that $\angle ABE = \angle CAD = \angle 3$ and $\angle BAE = \angle ACD = \angle 1$

Since $\Delta ABE \sim \Delta CAD$

$$\therefore \frac{AB}{CA} = \frac{AE}{CD} = \frac{BE}{AD}$$

Or $\dfrac{a}{m} = \dfrac{AE}{c} = \dfrac{BE}{d}$

$\therefore AE = \dfrac{ac}{m}$ and $BE = \dfrac{ad}{m}$ (1)

Construct a $\Delta ADF \sim \Delta CAB$ so that

$\angle ADF = \angle CAB = \angle 4$ and

$\angle DAF = \angle ACB = \angle 2$.

As $\Delta ADF \sim \Delta CAB$,

$\therefore \dfrac{AD}{CA} = \dfrac{AF}{CB} = \dfrac{DF}{AB}$

$\Rightarrow \dfrac{d}{m} = \dfrac{AF}{b} = \dfrac{DF}{a}$

$\Rightarrow AF = \dfrac{bd}{m}$ and $DF = \dfrac{ad}{m}$ (2)

$\therefore$ From Eqs. (1) and (2), we get,

$BE = DF = \dfrac{ad}{m}$

Also,

$\angle EBD + \angle BDF = \angle 3 + \angle ABD + \angle BDA + \angle 4$

$= \angle ABD + \angle BDA + \angle BAD$ (As $\angle 3 + \angle 4 = \angle BAD$)

$\therefore \angle EBD + \angle BDF = 180°$ (by ASP of ΔABD)

$\therefore BE \parallel DF$ and $BE = DF$

Since in a quadrilateral if one pair of opposite side is equal and parallel then it is a ||gm.

$\therefore EBDF$ is a parallelogram.

So, $EF = BD = n$

Further $\angle EAF = \angle 1 + \angle 2 + \angle 3 + \angle 4$

$\Rightarrow \angle EAF = \angle A + \angle C$

So applying cosine rule in ΔEAF

$EF^2 = AE^2 + AF^2 - 2AE \cdot AF \cos\angle EAF$

$n^2 = \dfrac{a^2c^2}{m^2} + \dfrac{b^2d^2}{m^2} - 2\left(\dfrac{ac}{m}\right)\left(\dfrac{bd}{m}\right)\cos(\angle A + \angle C)$ (From Eqs. (1), (2), and (3))

$m^2n^2 = a^2c^2 + b^2d^2 - 2abcd\cos(\angle A + \angle C)$

Now $\cos(\angle A + \angle C) \geq -1$

$\Rightarrow (mn)^2 \leq (ac)^2 + (bd)^2 - 2abcd(-1)$

$\Rightarrow (mn)^2 \leq (ac + bd)^2$

$\Rightarrow mn \leq ac + bd$

Also equality holds when $\cos(\angle A + \angle C) = -1 \Rightarrow \angle A + \angle C = 180°$.

And we get result of Ptolemy's theorem.

Hence, the product of the diagonals is less than or equal to the sum of the products of the pairs of opposite sides in any quadrilateral.

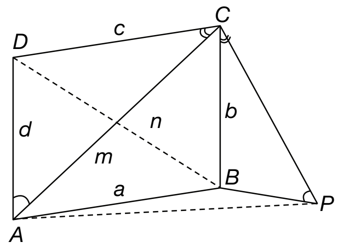
Aliter: $ABCD$ is a quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = m$ and $BD = n$.

On side BC of the quadrilateral outwardly construct a ΔPBC directly similar to ΔADC. Join PA.

As $\Delta PBC \sim \Delta ADC$,

$$\frac{PB}{AD} = \frac{BC}{DC} = \frac{CP}{CA}$$

$$\Rightarrow PB = \frac{d \cdot b}{c} \quad (1)$$



We can easily prove that $\Delta CDB \sim \Delta CAP$, as follows:

Since $\Delta PBC \sim \Delta ADC$

$$\frac{BC}{DC} = \frac{PC}{AC} \Rightarrow \frac{BC}{PC} = \frac{DC}{AC}$$

Also $\angle BCD = \angle PCA$

By SAS, $\Delta CDB \sim \Delta CAP$

So $\dfrac{CD}{CA} = \dfrac{DB}{AP}$

Or $\dfrac{c}{m} = \dfrac{n}{AP} \Rightarrow AP = \dfrac{m \cdot n}{c}$ (2)

Consider triangle inequality in ΔABP,

$AB + BP \geq PA$

Or $a + \dfrac{b \cdot d}{c} \geq \dfrac{mn}{c}$ (from Eqs. (1) and (2))

$\Rightarrow a \cdot c + b \cdot d \geq m \cdot n$

Equality occurs if and only if points A, B, P are collinear,

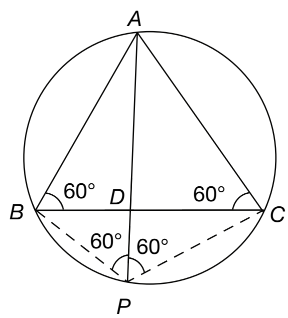
i.e., $\angle CBA = 180° - \angle PBC = 180° - \angle ADC$

$\Rightarrow$ Quadrilateral $ABCD$ is cyclic.

Example 112 *A line drawn from the vertex A of an equilateral triangle ABC meets BC at D and the circumcircle at P. Prove that*

(i) $PA = PB + PC$

(ii) $\dfrac{1}{PD} = \dfrac{1}{PB} + \dfrac{1}{PC}$



Solution:

(i) Since $ABPC$ is a cyclic quadrilateral, by Ptolemy's Theorem,

$\therefore AB \cdot PC + AC \cdot PB = BC \cdot AP$

Since $AB = BC = AC = a$,

$\therefore a \cdot PC + a \cdot PB = a \cdot PA$

$\Rightarrow PA = PB + PC$ (1)

(ii) Now divide Eq. (1) by $PB \cdot PC$ we get

$$\frac{PA}{PB \cdot PC} = \frac{1}{PC} + \frac{1}{PB} \quad (2)$$

Now it is enough to prove $\dfrac{PA}{PB \cdot PC} = \dfrac{1}{PD}$ or $\dfrac{PA}{PB} = \dfrac{PC}{PD}$

In ΔAPB and ΔCPD

$\angle APB = \angle ACB = 60° = \angle CBA = \angle CPD$

i.e., $\angle APB = \angle CPD = 60°$

Also $\angle PAB = \angle PCD$ (Angles in a same segment)

$\therefore$ By AA similarity

$\Delta APB \sim \Delta CPD$

$$\Rightarrow \frac{PA}{PC} = \frac{PB}{PD}$$

$$\Rightarrow \frac{PA}{PB \cdot PC} = \frac{1}{PD} \qquad (3)$$

$\therefore$ From Eqs. (2) and (3), we get

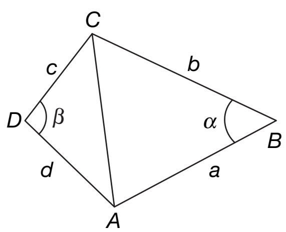
$$\frac{1}{PD} = \frac{1}{PB} + \frac{1}{PC}.$$

Example 113 *Given that a, b, c, d are the measures of the sides of a quadrilateral in clockwise direction, prove the inequalities,*

(i) $[ABCD] \le \frac{1}{2}(ab + cd).$

(ii) $[ABCD] \le \frac{1}{2}(ad + bc)$

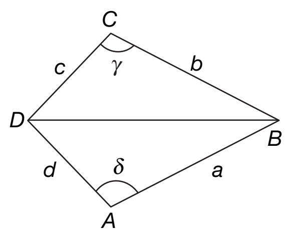
(iii) $[ABCD] \le \frac{1}{4}(a + b)(c + d).$



Solution: Area of the quadrilateral $ABCD$, i.e.,

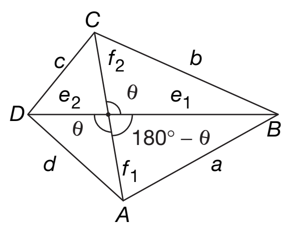
$$[ABCD] = \frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin\beta \qquad \text{(in The frist figure (1))}$$

$$[ABCD] \le \frac{1}{2}(ab + cd) \qquad (\text{As } \sin\alpha,\ \sin\beta \le 1) \qquad (1)$$



From the second figure, $[ABCD] = \frac{1}{2}ad\sin\delta + \frac{1}{2}bc\sin\gamma$

$$\Rightarrow [ABCD] \le \frac{1}{2}(ad + bc) \quad (\text{As } \sin\gamma, \sin\delta \le 1) \qquad (2)$$



In the third figure, let $AC = f_1 + f_2 = f$ and $BD = e_1 + e_2 = e$

Now,

$$[ABCD] = \frac{1}{2}e_1 f_1 \sin(180° - \theta) + \frac{1}{2}e_1 f_2 \sin\theta + \frac{1}{2}f_2 e_2 \sin(180° - \theta) + \frac{1}{2}e_2 f_1 \sin\theta$$

$$\text{i.e., } [ABCD] < \frac{1}{2}(e_1 f_1 + e_1 f_2 + f_2 e_2 + e_2 f_1) \quad (\text{as } \sin\theta \le 1)$$

$$\text{i.e., } [ABCD] \le \frac{1}{2}(e_1 + e_2)(f_1 + f_2) = \frac{1}{2}ef$$

$$\text{i.e., } [ABCD] \le \frac{1}{2}ef \qquad (3)$$

But $\frac{1}{2}ef \le \frac{1}{2}(ac + bd)$ (by Extended Ptolemy's theorem)

$$\therefore \ [ABCD] \le \frac{1}{2}(ac+bd) \tag{4}$$

Adding Eqs. (2) and (4), we get

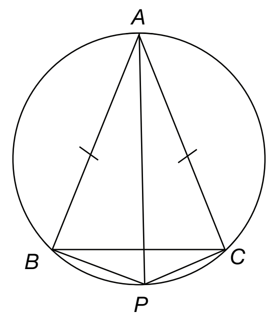
$$2[ABCD] < \frac{1}{2}(ad+bc+ac+bd)$$

$$\Rightarrow 2[ABCD] < \frac{1}{2}(a+b)(c+d)$$

$$\Rightarrow [ABCD] < \frac{1}{4}(a+b)(c+d)$$

Equality happens when the quadrilateral is a square. (*i.e.*, $\sin\theta = 1 \Rightarrow \theta = 90°$).

Example 114 *If isosceles ΔABC ($AB = AC$) is inscribe in a circle and a point P is on arc BC prove that* $\frac{PA}{PB+PC} = \frac{AC}{BC}$.



Solution: By Ptolemy's theorem

$PA \cdot BC = PB \cdot AC + PC \cdot AB$

$= PB \cdot AC + PC \cdot AC$ (As $AB = AC$)

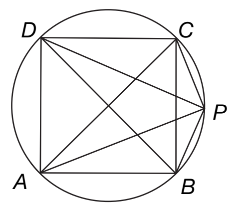
$\Rightarrow \quad PA \cdot BC = (PB + PC)\ AC$

$$\Rightarrow \quad \frac{PA}{PB+PC} = \frac{AC}{BC}$$

Example 115 *A square $ABCD$ is inscribed in a circle and a point P is on arc BC then prove that* $\frac{PA+PC}{PB+PD} = \frac{PD}{PA}$.

Solution: Since in a square $ABCD$, $AB = BC = CD = DA = a$ (Say) and $AC = BD = a\sqrt{2}$

In cyclic quadrilateral $APCD$, by Ptolemy's theorem



$PA \cdot CD + AD \cdot PC = PD \cdot AC$

$\Rightarrow \quad (PA + PC)a = PD \cdot a\sqrt{2}$

$$\Rightarrow PA + PC = PD\sqrt{2} \tag{1}$$

In cyclic quadrilateral $ABPD$, by using Ptolemy's theorem

$PD \cdot AB + PB \cdot AD = PA \cdot BD$

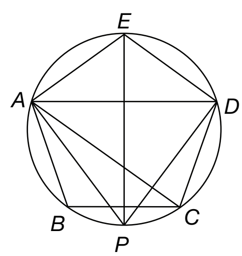
$\Rightarrow \ (PD + PB)a = PA \cdot a\sqrt{2}$

$$\Rightarrow PB + PD = PA \cdot \sqrt{2} \tag{2}$$

From Eq. (1)/ Eq. (2) we get, $\frac{PA+PC}{PB+PD} = \frac{PD}{PA}$.

Example 116 *A regular pentagon $ABCDE$ is inscribed in a circle and point P is chosen on arc BC. Prove that $PA + PD = PB + PC + PE$.*

Solution: In cyclic quadrilateral $ABPC$, $ABPD$ and $ABPE$ by using Ptolemy's theorem we get



$$AB \cdot PC + AC \cdot PB = AP \cdot BC \tag{1}$$

$$AB \cdot PD + AD \cdot PB = AP \cdot BD \tag{2}$$

$$AB \cdot PE + AE \cdot PB = AP \cdot BE \tag{3}$$

From Eq. (1) + Eq. (3) – Eq. (2)

$$AB(PC + PE - PD) + (AC + AE - AD)PB = (BC + BE - BD)AP$$

$AB[PC + PE - PD] + AE \cdot PB = BC \cdot AP$ (As $AC = AD$; $BE = BD$)

$\Rightarrow \quad PC + PE - PD + PB = PA$ (As $AB = AE = BC$)

$\Rightarrow \quad PC + PE + PB = PA + PD$.

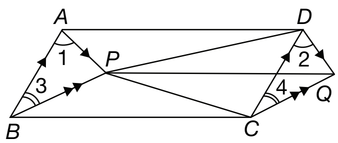
Example 117 *A point P is chosen inside a parallelogram ABCD such that $\angle APB$ is supplementary to $\angle CPD$. Prove that $AB \cdot AD = BP \cdot DP + AP \cdot CP$.*

Solution:

Given: $\angle APB + \angle CPD = 180°$

Construction: Draw $DQ \parallel AP$, $CQ \parallel BP$

Proof: Since $AB \parallel DC$, $AP \parallel DQ$



$\therefore \angle 1 = \angle 2$

$AB = DC$

Also $\angle 3 = \angle 4$ [$\because$ $AB \parallel DC$, $PB \parallel CQ$]

$\therefore$ By ASA congruency, $\Delta APB \cong \Delta DQC$

$\therefore \angle APB = \angle DQC$

And $AP = DQ$ and $BP = CQ$

Since $\angle APB + \angle DPC = 180°$ (Given)

$\therefore \angle DQC + \angle DPC = 180°$

$\therefore$ P, D, Q, C are concyclic

By Ptolemy's theorem in quadrilateral $PDQC$

$PD \cdot CQ + PC \cdot DQ = PQ \cdot CD$

Since $AP = DQ$ *and* $AP \parallel DQ$

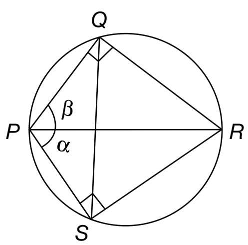
$\therefore$ $APQD$ is a parallelogram

$\therefore PD \cdot PB + PC \cdot PA = AD \cdot CD$

$\Rightarrow \quad PD \cdot PB + PC \cdot PA = AD \cdot AB$.

Example 118 *Prove that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ using Ptolemy's theorem or otherwise.*

Solution: Let PR be a diameter of the circle and $\angle SPR = \alpha$ and $\angle RPQ = \beta$.
In ΔPQR $\angle Q = 90°$



$$\cos\beta = \frac{PQ}{PR}$$

$\Rightarrow \quad PQ = PR\cos\beta$

$$\sin\beta = \frac{QR}{PR} \Rightarrow QR = PR\sin\beta$$

Similarly In ΔPSR, we get, $SR = PR\sin\alpha$ and $SP = PR\cos\alpha$

In ΔSPQ, by Sine rule, we get,

$$\frac{SQ}{\sin(\alpha + \beta)} = PR \Rightarrow \quad SQ = PR\sin(\alpha + \beta)$$

By Ptolemy's theorem in quadrilateral $PQRS$,

$PR \cdot SQ = PS \cdot RQ + PQ \cdot SR$

$\Rightarrow \quad PR \cdot PR\sin(\alpha + \beta) = PR \cdot \cos\alpha \cdot PR\sin\beta + PR\cos\beta \cdot PR\sin\alpha$

$\Rightarrow \quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Example 119 *Prove that* $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ *using Ptolemy's theorem or otherwise.*

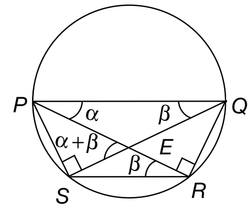
Solution: Let PQ is a diameter

$\angle QPR=\alpha$ and $\angle PQS=\beta$

Let PR and QS intersect at E

$\therefore\ \angle PES=\alpha+\beta$

In ΔPQR,



$$\sin\alpha=\frac{QR}{PQ}\Rightarrow QR=PQ\sin\alpha$$

$$\cos\alpha=\frac{PR}{PQ}\Rightarrow PR=PQ\cos\alpha$$

In ΔPSQ,

$$\sin\beta=\frac{PS}{PQ}\Rightarrow PS=PQ\sin\beta$$

$$\cos\beta=\frac{QS}{PQ}\Rightarrow QS=PQ\cos\beta$$

also In ΔRES and ΔQEP

$\angle RES=\angle QEP$ (VOA)

$\angle ERS=\angle EQP=\beta$ (Angle in a same segment)

$\therefore$ By AA similarity

$\Delta RES\sim\Delta QEP$

$$\Rightarrow\quad\frac{RS}{PQ}=\frac{SE}{PE}\tag{1}$$

In ΔSEP,

$$\cos(\alpha+\beta)=\frac{SE}{PE}\tag{2}$$

$$\Rightarrow\quad\frac{RS}{PQ}=\cos(\alpha+\beta)\qquad\text{(from Eqs. (1) and (2))}$$

$\Rightarrow\quad RS=PQ\cos(\alpha+\beta)$

Now by using Ptolemy's theorem

$PQ\cdot RS+PS\cdot QR=PR\cdot QS$

$\Rightarrow\quad PQ\cdot PQ\cos(\alpha+\beta)+PQ\sin\beta\cdot PQ\sin\alpha=PQ\cos\alpha\cdot PQ\cos\beta$

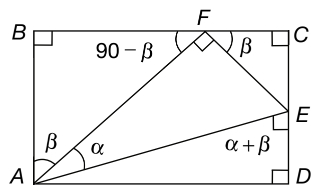
$\Rightarrow\quad\cos(\alpha+\beta)+\sin\alpha\sin\beta=\cos\alpha\cos\beta$

$\Rightarrow\quad\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Aliter: (Without using Ptolemy's theorem)

Let $ABCD$ is a rectangle and ΔAFE is a right angle triangle with $AE=1$

In ΔAEF,



$$\sin\alpha=\frac{EF}{AE}\quad\Rightarrow\quad EF=\sin\alpha\text{(As }AE=1)$$

$$\cos\alpha=\frac{AF}{AE}\Rightarrow AF=\cos\alpha\text{ (As }AE=1)$$

In ΔABF,

$$\cos\beta = \frac{AB}{AF} \Rightarrow AB = AF\cos\beta = \cos\alpha\cos\beta \qquad (1)$$

$$\sin\beta = \frac{BF}{AF} \Rightarrow BF = AF\sin\beta = \cos\alpha\sin\beta \qquad (2)$$

In ΔCEF

$$\cos\beta = \frac{CF}{FE} \Rightarrow CF = FE\cos\beta = \sin\alpha\cos\beta \qquad (3)$$

$$\sin\beta = \frac{CE}{EF} \Rightarrow CE = EF\sin\beta = \sin\alpha\sin\beta \qquad (4)$$

Since $AB \parallel CD$

$\angle EAB = \angle AED = \alpha + \beta$

$\therefore$ In ΔADE

$$\cos(\alpha+\beta) = \frac{ED}{AE} \Rightarrow ED = \cos(\alpha+\beta) \qquad (5)$$

$$\sin(\alpha+\beta) = \frac{AD}{AE} \Rightarrow AD = \sin(\alpha+\beta) \qquad (6)$$

Since $ABCD$ is a rectangle

$\therefore\ AD = CB$

$AD = CF + FB$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ (From Eqs. (2), (3) and (6))

Also $AB = DC = DE + EC$

$\Rightarrow \quad DE = AB - EC$

$\Rightarrow \quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ (From Eqs. (1), (4) and (5)).

Henri Pitot

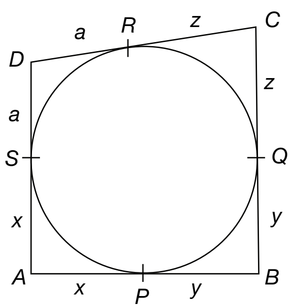
3 May 1695–27 Dec 1771
Nationality: French

8.12.5 Tangential Quadrilateral

A convex quadrilateral whose sides are all tangent to a single circle within the quadrilateral is called tangential quadrilateral (or circumscribed quadrilateral or inscriptible quadrilateral). This circle is called the incircle of the quadrilateral or its inscribed circle, its centre is the incentre and its radius is called the inradius.

8.12.5.1 Pitot Theorem

Let $ABCD$ be a tangential quadrilateral. Then the sum of the opposite sides are equal. That is, $AB + CD = AD + BC$.



Proof: Let the incircle C touches the sides AB, BC, CD, DA at P, Q, R and S respectively. Since the lengths of the tangents drawn from an external point to a circle are equal

$\therefore\ AP = AS = x$ (Say)

$BP = BQ = y$ (Say)

$CQ = CR = z$ (Say)

$DR = DS = a$ (Say)

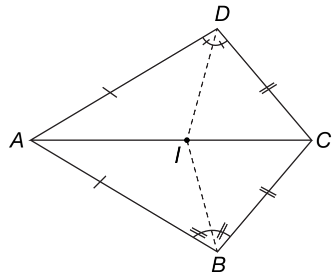
Also $AB + CD = AP + PB + CR + RD = x + y + z + a$ (1)

$AD + BC = AS + SD + CQ + QB = x + a + z + y$ (2)

Then from Eqs. (1) and (2), we get, $AB + CD = AD + BC$

8.12.5.2 Converse of Pitot Theorem

Any convex quadrilateral that satisfies $AB + CD = AD + BC$ is tangential.



Proof: Let us consider two cases as given quadrilateral is a kite or not a kite.

Case 1: Consider $ABCD$ is a kite with $AB = AD$ and $BC = CD$.
Observe AC is angle bisector of $\angle A$ and $\angle C$.

By symmetry angle bisectors of $\angle B$ and $\angle D$ will meet each other at I on AC.

So I is equidistant from all four sides of the quadrilateral.

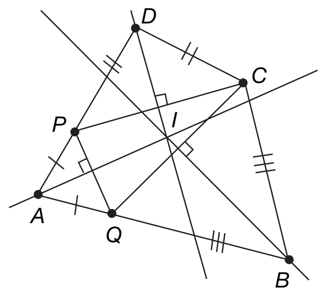
Hence quadrilateral must be tangential.

Case 2: Consider $ABCD$ is not a kite. Thus either $AD > DC$ or $AD < DC$. *WLOG* let $AD > DC$.

Now $AD > DC \quad \Rightarrow \quad AB > BC$.

So we can locate a point P on AD and Q on AB such that $DP = CD$ and $BQ = BC$.

From $AB + CD = AD + BC$ we get, $AQ + QB + CD = AP + PD + BC$.



$\Rightarrow \quad AQ = AP \quad$ (As $CD = PD$ and $BC = BQ$)

$\Rightarrow \quad \Delta CBQ$, ΔCDP, and ΔAPQ are isosceles.

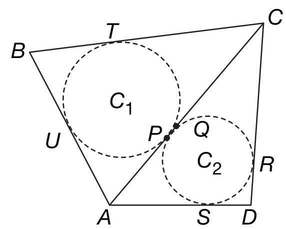
Now draw angle bisectors of $\angle A$, $\angle B$ and $\angle D$, and these angle bisector will be perpendicular bisector of PQ, QC and CP respectively as ΔCBQ, ΔCDP, and ΔAPQ are isosceles.

In ΔPQC, perpendicular bisectors of sides are concurrent. Let their point of concurrency be I.

I is equidistant from all sides of the quadrilateral. Hence quadrilateral is tangential.

Note: A quadrilateral which has both a circumcircle and an incircle is called a bicentric quadrilateral.

Example 120 *Let ABCD be a circumscribed (or tangential) quadrilateral. Prove that the circles in the two triangles ABC and ADC are tangent to each other.*



Solution: Let the incircle of ΔABC be C_1 and that of ΔADC be C_2.

Since C_1 and C_2 lie on either side of AC, the diagonal, if they touch each other, then, they must touch at a point only on AC.

If possible let C_1 touch AC at P and C_2 touch AC at a point Q. (We assume to the contrary).

Then, $PQ = AQ - AP$ (1)

Now $AQ = AC - CQ = AC - CR = AC - CD + DR$ (Equal tangent property)

$= AC - CD + DS = AC - CD + DA - SA$ (Equal tangent property)

$= AC - CD + DA - AQ$ (Equal tangent property)

$\therefore\ 2AQ = AC - CD + AD$ (2)

Similarly, $2AP = AC - BC + AB$ (3)

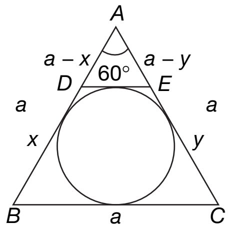
$\therefore\ 2PQ = (AC - CD + AD) - (AC - BC + AB)$ (From Eqs. (1), (2) and (3))

i.e., $2PQ = (AD + BC) - (AB + CD) = 0$ (by Pitot's theorem)

Therefore the points P and Q must coincide with each other, *i.e.*, the two circles touch AC at the same point.

Example 121 *Triangle ABC is equilateral. D is on AB and E is on AC, such that, DE is tangent to the incircle. Prove the result:*

$$\frac{AD}{DB}+\frac{AE}{CE}=1.$$



Solution: Let $AB = AC = BC = a$.

Let $BD = x$ and $CE = y$, so that, $AD = a - x$ and $AE = a - y$

By Pitot's theorem for circumscribed quadrilateral $BDEC$.

$$BC + DE = BD + CE \Rightarrow DE = x + y - a \tag{1}$$

$$\therefore DE^2 = (x + y - a)^2$$

$$= x^2 + y^2 + a^2 + 2xy - 2ax - 2ay \tag{2}$$

Also, by cosine rule applied to ΔADE, we have

$$DE^2 = (a-x)^2 + (a-y)^2 - 2(a-x)(a-y)\cos 60°$$

$$\therefore DE^2 = a^2 + x^2 - 2ax + a^2 + y^2 - 2ay - (a^2 - ay - ax + xy) \quad (\text{As } \cos 60° = \frac{1}{2})$$

$$i.e., DE^2 = x^2 + y^2 + a^2 - ax - ay - xy \tag{3}$$

$\therefore$ Equating Eqs. (2) and (3), we have

$$x^2 + y^2 + a^2 - 2ax - 2ay + 2xy = x^2 + y^2 + a^2 - ax - ay - xy \tag{4}$$

$$\Rightarrow 3xy = ax + ay$$

$$\Rightarrow a = \frac{3xy}{x+y}$$

Substituting this value of 'a' for AD and AE, we have

$$AD = a - x = \frac{3xy}{x+y} - x = \frac{3xy - x^2 - xy}{x+y} = \frac{x(2y-x)}{x+y}$$

$$\Rightarrow \frac{AD}{DB} = \frac{2y-x}{x+y} \qquad (\text{As } x = DB) \tag{5}$$

$$AE = a - y = \frac{3xy}{x+y} - y = \frac{3xy - xy - y^2}{x+y} = \frac{y(2x-y)}{x+y}$$

$$\Rightarrow \frac{AE}{EC} = \frac{2x-y}{x+y} \qquad (\text{As } y = EC) \tag{6}$$

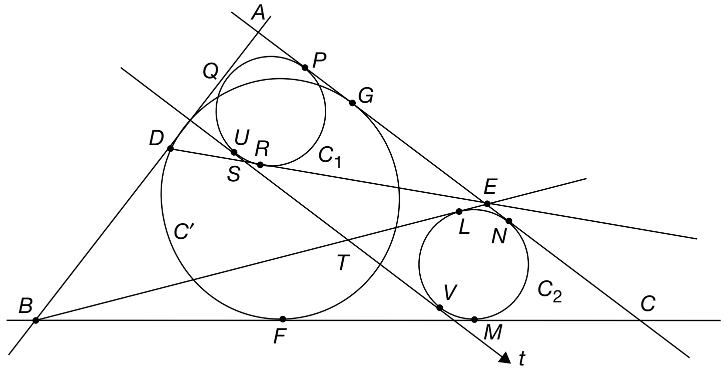
From Eq. (5) + Eq. (6), we get,

$$\frac{AD}{DB}+\frac{AE}{EC} = \frac{2y-x}{x+y}+\frac{2x-y}{x+y} = \frac{x+y}{x+y} = 1.$$

Example 122 *Let the incircles of ΔABC touch AB at D and let E be a point on the side AC. Prove that the incircles of triangles ADE, BCE and BDE have common tangents.*

Solution: Let the incircle C' of ΔABC touch AB at D, BC at F and AC at G respectively. Let the incircle C_1, of ΔADE touch the sides EA, AD and DE at P, Q and R respectively.

Let the incircle C_2, of ΔBCE touch the sides BC, CE, EB at M, N and L respectively.



Let 't' be the common tangent of circles C_1 and C_2 respectively meeting the lines DE, BE at S and T respectively and touching C_1 at U and C_2 at V respectively.

We are required to prove that t is a tangent to the incircle of ΔBDE, *i.e.*, to prove that quadrilateral B, D, S, T is a tangential quadrilateral, *i.e.*, prove $BD + ST = DS + BT$ (As incircle of $BDST$ is the incircle of ΔBDE)

$\therefore\ BD + ST$

$= BF + UV - SU - TV$ (as $BD = BF$ and $ST = UV - SU - TV$)

$= BF + PN - SU - TV$ (As UV and PN are direct common tangent to C_1 and C_2)

$= BF + PG + GN - SR - TL$ (As $PN = PG + GN$, $SU = SR$, $TV = TL$)

$= BF + DQ + FM - SR - TL$ (As PG and DQ are direct common tangent to C and C_1, $PG = DQ$ and similarly $GN = FM$)

$= BF + DR + FM - SR - TL$

$= (BF + FM) + (DR - SR) - TL$

$= BM - TL + DS = BL - TL + DS$ (As $BM = BL$)

$\Rightarrow\quad BD + ST = BT + DS.$

Build-up Your Understanding 13

1. In the ΔABC, $AB = AC$. The altitude AD of the triangle meets the circumcircle at P. Prove that $AP \cdot BC = 2AB \cdot BP$.
2. In a parallelogram $ABCD$, If a circle passing through point A cuts two sides AB and AD at P and R respectively and diagonal AC at Q, then prove that $AP \times AB + AR \times AD = AQ \times AC$.
3. Let P and Q be points on the circumcircle of ΔABC such that PQ is parallel to BC. Prove that QA is perpendicular to the Simson–Wallace line of P.
4. Suppose four lines intersect with each other and therefore any three lines among them determine a triangle. There are four such triangles. Prove that the circumcircles of these triangles have a common point.
5. Let A, B, C, D be adjacent vertices of a regular 7-sided polygon, in that order. Prove that $\dfrac{1}{AB} = \dfrac{1}{AC} + \dfrac{1}{AD}$
6. Let $ABCD$ be a square. If P is a point on the circumcircle of $ABCD$ which lies on the arc AD, prove that the value $(PA + PC)/PB$ does not depend on the position of P.
7. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$ and $\angle BCD = \angle EFA = 60°$. Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120°$. Show that $AG + GB + GH + DH + HE \geq CF$. **[IMO, 1995]**

8. Diagonals AC and BD of a cyclic quadrilateral $ABCD$ meets at P. Let the circumcentres of $ABCD$, ABP, BCP, CDP and DAP be O, O_1, O_2, O_3 and O_4, respectively. Prove that OP, O_1O_3, O_2O_4 are concurrent.

9. ABC is a triangle with $BC > CA > AB$. D is a point on side BC, and E is a point on BA produced beyond A so that $BD = BE = CA$. Let P be a point on side AC such that E, B, D, P are concyclic, and let Q be the second intersection point of BP with the circumcircle of ΔABC. Prove that $AQ + CQ = BP$. **[Iranian MO, 1998-99]**

10. Let D, E, F be respectively the feet of perpendicular from A to BC, B to CA, and C to AB. Draw perpendicular lines from D to AB, AC, BE, CF and let P, Q, M, N be the feet of perpendiculars respectively. Prove that P, Q, M, N are collinear.

11. Let ABC be a triangle, H its orthocentre, O its circumcentre, and R its circumradius. Let D be the reflection of A across BC, E be that of B across CA, and F that of C across AB. Prove that D, E and F are collinear if and only if $OH = 2R$.
[IMO Shortlisted Problem, 1998]

12. The incircle of triangle ABC touches BC, CA and AB at D, E and F respectively. X is a point inside triangle ABC such that the incircle of triangle XBC touches BC at D also, and touches CX and XB at Y and Z respectively. Prove that $EFZY$ is a cyclic quadrilateral. **[IMO Shortlisted Problem, 1995]**

13. $ABCDE$ is a cyclic pentagon. It is symmetric about the diameter through A. The chord CD is twice as far from A as the chord BE. Prove $BC + BD = BE$.

14. A circle has centre on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$. **[IMO, 1985]**

15. $ABCD$ is a cyclic quadrilateral. AB produced meets DC produced at F. AD produced meets BC produced at E. Prove that

(i) the angle bisectors of $\angle AEB$ and $\angle AFD$ are at right angles

(ii) Also show that the circumcircles of ΔBCF and ΔCDE meet on the straight line joining E and F.

16. Let P be a point inside an acute triangle ABC. Then prove that

$$PA \cdot PB \cdot AB + PB \cdot PC \cdot BC + PC \cdot PA \cdot CA \geq AB \cdot BC \cdot CA$$

With equality iff P is the orthocentre of ΔABC.

17. Let, $ABCD$ be a cyclic quadrilateral which has, its incentre as I. A line through I, parallel to AB, meets the sides AD and BC at P and R. Prove that length of PR is $\frac{1}{4}$ the perimeter of quadirlateral $ABCD$.

18. $ABCD$ is a fixed cyclic quadrilateral. Two circles PAB, PCD are drawn to touch at P. Prove that the locus of P is a circle.

19. $ABCD$ is a quadrilateral whose sides touch a circle. If the of ΔABD, touches AB, AD in P, Q, and the incircle of ΔBCD touches CB, CD in R, S, then prove that P, Q, R, S are concyclic.

20. The tangents at B and C to the circumcircle of an acute angled ΔABC meet in K. If the line through K parallel to AC meets the circumcircle in P and Q and AB in M, then prove that $PM = MQ$.

21. If the Simson–Wallace line of P, a point on the circumcircle of ΔABC, is parallel to AO, where O is the circumcentre of ΔABC, then prove that $PA \parallel BC$.

22. Prove that the Simson–Wallace line of the point at which the altitude through A of ΔABC meets the circumcircle is parallel to the tangent at A.

23. From vertex A of ΔABC, perpendiculars are dropped to the internal and external bisectors of $\angle B$ and $\angle C$, prove that the feet of those four perpendiculars lie on a straight line.

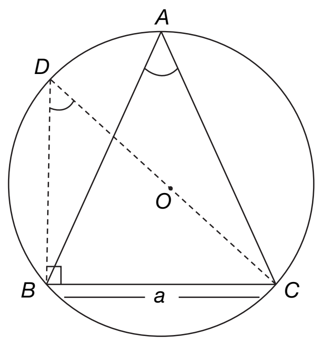
8.13 Application of Trigonometry in Geometry

8.13.1 Some Standard Notations

In a ΔABC, the angles are denoted by capital letters A, B and C; and the lengths of the sides opposite to these angle are denoted by small letters a, b and c respectively. Semi-perimeter of ΔABC is given by $s = \dfrac{a+b+c}{2}$. Its area and circumradius is denoted by Δ and R respectively. h_a, h_b, and h_c represent the lengths of the altitudes from A, B, and C, respectively. m_a, m_b, and m_c represent the lengths of the medians through A, B, and C respectively. t_a, t_b, and t_c represent the lengths of the internal angle bisectors of $\angle A$, $\angle B$, and $\angle C$ respectively.

8.13.2 Sine Rule

In a ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$



Proof: In acute angle triangle ABC, circumcentre lies inside the triangle.
Let O be the circumcentre of ΔABC. Join CO and produce it to cut the circumcircle at D.

So CD is a diameter of a circle, $CD = 2R$

By angle in a same segment property $\angle BDC = \angle BAC = \angle A$

and $\angle DBC = 90°$ (Angle in a semi-circle)

In ΔDBC,

$$\sin \angle BDC = \frac{BC}{CD}$$

$\Rightarrow \sin A = \dfrac{a}{2R}$ (As $\angle BDC = \angle A$)

$\Rightarrow \dfrac{a}{\sin A} = 2R$

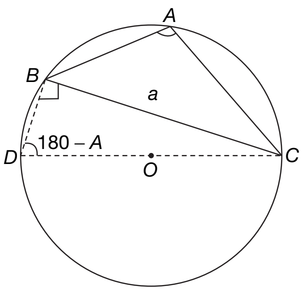
Similarly, $\dfrac{b}{\sin B} = 2R$, $\dfrac{c}{\sin C} = 2R$

Hence $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

When $\angle A > 90°$, then circumcentre O lies outside the ΔABC. Again join CO and produce it to cut the circumcircle at D. Join DB

Now $ABDC$ is a cyclic quadrilateral

$\therefore \angle BDC = 180° - \angle A$



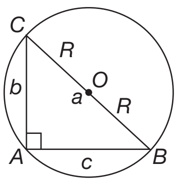
In $\angle BDC$,

$$\sin\angle BDC = \frac{BC}{CD} \Rightarrow \sin(180^\circ - A) = \frac{a}{2R}$$

$$\Rightarrow \sin A = \frac{a}{2R} \quad \Rightarrow \quad \frac{a}{\sin A} = 2R$$

Similarly $\dfrac{b}{\sin b} = 2R$; $\dfrac{c}{\sin c} = 2R$

Again $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$



When $\angle A = 90^\circ$, then $2R = BC = a$

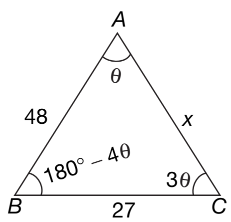
$$\sin B = \frac{AC}{BC} = \frac{b}{2R};\ \sin C = \frac{AB}{BC} = \frac{c}{2R}$$

$$\sin A = \sin 90^\circ = 1 = \frac{BC}{BC} = \frac{a}{2R}$$

Again $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$

Example 123 *In a non-degenerate triangle ABC, $\angle C = 3\angle A$; $BC = 27$; $AB = 48$; prove that the side AC has an integer measure.*

Solution:



Let $\angle A = \theta \quad \Rightarrow \quad \angle C = 3\theta$ and $\angle B = (180^\circ - 4\theta)$.

Applying sine rule in ΔABC,

$$\frac{48}{\sin 3\theta} = \frac{27}{\sin\theta}$$

$$\Rightarrow 48 = \frac{27\sin 3\theta}{\sin\theta} = 27(3 - 4\sin^2\theta)$$

$$\Rightarrow \sin^2\theta = \frac{11}{36} \quad \text{(on simplification)} \qquad (1)$$

Also, $\dfrac{27}{\sin\theta} = \dfrac{x}{\sin 4\theta}$

$$\Rightarrow AC = x = \frac{27(\sin 4\theta)}{\sin\theta}$$

$$= \frac{27}{\sin\theta}(2\sin 2\theta\cos 2\theta)$$

$$= \frac{27}{\sin\theta}[2\cdot 2\sin\theta\cos\theta\cdot(1 - 2\sin^2\theta)]$$

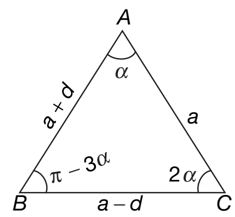
$$= (27)(4)\left(\sqrt{1 - \frac{11}{36}}\right)\left(1 - 2\times\frac{11}{36}\right) \qquad \text{(from Eq. (1))}$$

$$= 35 \qquad \text{(on simplification)}$$

Thus the measure of AC is 35 units, an integer.

Example 124 *The sides of a triangle are in AP and the greatest angle of the triangle is double the least. Prove that, this triangle is acute angled triangle.*

Solution:



Let the sides be $a - d$, a, $a + d$ ($a > 0$, $d > 0$).

Let α be the smallest angle of the triangle opposite to $(a - d)$; then the greatest angle 2α is opposite to $(a + d)$.

Applying sine rule for ΔABC,

$$\frac{a-d}{\sin\alpha} = \frac{a}{\sin(\pi - 3\alpha)} = \frac{a+d}{\sin 2\alpha} \tag{1}$$

Now, $\dfrac{a-d}{a+d} = \dfrac{\sin\alpha}{\sin 2\alpha} = \dfrac{\cancel{\sin\alpha}}{2\cancel{\sin\alpha}\cos\alpha} = \dfrac{1}{2\cos\alpha}$

$$\therefore 2\cos\alpha = \frac{a+d}{a-d}$$

And so, $$4\cos^2\alpha = \left(\frac{a+d}{a-d}\right)^2 \tag{2}$$

Also, $\dfrac{a-d}{a} = \dfrac{\sin\alpha}{\sin 3\alpha} = \dfrac{\sin\alpha}{3\sin\alpha - 4\sin^3\alpha} = \dfrac{1}{3-4\sin^2\alpha}$

$$\therefore 3 - 4\sin^2\alpha = \frac{a}{a-d}$$

$$\Rightarrow 3 - 4 + 4\cos^2\alpha = \frac{a}{a-d}$$

$$\Rightarrow 4\cos^2\alpha = \frac{a}{a-d} + 1 = \frac{2a-d}{a-d} \tag{3}$$

Thus, $\left(\dfrac{a+d}{a-d}\right)^2 = \left(\dfrac{2a-d}{a-d}\right)$

$\Rightarrow (a+d)^2 = (a-d)(2a-d)$

$\Rightarrow a = 5d$ (on simplification)

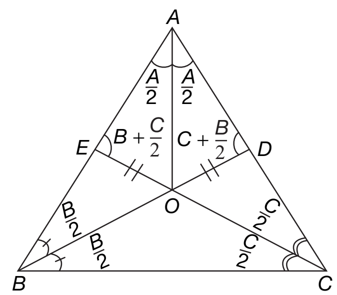
$\therefore$ Ratio of the sides is $(a - d) : a : (a + d) = 4d : 5d : 6d$, *i.e.*, $4 : 5 : 6$.

Here $6^2 < 4^2 + 5^2 \Rightarrow$ the triangle is acute. (By acute angle theorem)

Example 125 *ΔABC is an arbitrary triangle. The bisector of $\angle B$ and $\angle C$ meet AC and AB at D and E respectively. BD and CE intersect at 'O'. If $OD = OE$, prove that, either $\angle BAC = 60°$ or the triangle is isosceles.*

Solution:

Join AO.



In ΔAOD, $\angle OAD = \dfrac{A}{2}$.

$\therefore \angle ODA = \angle BDA = C + \dfrac{B}{2}$

$$\Rightarrow \angle AOD = 180° - \left(C + \frac{B}{2} + \frac{A}{2}\right) = 90° - \frac{C}{2}$$

Similarly in ΔAOE, $\angle AOE = 90° - \frac{B}{2}$.

Use the sine rule for triangles AOD and AOE,

$$\frac{OD}{\sin\frac{A}{2}} = \frac{OA}{\sin\angle ADO} \Rightarrow OD = \frac{OA\sin\frac{A}{2}}{\sin\left(C+\frac{B}{2}\right)} \tag{1}$$

Similarly,

$$\frac{OE}{\sin\frac{A}{2}} = \frac{OA}{\sin\angle OEA} \Rightarrow OE = \frac{OA\sin\frac{A}{2}}{\sin\left(B+\frac{C}{2}\right)} \tag{2}$$

As, $OD = OE$ (given), $\sin\left(B+\frac{C}{2}\right) = \sin\left(C+\frac{B}{2}\right)$ (3)

$$\therefore B+\frac{C}{2} = C+\frac{B}{2} \quad \text{or} \quad B+\frac{C}{2}+C+\frac{B}{2} = 180°$$

$$\Rightarrow \angle B = \angle C \quad \text{or} \quad B+C = \frac{2\times 180}{3} = 120°$$

$$\Rightarrow \Delta ABC \text{ is an isosceles triangle or } \angle A = 60°.$$

Example 126 *In any triangle ABC, prove the inequality:*

$$\sum_{A,B,C} \frac{\sqrt{\sin A}}{\sqrt{\sin B}+\sqrt{\sin C}-\sqrt{\sin A}} \geq 3$$

When does the equality hold?

Solution: By application of sine rule to ΔABC in the usual notation, the problem reduces to

$$\sum_{a,b,c} \frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} \geq 3;$$

Let, $x = \sqrt{b}+\sqrt{c}-\sqrt{a}$; $y = \sqrt{c}+\sqrt{a}-\sqrt{b}$; $z = \sqrt{a}+\sqrt{b}-\sqrt{c}$.

Thus, $(\sqrt{b}+\sqrt{c})^2 > b+c > 0$; x is a positive number and similarly y and z.

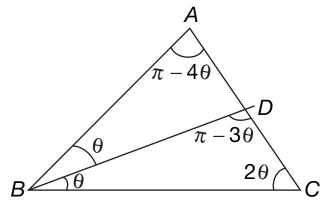
Now,

$$\text{LHS} = \frac{y+z}{2x}+\frac{z+x}{2y}+\frac{x+y}{2z}$$

$$= \frac{1}{2}\left(\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z}\right) \geq \frac{1}{2}(2+2+2) = 3. \qquad \left(\text{As } t+\frac{1}{t} \geq 2\ \forall t \in \mathbb{R}^+\right)$$

Equality holds when $a = b = c$, *i.e.*, when the triangle is equilateral.

Example 127 *ABC is an isosceles triangle in which AB = AC. The bisector of ∠B meets AC at D. Also BC = BD + AD. Find the size of ∠A.*



Solution:

Let $\angle DBC=\theta$, so that $\angle ACB=2\theta$ and $\angle BDC=\pi-3\theta$, also $\angle BAC=\pi-4\theta$.

Now by sine rule, in $\angle BDC$ and ΔABD respectively, we get,

$$\frac{BC}{\sin 3\theta}=\frac{BD}{\sin 2\theta} \quad \text{and} \quad \frac{AD}{\sin\theta}=\frac{BD}{\sin 4\theta}$$

It is given that $BC=BD+AD$.

$$\therefore \frac{BC}{BD}=1+\frac{AD}{BD} \Rightarrow \frac{\sin 3\theta}{\sin 2\theta}=1+\frac{\sin\theta}{\sin 4\theta}$$

$$i.e., \frac{\sin 3\theta}{\cancel{\sin 2\theta}}=\frac{\sin 4\theta+\sin\theta}{2\cancel{\sin 2\theta}\cos 2\theta} \quad (\text{As } \sin 4\theta=2\sin 2\theta\cos 2\theta)$$

$$\Rightarrow 2\sin 3\theta\cos 2\theta=\sin 4\theta+\sin\theta$$

$$i.e., \sin 5\theta+\cancel{\sin\theta}=\sin 4\theta+\cancel{\sin\theta}$$

$$\Rightarrow \sin 5\theta=\sin 4\theta$$

$$\Rightarrow 5\theta=4\theta \quad \text{or} \quad 5\theta+4\theta=180°; \text{ But } 5\theta\neq 4\theta$$

$$\therefore 9\theta=180° \Rightarrow \theta=20°, \text{ which gives } \angle BAC=\pi-4\theta=100°.$$

Build-up Your Understanding 14

1. In any triangle ABC, prove that $\dfrac{a^2\sin(B-C)}{\sin B+\sin C}+\dfrac{b^2\sin(C-A)}{\sin C+\sin A}+\dfrac{b^2\sin(C-A)}{\sin C+\sin A}$ $\dfrac{c^2\sin(A-B)}{\sin A+\sin B}=0$
2. If in a ΔABC, $\dfrac{\sin A}{\sin C}=\dfrac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in AP.
3. $ABCD$ is a trapezium such that AB and CD are parallel and CB is perpendicular to them. If $\angle ADB=60°$, $BC=4$ and $CD=3$, then find the length of side AB.
4. If the sides of a triangle are in arithmetic progression, and if its greatest angle exceeds the least angle by α, show that the sides are in the ration $1-x:1:1+x$, where $x=\sqrt{\dfrac{1-\cos\alpha}{7-\cos\alpha}}$.
5. If a, b, c be the sides of a triangle, $\lambda a, \lambda b, \lambda c$ the sides of a similar triangle inscribed in the former and θ the angle between the sides a and λa, prove that $2\lambda \cos\theta=1$.
6. Let ABC be an arbitrary acute-angled triangle. Let D, E, F denote the feet of the perpendiculars from P onto the sides AB, BC, CA respectively. Determine the set of all possible positions of P, for which, the triangle DEF is isosceles. For what position of P, will triangle DEF be equilateral? Why?
7. The sides a, b, c of ΔABC satisfy the equality $c^2b=(a+b)(a-b)^2$. Prove that $\angle A=3\angle B$.
8. A triangle has circumradius R and sides a, b, c with the relation: $R(b+c)=a$. Prove that, such a triangle is right angled.

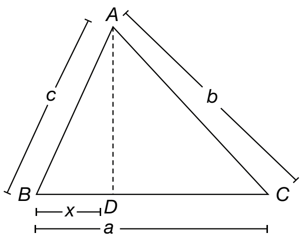
9. Given a circle of radius unity and AB is a chord of the circle, with length unity. If C is any point in the major segment, prove that, $AC^2 + BC^2 \le 2\ (2+\sqrt{3})$. When does the equality hold?
10. Let ABC be a triangle inscribed in a circle and let $l_a = m_a/M_a$, $l_b = m_b/M_b$, $l_c = m_c/M_c$, where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that $\dfrac{l_a}{\sin^2 A}+\dfrac{l_b}{\sin^2 B}+\dfrac{l_c}{\sin^2 C}\ge 3$ and that equality holds iff ABC is equilateral. **[APMO, 1997]**

8.13.3 Cosine Formula

In ΔABC, we have following cosine rules:

$$\cos A = \frac{b^2+c^2-a^2}{2bc};\ \cos B = \frac{a^2+c^2-b^2}{2ac};\ \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Proof:



Case 1: If $\angle B < 90°$, then by acute angle triangle theorem

$$AC^2 = AB^2 + BC^2 - 2BD\cdot BC$$
$$\Rightarrow b^2 = c^2 + a^2 - 2x\cdot a$$

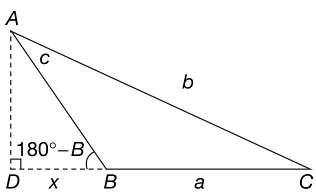
In ΔABD, $\cos B = \dfrac{BD}{AB} = \dfrac{x}{c} \quad \Rightarrow \quad x = c\cos B$

$$\therefore\quad b^2 = c^2 + a^2 - 2ac\cos B \ \Rightarrow\ 2ac\cos B = a^2 + c^2 - b^2$$

$$\Rightarrow \cos B = \frac{a^2+c^2-b^2}{2ac}$$

Case 2: If $\angle B > 90°$, then by obtuse angle triangle theorem

$$AC^2 = AB^2 + BC^2 + 2BD\cdot BC \quad\Rightarrow\quad b^2 = c^2 + a^2 + 2x\cdot a$$



In ΔABD,

$$\cos\angle ABD = \frac{BD}{AC} \quad\Rightarrow\quad \cos(180° - B) = \frac{x}{c}$$

$$\Rightarrow -\cos B = \frac{x}{c} \quad\Rightarrow\quad x = -c\cos B$$

$$\therefore\quad b^2 = c^2 + a^2 - 2ac\cos B$$
$$\Rightarrow\quad 2ac\cos B = a^2 + c^2 - b^2$$
$$\Rightarrow\quad \cos B = \frac{a^2+c^2-b^2}{2ac}$$

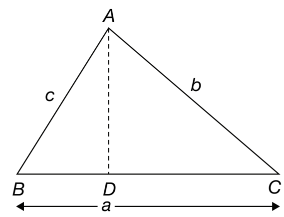
Similarly $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ and $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

8.13.4 Projection Formula

In ΔABC, we have following projection formulas:

$a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$ and $c = a\cos B + b\cos A$

Proof: For both base angles being acute, *i.e.*, $\angle B < 90°$, $\angle C < 90°$
In ΔABD,



$$\cos B = \frac{BD}{AB} = \frac{BD}{c}$$

$$\Rightarrow \quad BD = c\cos B$$

In ΔADC

$$\cos C = \frac{CD}{AC} = \frac{CD}{b}$$

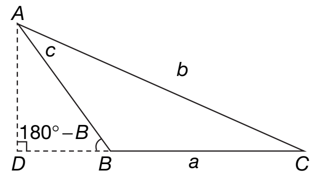
$$\Rightarrow \quad CD = b\cos C$$

Now $\quad a = BC = BD + DC$

$$\Rightarrow \quad a = c\cos B + b\cos C$$

or $\quad a = b\cos C + c\cos B$

For one base angle obtuse, WLOG let $\angle B > 90°$ and $\angle C < 90°$.

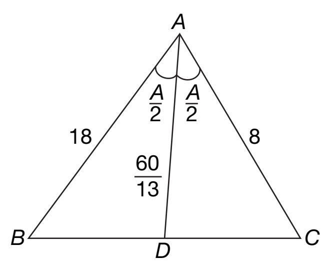


$$DC = b\cos C$$
$$DB = c\cos(180° - B) = -c\cos B$$
$$BC = DC - DB = b\cos C - (-c\cos B)$$
$$\Rightarrow a = b\cos C + c\cos B$$

Similarly $b = c\cos A + a\cos C$ and $c = a\cos B + b\cos A$

Example 128 *Two sides of a triangle are* 8 *cm and* 18 *cm and the bisector of the angle formed by them is of length* $\frac{60}{13}$ *cm. Find the perimeter of the triangle.*

Solution:
Let ABC be the triangle with $AC = 8$ cm. Let AD be the bisector of $\angle A$; $AD = \frac{60}{13}$ cm.



$AD = \left(\frac{2bc}{b+c}\right)\cos\frac{A}{2}$ (From the note 4 on page number 8.47)

Using the measures of AB, AC and AD in above formula, we get,

$$\cos\frac{A}{2} = \frac{60}{13} \times \frac{26}{2\cdot 18\cdot 8}, \ i.e., \ \cos\frac{A}{2} = \frac{5}{12}$$

$$\therefore \cos A = 2\cos^2\frac{A}{2} - 1 = 2\left(\frac{5}{12}\right)^2 - 1 = -\frac{47}{72}$$

Thus, $BC^2 = AB^2 + AC^2 - 2\cdot AB\cdot AC\cdot\cos A$ (Using cosine rule)

$$i.e., \ BC^2 = 18^2 + 8^2 + 2\cdot 8\cdot 18\cdot\frac{47}{72} = 576$$

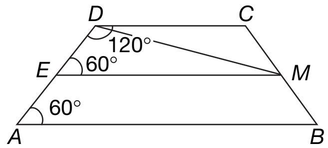
$$\Rightarrow BC = 24;$$

$\Rightarrow$ Perimeter $= 18 + 8 + 24 = 40$ cm.

Example 129 *ABCD is a convex quadrilateral in which*

$AD = 2\sqrt{3}$; $\angle A = 60°$; $\angle D = 120°$ and $AB + CD = 2AD$.

M is the mid-point of BC. Find DM.



Solution:

Since $\angle A + \angle D = 180°$, $AB \parallel CD$. Draw $ME \parallel BA$ to meet AD at E. As M is the mid-point of BC, E is the mid-point of AD. Also $EM = \frac{1}{2}(AB + CD) = AD$ (given).

$\therefore EM = AD.$

From ΔEDM, using cosine rule,

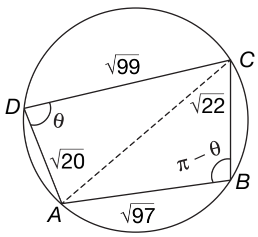
$$DM^2 = DE^2 + EM^2 - 2 \cdot DE \cdot EM \cdot \cos 60°$$

$$\text{i.e., } DM^2 = \left(\frac{1}{2}DA\right)^2 + (AD)^2 - 2\left(\frac{AD}{2}\right)(AD)\left(\frac{1}{2}\right)$$

$$\text{i.e., } DM^2 = \frac{1}{4}DA^2 + AD^2 - \frac{1}{2}AD^2 = \frac{3}{4}AD^2$$

$$\therefore DM^2 = \frac{3}{4}(2\sqrt{3})^2 = 9 \Rightarrow DM = 3.$$

Example 130 *A quadrilateral inscribed in the circle has side lengths* $\sqrt{20}, \sqrt{99}, \sqrt{22}$, and $\sqrt{97}$ *in that order. Taking* $\pi = \frac{22}{7}$ *show that the area of the circle is rational.*



Solution:

Let $\angle D = \theta$; then $\angle B = 180 - \theta$ (cyclic quadrilateral).

$$AC^2 = 20 + 99 - 2(\sqrt{20})(\sqrt{99})\cos\theta \qquad \text{(cosine rule in } \Delta ADC)$$

$$\text{Also, } AC^2 = 22 + 97 - 2(\sqrt{22})(\sqrt{97})\cos(180° - \theta) \qquad \text{(cosine rule in } \Delta ACB)$$

Equating for AC^2, we get

$$2\cos\theta(\sqrt{22} \cdot \sqrt{97} + \sqrt{20} \cdot \sqrt{99}) = 0$$

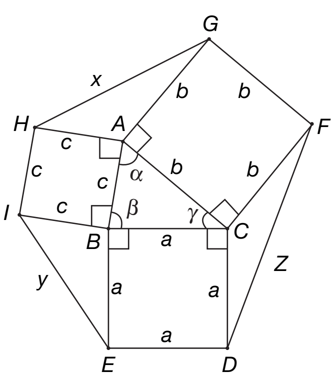
$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = 90°.$$

Thus,

$$AC^2 = 20 + 99 = 119$$

$$\therefore (2R)^2 = 119 \Rightarrow R^2 = \frac{119}{4}$$

$$\therefore A = \pi R^2 = \frac{22}{7} \times \frac{119}{4} = \frac{11 \times 17}{2} = \frac{187}{2}.$$



Example 131 *Squares are drawn on the sides of an arbitrary triangle and the vertices of the squares are connected to form a six sided figure. If the sides of the triangle are a, b, c and outside lengths are x, y, z, prove that,* $x^2 + y^2 + z^2 = 3(a^2 + b^2 + c^2)$.

Solution:

Applying cosine rule in ΔAGH, we get,

$$x^2 = b^2 + c^2 - 2bc\cos(180° - \alpha)$$

$$\Rightarrow x^2 = b^2 + c^2 + 2bc\cos\alpha \quad (1)$$

Also from ΔABC,

$$2bc\cos\alpha = b^2 + c^2 - a^2 \quad (2)$$

From Eqs. (1) and (2) we get,

$$x^2 = 2b^2 + 2c^2 - a^2 \quad (3)$$

Similarly,

$$y^2 = 2c^2 + 2a^2 - b^2 \quad (4)$$

And $z^2 = 2a^2 + 2b^2 - c^2$ (5)

Thus, by adding Eqs. (3), (4) and (5), we get,

$$x^2 + y^2 + z^2 = 3(a^2 + b^2 + c^2).$$

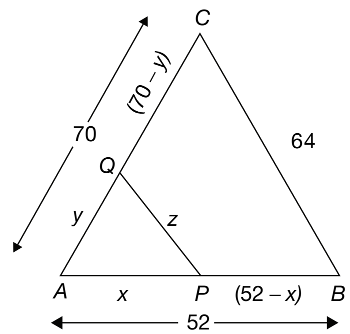
Example 132 *In ΔABC, $AB = 52$; $BC = 64$; $CA = 70$ and assume P, Q as points chosen in AB, AC respectively such that the triangle APQ and quadrilateral PBCQ have the same area and same perimeter. Prove that $PQ^2 = 3255$.*

Solution:

Let $AP = x$; $AQ = y$ and $PQ = z$

$\therefore (52 - x) + z + (70 - y) + 64 = x + y + z$

i.e., $2(x + y) = 186 \quad \Rightarrow \quad x + y = 93$ (1)



Also,

$$[APQ] = [PBCQ] \Rightarrow [APQ] = [ABC] - [APQ] \Rightarrow 2[APQ] = [ABC] \quad (2)$$

$$\therefore\ \cancel{2} \times \frac{1}{\cancel{2}} \times x \times y \times \cancel{\sin A} = \frac{1}{2} \times 52 \times 70 \times \cancel{\sin A}$$

$$\therefore\ 2xy = 52 \times 70 \quad (3)$$

Using cosine rule for ΔAPQ, $PQ^2 = z^2 = x^2 + y^2 - 2xy\cos A$ (4)

$$\left(\text{where } \cos A = \frac{52^2 + 70^2 - 64^2}{2 \times 52 \times 70}\right)$$

writing $(x + y)^2 = 93^2$ and $x^2 + y^2 = (x + y)^2 - 2xy = 93^2 - (52 \times 70)$ and $\cos A$

$\frac{52^2 + 70^2 - 64^2}{2 \times 52 \times 70}$ in Eq. (4), we get, $z^2 = PQ^2 = 3255$ (on simplification).

Build-up Your Understanding 15

1. In any ΔABC, prove that $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$.

2. Let ABC be a triangle such that $2b = (m + 1)a$ and $\cos A = \frac{1}{2}\sqrt{\frac{(m-1)(m+3)}{m}}$, where $m \in (1, 3)$. Prove that there are two values of the third side one of which is m times the other.

3. In a triangle ABC, $\angle C = 60°$, then prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$.

4. If in a triangle ABC, $\dfrac{\cos A+2\cos C}{\cos A+2\cos B}=\dfrac{\sin B}{\sin C}$, prove that the triangle is either isosceles or right angled.

5. A ring, 10 cm, in diameter, is suspended from a point 12 cm, above its centre by 6 equal strings attached to its circumference at equal intervals. Find the cosine of the angle between consecutive strings.

6. Let AC be a line segment in a plane and B, a point between A and C. Construct isosceles triangles PAB and QBC on one side of the segment AC, such that $APB = BAQ = 120°$; Construct an isosceles triangles RAC on the other side of AC, such that $ARC = 120°$. Prove that ΔPQR is equilateral.

7. If α, β, γ are the altitudes of ΔABC from the vertices A, B, C respectively, prove the following equality: $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2}=\left(\dfrac{1}{\Delta}\right)$ (cot$A$ + cot$B$ cot$C$).

8. Determine all triples (a, b, c) of positive integers which are the lengths of the sides of a triangle inscribed in a circle of diameter 6.25 units.

9. The sides of a triangle are of lengths a, b, and c where a, b, c, are integers and $a > b$. Also $\angle C$ is 60°. Show that the measure of side BC is not prime.

10. Let the angle bisectors of $\angle A$, $\angle B$, $\angle C$ of triangle ABC intersect its circumcircle at P, Q, R, respectively. Prove that $AP + BQ + CR > BC + CA + AB$.

John Napier

1 Feb 1550–4 Apr 1617
Nationality: Scottish

8.13.5 Napier's Analogy (Tangent's Rule)

In a ΔABC,

1. $\tan\left(\dfrac{A-B}{2}\right)=\left(\dfrac{a-b}{a+b}\right)\cot\dfrac{C}{2}$

2. $\tan\left(\dfrac{B-C}{2}\right)=\left(\dfrac{b-c}{b+c}\right)\cot\dfrac{A}{2}$

3. $\tan\left(\dfrac{C-A}{2}\right)=\left(\dfrac{c-a}{c+a}\right)\cot\dfrac{B}{2}$

Proof: For (1)

$$\frac{a-b}{a+b}=\frac{2R(\sin A-\sin B)}{2R(\sin A+\sin B)}=\frac{2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)}{2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}$$

$$\therefore\quad \frac{a-b}{a+b}=\frac{\tan\left(\dfrac{A-B}{2}\right)}{\tan\left(\dfrac{A+B}{2}\right)}=\frac{\tan\left(\dfrac{A-B}{2}\right)}{\tan\left(\dfrac{\pi}{2}-\dfrac{C}{2}\right)}=\frac{\tan\left(\dfrac{A-B}{2}\right)}{\cot\dfrac{C}{2}}$$

$$\Rightarrow\quad \tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$$

Similarly for others.

8.13.6 Mollweide's Formula

Karl Brandan Mollweide

3 Feb 1774–10 March 1825
Nationality: German

In a ΔABC, we have following:

1. $\dfrac{a+b}{c}=\dfrac{\cos\left(\dfrac{A-B}{2}\right)}{\sin\dfrac{C}{2}}$; $\dfrac{b+c}{a}=\dfrac{\cos\left(\dfrac{B-C}{2}\right)}{\sin\dfrac{A}{2}}$; $\dfrac{c+a}{b}=\dfrac{\cos\left(\dfrac{C-A}{2}\right)}{\sin\dfrac{B}{2}}$

2. $\dfrac{a-b}{c}=\dfrac{\sin\left(\dfrac{A-B}{2}\right)}{\cos\dfrac{C}{2}}$; $\dfrac{b-c}{a}=\dfrac{\sin\left(\dfrac{B-C}{2}\right)}{\cos\dfrac{A}{2}}$; $\dfrac{c-a}{b}=\dfrac{\sin\left(\dfrac{C-A}{2}\right)}{\cos\dfrac{B}{2}}$

Proof: For (1)

$$\frac{a+b}{c}=\frac{2R(\sin A+\sin B)}{2R\sin C}=\frac{2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}{\sin C}$$

$$=\frac{2\sin\left(\dfrac{\pi-C}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}{2\sin\dfrac{C}{2}\cos\dfrac{C}{2}}=\frac{2\cos\dfrac{C}{2}\cos\left(\dfrac{A-B}{2}\right)}{2\sin\dfrac{C}{2}\cdot\cos\dfrac{C}{2}}$$

$$\therefore\quad \frac{a+b}{c}=\frac{\cos\left(\dfrac{A-B}{2}\right)}{\sin\dfrac{C}{2}}$$

Similarly for others.

For (2)

$$\frac{a-b}{c}=\frac{2R(\sin A-\sin B)}{2R\sin C}=\frac{2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)}{\sin C}$$

$$=\frac{2\cos\left(\dfrac{\pi-C}{2}\right)\sin\left(\dfrac{A-B}{2}\right)}{2\sin\dfrac{C}{2}\cos\dfrac{C}{2}}$$

$$=\frac{2\sin\dfrac{C}{2}\sin\left(\dfrac{A-B}{2}\right)}{2\sin\dfrac{C}{2}\cdot\cos\dfrac{C}{2}}$$

$$\therefore\quad \frac{a-b}{c}=\frac{\sin\left(\dfrac{A-B}{2}\right)}{\cos\dfrac{C}{2}}$$

Similarly for others.

8.13.7 Half Angle Formulae's

1. $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

2. $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

3. $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$; $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$; $\tan\frac{c}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

For (1)

Since $2\cos^2\frac{A}{2}=1+\cos A$

$$\therefore\quad 2\cos^2\frac{A}{2}=1+\frac{b^2+c^2-a^2}{2bc}=\frac{2bc+b^2+c^2-a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}=\frac{(b+c+a)(b+c-a)}{2bc}$$

$$\Rightarrow\quad 2\cos^2\frac{A}{2}=\frac{2s(2s-2a)}{2bc}$$

$$\Rightarrow\quad \cos^2\frac{A}{2}=\frac{s(s-a)}{bc}$$

$$\Rightarrow\quad \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}\qquad\left(\text{As }\cos\frac{A}{2}>0\right)$$

Similarly for others.

For (2)

$$2\sin^2\frac{A}{2}=1-\cos A=1-\frac{b^2+c^2-a^2}{2bc}=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$=\frac{a^2-(b^2+c^2-2bc)}{2bc}=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a+b-c)(a-b+c)}{2bc}$$

$$2\sin^2\frac{A}{2}=\frac{(a+b+c-2c)(a+b+c-2b)}{2bc}$$

$$\Rightarrow 2\sin^2\frac{A}{2}=\frac{(2s-2c)(2s-2b)}{2bc}$$

$$\Rightarrow \sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$$

$$\Rightarrow \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}\qquad\left(\text{As }\sin\frac{A}{2}>0\right)$$

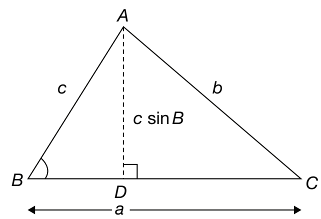
Similarly for others.

For (3)

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly for others.

8.13.8 Area of Triangle



Since in ΔABC, area of $\Delta ABC = \frac{1}{2} \times BC \times AD$

$$\Rightarrow \quad \Delta = \frac{1}{2} ac \sin B$$

Similarly, $\Delta = \frac{1}{2} ab \sin C$; $\Delta = \frac{1}{2} bc \sin A$

Hence, $\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B$

Thus area of any triangle

$= \frac{1}{2} \times$ Product of the two sides of a triangle $\times$ Sine of the included angle.

8.12.8.1 Heron's Formula

Heron of Alexandria

c. 10 AD–c. 70 AD
Nationality: Greek

Since area of $\Delta ABC = \frac{1}{2} bc \sin A = \frac{1}{2} bc \times 2 \sin\frac{A}{2} \cos\frac{A}{2}$

$$\Rightarrow \Delta = bc\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}$$

$$\Rightarrow \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Aliter (without trigonometry): In ΔADB and ΔADC

$$h^2 = c^2 - x^2 = b^2 - (a-x)^2 \Rightarrow \quad 2ax = a^2 + c^2 - b^2$$

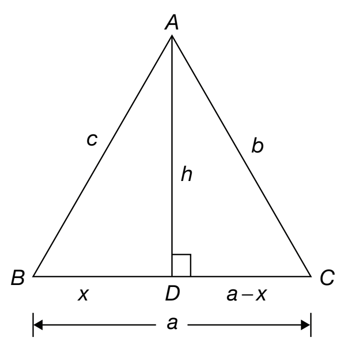
$$\Rightarrow x = \frac{a^2 + c^2 - b^2}{2a}$$

Also $h^2 = c^2 - x^2 = c^2 - \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 = c^2 - \frac{(a^2 + c^2 - b^2)^2}{4a^2}$

$$\Rightarrow h^2 = \frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4a^2} = \frac{(2ac)^2 - (a^2 + c^2 - b^2)^2}{4a^2}$$

$$= \frac{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}{4a^2} = \frac{\left[(a+c)^2 - b^2\right]\left[b^2 - (a^2 + c^2 - 2ac)\right]}{4a^2}$$

$$= \frac{\left[(a+c)^2 - b^2\right]\left[b^2 - (a-c)^2\right]}{4a^2}$$



$$= \frac{(a+c+b)(a+c-b)(b+a-c)(b-a+c)}{4a^2}$$

$$= \frac{(a+b+c)(a+b+c-2b)(a+b+c-2c)(a+b+c-2a)}{4a^2}$$

$$= \frac{2s(2s-2b)(2s-2c)(2s-2a)}{4a^2}$$

$$\Rightarrow h^2 = \frac{4s(s-a)(s-b)(s-c)}{a^2}$$

$$\Rightarrow h = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow \frac{1}{2}ah = \sqrt{s(s-a)(s-b)(s-c)} = \Delta.$$

8.13.9 *m-n* Theorem

Let D be a point on the side BC of a ΔABC such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$. Prove that

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) $(m+n)\cot\theta = n\cot B - m\cot C$

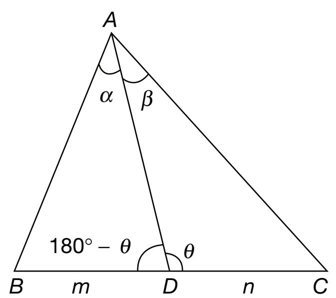
Proof:

Given $\dfrac{BD}{DC} = \dfrac{m}{n}$ and $\angle ADC = \theta = \angle ABD + \alpha$

$\therefore \quad \angle ABD = \theta - \alpha$

Also $\quad \angle ACD = 180° - (\theta + \beta)$

In ΔABD by using sine rule



$$\frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta - \alpha)} \tag{1}$$

In ΔADC

$$\frac{DC}{\sin\beta} = \frac{AD}{\sin\left(180° - (\theta + \beta)\right)} = \frac{AD}{\sin(\theta + \beta)} \tag{2}$$

From Eq. (1) ÷ Eq. (2) we get,

$$\frac{m \cdot \sin\beta}{n\sin\alpha} = \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} \qquad \left(\text{As } \frac{BD}{DC} = \frac{m}{n}\right)$$

$$\Rightarrow \frac{m(\sin\theta\cos\alpha - \cos\theta\sin\alpha)}{\sin\theta\sin\alpha} = \frac{n(\sin\theta\cos\beta + \cos\theta\sin\beta)}{\sin\theta\sin\beta}.$$

$$\Rightarrow m\cot\alpha - m\cot\theta = n\cot\beta + n\cot\theta$$

$$\Rightarrow (m+n)\cot\theta = m\cot\alpha - n\cot\beta.$$

(ii) In ΔABD, $\alpha = \theta - B$. Also In ΔADC, $\beta = 180° - (\theta + C)$

In ΔABD, $\dfrac{BD}{\sin(\theta - B)} = \dfrac{AD}{\sin B}$ (3)

In ΔADC, $\dfrac{DC}{\sin(180° - (\theta + C))} = \dfrac{AD}{\sin C} \Rightarrow \dfrac{DC}{\sin(\theta + C)} = \dfrac{AD}{\sin C}$ (4)

From Eq. (3) ÷ Eq. (4) we get $\dfrac{m}{n}\dfrac{\sin(\theta + C)}{\sin(\theta - B)} = \dfrac{\sin C}{\sin B}$

$$\Rightarrow \frac{m(\sin\theta\cos C + \cos\theta\sin C)}{\sin C\sin\theta} = \frac{n(\sin\theta\cos B - \cos\theta\sin B)}{\sin B\sin\theta}$$

$$\Rightarrow m\cot C + m\cot\theta = n\cot B - n\cot\theta$$

$$\Rightarrow (m+n)\cot\theta = n\cot B - m\cot C.$$

Build-up Your Understanding 16

1. If the medians of a ΔABC make angles α, β, γ with each other, prove that $\cot\alpha + \cot\beta + \cot\gamma + \cot A + \cot B + \cot C = 0$.
2. In an isosceles right angled triangle a straight line is drawn from the mid-point of one of the equal sides to the opposite angle. Show that it divides the angle into parts whose cotangents are 2 and 3.
3. Prove that the median through A divides it into angles whose cotangents are $2\cot A + \cot C$ and $2\cot A + \cot B$, and makes with the base an angle whose cotangent is $\dfrac{1}{2}(\cot C \sim \cot B)$.
4. Prove that the distance between the mid-point of BC and the foot of the perpendicular from A is $\dfrac{b^2 \sim c^2}{2a}$.
5. Through the angular points of a triangle are drawn straight lines which make the same angle α with the opposite sides of the triangle; prove that area of the triangle formed by them is to the area of the original triangle as $4\cos 2\alpha : 1$.
6. The measures of the sides of a triangle are integers and the area of the triangle is also an integer. One side is 21 and perimeter 48. Find the shortest side as well as the area of the triangle.
7. Find a point P, in the interior of ΔABC, such that, the product of its distances from the sides is maximum.
8. Consider the following statements about a triangle.
   (i) The sides a, b, c and area S are rational.
   (ii) a, $\tan\dfrac{B}{2}, \tan\dfrac{C}{2}$ are rational
   (iii) a, $\sin A$, $\sin B$, $\sin C$ are rational.
   Prove the following chain of results:
   Statement (i) ⇒ Statement (ii) ⇒ Statement (iii) ⇒ Statement (i).
9. Given a triangle ABC, define the quantities x, y, z as follows:
   $$x = \tan\frac{B-C}{2}\tan\frac{A}{2}, y = \tan\frac{C-A}{2}\tan\frac{B}{2}, z = \tan\frac{A-B}{2}\tan\frac{C}{2}.$$
   Prove that, $x + y + z + xyz = 0$.

10. Prove that if the Euler line passes through a vertex, then the Δ is either right-angled or isosceles.

11. If the Euler line is parallel to BC prove that $\tan B \cdot \tan C = 3$.

12. If $\angle BAC = 60°$, prove that the Euler line forms with AB, AC an equilateral triangle.

13. Six different points are given on a circle. The orthocentre of the triangle formed by three of these points are joined to the centroid of the triangle formed by the other three points by a line segment. Prove that the 20 line segments, so formed, are concurrent.

14. If D is the foot of the altitude from A in ΔABC and G is its centroid and DG is produced to meet the circumcircle at Q, then prove that $\angle QAD = 90°$.

15. If P is the mid-point of AH and if PG extended meets the circumcircle at Q prove that $PA' \parallel$ to AQ where A' is the mid-point of BC.

8.13.10 Circles, Centres and the Triangle

8.13.10.1 Circumcircle and Circumcentre

The circle which passes through the vertices of a triangle is called circumcircle.

The centre of this circle is the point of intersection of perpendicular bisectors of the sides and called the circumcentre. Its radius is always denoted by R and is called circumradius.

Circumradius (*R*):

Circumradius R of the ΔABC is equal to $\dfrac{abc}{4\Delta}$

Proof: From sine rule, $2R = \dfrac{c}{\sin C}$

$$\Rightarrow R = \frac{abc}{2ab\sin C} \qquad \text{(Multiplying '}ab\text{' in numerator and denominator)}$$

$$\Rightarrow R = \frac{abc}{4\Delta} \qquad \left(\text{As } \Delta = \frac{1}{2}ab\sin C\right)$$

Notes:

1. Circumcentre is a point which is always equidistant from the vertices of the triangle.
2. Circumcentre of an obtuse angled triangle lies outside the triangle.
3. Circumcentre of an acute angled triangle lies inside the triangle.
4. Circumcentre of a right angled triangle is the mid-point of the hypotenuse.

Example 133 *If the internal bisector of $\angle A$ of a triangle ABC meets the base BC at D and the circumcircle at E, show that $AB \cdot AC = AD \cdot AE$. Hence find an expression for the circumradius of ΔABC in terms of sides.*

Solution:

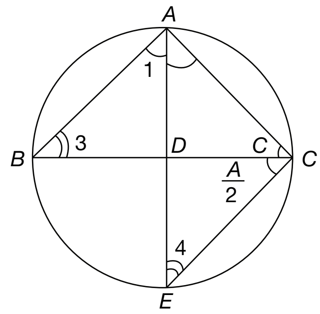
In ΔABD and ΔAEC,

$\angle BAE = \angle EAC$ (since AE bisects $\angle BAC$)

Also $\angle ABD = \angle AEC$ (Angles in the same segment of a circle)

$\therefore\ \Delta ABD \sim \Delta AEC$ (AA criterion)

$$\therefore \frac{AB}{AE} = \frac{AD}{AC} \quad \Rightarrow \quad AB \cdot AC = AD \cdot AE. \qquad (1)$$



$$\angle ACE = C + \frac{A}{2} = \frac{\pi}{2} + \frac{C-B}{2}$$

By sine rule in ΔAEC, we get

$$AE = 2R\sin\left(\frac{\pi}{2} + \frac{C-B}{2}\right) = 2R\cos\left(\frac{C-B}{2}\right)$$

$$= \frac{2R\cos\left(\frac{C-B}{2}\right)\sin\left(\frac{C+B}{2}\right)}{\sin\left(\frac{C+B}{2}\right)} = \frac{R(\sin B + \sin C)}{\cos\left(\frac{A}{2}\right)}$$

$$= \frac{b+c}{2\cos\left(\frac{A}{2}\right)}$$

$$\Rightarrow AD \cdot AE = \frac{AD \cdot b + AD \cdot c}{2\cos\frac{A}{2}} = \frac{AD \cdot b\sin\frac{A}{2} + AD \cdot c\sin\frac{A}{2}}{2\cos\frac{A}{2}\sin\frac{A}{2}}$$

$$\Rightarrow AD \cdot AE = \frac{2\Delta}{\sin A} = \frac{4\Delta R}{a} \qquad (2)$$

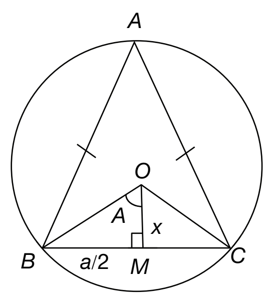
Hence from Eqs. (1) and (2), we get

$$\Rightarrow bc = \frac{4\Delta R}{a} \Rightarrow R = \frac{abc}{4\Delta}.$$

Example 134 *If x, y, z are perpendicular from the circumcentre of the sides of the ΔABC respectively. Prove that* $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.

In ΔOBM, $\tan A = \frac{BM}{OM} = \frac{a}{2x}$

Similarly, $\tan B = \frac{b}{2y}$ and $\tan C = \frac{C}{2z}$



Also, $A + B = \pi - C$

$\tan(A + B) = \tan(\pi - C)$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\Rightarrow \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$

$$\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = \frac{abc}{8xyz}$$

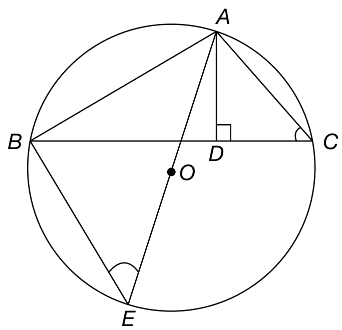
$$\Rightarrow \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}.$$

Bramhagupta

598 AD–670 AD
Nationality: Indian

8.13.10.2 Bramhagupta's Theorem

In any triangle product of any two sides is equal to the product of the perpendicular drawn to the third side with circum-diameter



In ΔABC, $AD \perp BC$. Let O be the circumcentre. Join AO and produced it to cut circum-circle at E, then AE is the diameter and $\angle ABE = 90°$

In ΔABE and ΔADC

$\angle ABE = \angle ADC = 90°$

$\angle AEB = \angle ACD$ (Angles in the same segment)

$\therefore$ By AA similarly, $\Delta ABE \sim \Delta ADC$

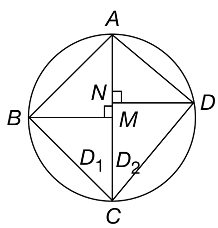
$$\therefore \quad \frac{AB}{AD} = \frac{AE}{AC}$$

$$\Rightarrow \quad AB \cdot AC = AE \cdot AD$$

$$\Rightarrow \quad AB \cdot AC = 2R \cdot AD.$$

Example 135 *ABCD is a cyclic quadrilateral. Prove the result:*

$$AC[AB \cdot BC + CD \cdot DA] = BD[AB \cdot AD + CB \cdot CD]$$



Solution: Let R be the circumradius to ΔABC.

Draw $BM \perp AC$ and $DN \perp AC$

From ΔABC, $BA \cdot BC = 2R \cdot BM$ (Bramhagupta's theorem)

From ΔADC, $DA \cdot DC = 2R \cdot DN$ (Bramhagupta's theorem)

$BA \cdot BC + DA \cdot DC = 2R\ (BM + DN)$

$\therefore AC[BA \cdot BC + DA \cdot DC] = 2R[AC \cdot BM + AC \cdot DN]$

i.e., $AC[BA \cdot BC + DA \cdot DC] = 2R[2\Delta_1 + 2\Delta_2]$

Where Δ_1 and Δ_2 are the areas of ΔABC and ΔADC respectively.

Thus, $AC[BA \cdot BC + DA \cdot DC] = 4R[\Delta_1 + \Delta_2] = 4R\ [ABCD]$

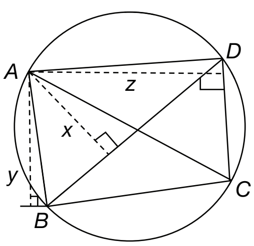
In the same way, we can show, by drawing the other diagonal BD and the perpendiculars from A and C to BD, that,

$BD[AB \cdot AD + CB \cdot CD] = 4R[ABCD]$

Thus, $AC[BA \cdot BC + DA \cdot DC] = BD[AB \cdot AD + CB \cdot CD]$.

Example 136 *ABCD is a cyclic quadrilateral, x, y, z are the distances of A from the lines BD, BC, CD respectively. Prove that* $\dfrac{BD}{x} = \dfrac{BC}{y} + \dfrac{CD}{z}$.

Solution:



In ΔABD by using Bramhagupta's theorem

$$AB \cdot AD = 2R \cdot x \tag{1}$$

In ΔABC

$$AB \cdot AC = 2R \cdot y \tag{2}$$

In ΔACD

$$AC \cdot AD = 2R \cdot z \tag{3}$$

By applying Ptolemy's theorem in $ABCD$

$$AC \cdot BD = AD \cdot BC + AB \cdot CD$$

Divide by AC

$$BD = BC\left(\frac{AD}{AC}\right) + CD\left(\frac{AB}{AC}\right) \tag{4}$$

From Eq. (1)/ Eq. (2), we get, $\dfrac{AD}{AC} = \dfrac{x}{y}$ (5)

From Eq. (1)/ Eq. (3), we get $\dfrac{AB}{AC} = \dfrac{x}{z}$ (6)

$\therefore$ From Eqs. (4), (5) and (6), we get $BD = BC \cdot \dfrac{x}{y} + CD \cdot \dfrac{x}{y}$.

$\Rightarrow \dfrac{BD}{x} = \dfrac{BC}{y} + \dfrac{CD}{z}$.

8.13.10.3 Incircle and Incentre

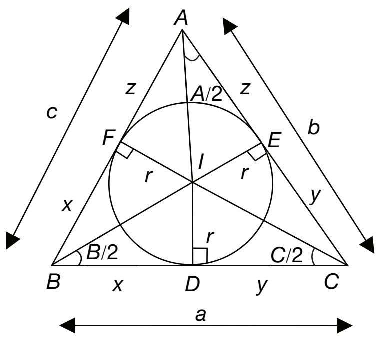
The circle that can be inscribed within triangle so as to touch each of its sides is called its inscribed circle or incircle. The centre of this circle is the point of intersection of angle bisectors of the triangle and hence it is equidistant from the sides of a triangle. The radius of the circle is always denoted by 'r' and is equal to the length of perpendicular from its centre to any one of the sides of the triangle.

Some standard results:

1. $r = \dfrac{\Delta}{s}$
2. $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$
3. $r = 4R\sin\dfrac{A}{2}.\sin\dfrac{B}{2}.\sin\dfrac{C}{2}$

Proof: Let the internal bisectors of the angles of the ΔABC meet at I. Suppose the circle touches the sides BC, CA, AB at D, E and F respectively.

Then ID, IE, IF are perpendiculars to these sides and $ID = IE = IF = r$



1. Now

$[IBC] + [ICA] + [IAB] = [ABC]$

$\dfrac{1}{2}ar + \dfrac{1}{2}br + \dfrac{1}{2}cr = \Delta$

$\Rightarrow \dfrac{1}{2}(a+b+c)\cdot r = \Delta$

$\Rightarrow r = \dfrac{\Delta}{s}$. (As $a + b + c = 2s$)

2. Since the lengths of the tangents drawn from an external point to the circle are equal

$\therefore\ BD = BF = x$ (Say)

$CD = CE = y$ (Say)

$AE = AF = z$ (Say)

$\therefore\ x + y = a$ (1)

$y + z = b$ (2)

$z + x = c$ (3)

Adding $2(x + y + z) = a + b + c = 2s$

$x + y + z = s$ (4)

From Eq. (4) – Eq. (1), $z = s - a = AE = AF$

From Eq. (4) – Eq. (2), $x = s - b = BD = BF$

From Eq. (4) – Eq. (3), $y = s - c = CD = CE$

In ΔIAE $\tan\frac{A}{2} = \frac{IE}{AE} = \frac{r}{s-a}$

$\Rightarrow\ r = (s - a)\tan\frac{A}{2}$

Similarly, $r = (s - b)\tan\frac{B}{2}$ and $r = (s-c)\tan\frac{C}{2}$

3. In ΔIBD and ΔICD

$\cot\frac{B}{2} = \frac{BD}{r}$ and $\cot\frac{C}{2} = \frac{CD}{r}$

$$a = BD + CD = r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = r\left(\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right)$$

$$= \frac{r\left(\cos\frac{B}{2}\sin\frac{C}{2} + \sin\frac{B}{2}.\cos\frac{C}{2}\right)}{\sin\frac{B}{2}.\sin\frac{C}{2}} = r\left(\frac{\sin\left(\frac{B+C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}\right) = \frac{r\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)}{\sin\frac{B}{2}.\sin\frac{C}{2}}$$

$$a = \frac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\cdot\sin\frac{C}{2}} \Rightarrow r = \frac{a\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

$$\therefore r = \frac{2R\sin A\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{4R\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} \quad (\text{As } a = 2R\sin A)$$

$$\Rightarrow r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Example 137 *If the incircle of a right angled triangle ABC, touches the hypotenuse AC at K. Then prove that the area of right angle triangle is the product of CK and AK. Also prove that inradius is* $\frac{AB + BC - AC}{2}$.

Solution: Since the length of the tangents drawn from an external point to the circle are equal

$\therefore\ BL = BM = x$ (Say)

$CL = CK = y$ (Say)

$AM = AK = z$ (Say)

In ΔABC, by using Baudhayana theorem

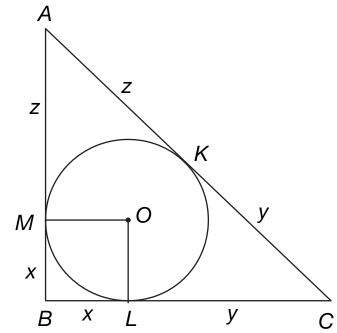
$AC^2 = AB^2 + BC^2$

$\Rightarrow (y+z)^2 = (x+z)^2 + (x+y)^2$

$y^2 + z^2 + 2yz = x^2 + z^2 + 2xz + x^2 + y^2 + 2xy$

$\Rightarrow 2yz = 2x^2 + 2xz + 2xy$

$\Rightarrow yz = x^2 + xz + xy$ (1)



Area of $\Delta ABC = \frac{1}{2} BC \cdot AB$

$= \frac{1}{2}\ (x+y)\ (x+z)$

$= \frac{1}{2}\ (x^2 + xz + xy + yz)$

$= \frac{1}{2}\ (yz + yz) = \frac{1}{2} \times 2yz$ (From Eq. (1))

$= yz$

$[ABC] = CK \cdot AK$

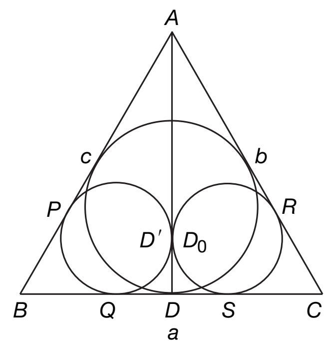
Also Inradius $= x$

And $AC = AK + KC = AM + LC = AB - x + BC - x$

$\Rightarrow x = \dfrac{AB + BC - AC}{2}$.

Example 138 *The incircle of ΔABC touch BC at D. Show that the circles inscribed in triangles ABD and CAD touch each other.*

Solution:



To proof: $AD' = AD_0$

We know that, $BD = s - b$ (where 's' is semi perimeter)

$\Rightarrow AD_0 = \dfrac{c + s - b + AD}{2} - (s - b) = \dfrac{c + b - s + AD}{2}$

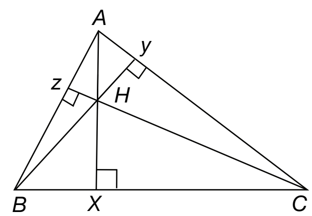
And $AD' = \dfrac{b + s - c + AD}{2} - (s - c) = \dfrac{c + b - s + AD}{2}$

$\Rightarrow AD_0 = AD'$

Hence we can say D_0 and D' are same points.

8.13.10.4 Orthocentre

Let ABC be any triangle and let AX, BY, CZ be the perpendiculars from A, B and C upon the opposite sides of the triangle. These are concurrent at H, which is called the orthocentre of the triangle

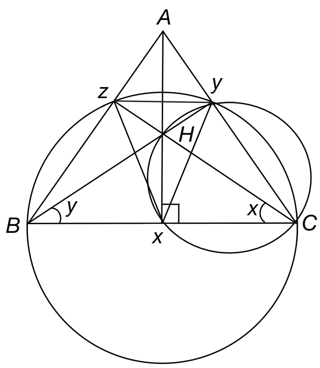


Some Standard Results:

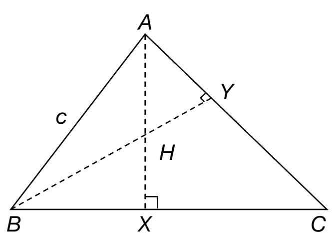
1. In an acute angled triangle orthocentre lies inside the triangle. In a right angled triangle, the orthocentre is at the right angled vertex. In an obtuse angled triangle orthocentre lies in the exterior of the triangle and behind the obtuse angle.

2. Out of four points A, B, C and H each point is the orthocentre of the triangle formed by other three.
   For ΔABC orthocentre is H
   For ΔABH orthocentre is C
   For ΔBCH orthocentre is A
   For ΔACH orthocentre is B
3. There are 6 Cyclic Quadrilaterals in above diagram namely, $BXHZ$, $CYHX$, $AZHY$, $BZYC$, $CXZA$, $AYXB$.
4. $\angle BHC = 180° - \angle A = \angle B + \angle C$
   $\angle AHC = 180° - \angle B = \angle A + \angle C$
   $\angle AHB = 180° - \angle C = \angle A + \angle B$

   Proof: In cyclic quadrilateral $AZHY$
   $\therefore\ \angle ZHY + \angle A = 180°$
   $\therefore\ \angle ZHY = 180° - \angle A$
   $\therefore\ \angle BHC = \angle ZHY = 180° - \angle A$
   Similarly others.
5. Since $HXCY$ is cyclic quadrilateral
   $\therefore\ BX \cdot BC = BH \cdot BY$ (Power of the point B)
   Also $AZXC$ is cyclic
   $\therefore\ BX \cdot BC = BZ \cdot BA$ (Power of the point B)
   Combing the above result we get
   $BX \cdot BC = BH \cdot BY = BZ \cdot BA$
   Similarly, $CX \cdot CB = CH \cdot CZ = CY \cdot CA$ and $AZ \cdot AB = AH \cdot AX = AY \cdot AC$.
6. The triangle XYZ formed by joining the feet's of these perpendiculars is called the orthic triangle of the ΔABC.
7. The orthocentre H of ΔABC is the incentre of Orthic triangle XYZ provided ABC is an acute angle triangle.



   Proof: Since $BZYC$ is cyclic quadrilateral
   $\therefore\ \angle BCZ = \angle BYZ = x$
   Also $HXCY$ is cyclic quadrilateral
   $\therefore\ \angle HCX = \angle HYX = x$
   $\Rightarrow \angle HYX = \angle HYZ = x$
   $\Rightarrow HY$ bisects the $\angle ZYX$.
   Similarly HX and HZ bisects the $\angle ZXY$ and $\angle YZX$ respectively.
   Hence the orthocentre H of ΔABC is the incentre of ΔXYZ. Also A, B, C will be Ex-centres of ΔXYZ. In case of ΔABC obtuse angle triangle say $\angle A$ be obtuse, then A will be incentre of orthic triangle and H, B, C will be Ex-centres of orthic triangle XYZ.
8. In ΔABC, if AX, BY, CZ are the altitudes and ΔXYZ is the Orthic triangle then
   $\angle ZXB = \angle YXC = \angle A$
   $\angle \mathrm{XYC} = \angle \mathrm{ZYA} = \angle B$
   $\angle XZB = \angle YZA = \angle A$
   i.e., $\Delta ABC \sim \Delta AYZ \sim \Delta XRZ \sim \Delta XYC$



Example 139 *In ΔABC, if H is the orthocentre then find AH, BH, CH respectively.*

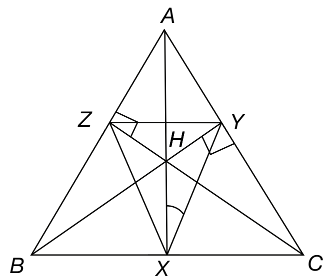
Solution: In ΔBAY, $AY = c \cos A$
In ΔAHY, $AH = AY \operatorname{cosec} C$

$$= c \cos A \ \frac{1}{\sin C}$$
$$= 2R \cos A$$

Similarly $BH = 2R\cos B$

$CH = 2R\cos C$

Example 140 *In ΔABC a, b and c represents the sides, find the sides and angles of the orthic triangle.*



Solution: From point 8 we have

$\angle YXZ = 180° - 2A$

Similarly $\angle XYZ = 180° - 2B$ and $\angle XZY = 180° - 2C$

For side of ΔXYZ,

Consider ΔAZY, as AH is diameter of circumcircle of ΔAZY, by sine rule

$$\frac{ZY}{\sin A} = AH = 2R\cos A$$

$$\Rightarrow ZY = 2R\cos A\sin A = R\sin 2A$$

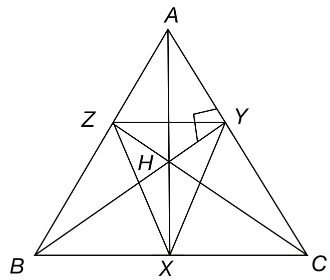
Similarly, $XY = R\sin 2C$, $ZX = R\sin 2B$

Thus sides of pedal triangle are

$a\cos A$, $b\cos B$, $c\cos C$ or $R\sin 2A$, $R\sin 2B$, $R\sin 2C$

Note: If given triangle is obtuse, say $\angle C$ is obtuse then angles of pedal triangle are represented by $2A$, $2B$, $2C - 180°$ and the sides are $a\cos A$, $b\cos B$, $-c\cos C$.

Example 141 *AX, BY, CZ are the perpendiculars from the angular points of a ΔABC upon the opposite sides, prove that the diameters of the circumcircles of triangles AYZ, BXZ, and CXY are respectively a $\cot A$, b $\cot B$ and c $\cot C$ and that the perimeters of the Δ XYZ and ΔABC are in the ratio r : R.*



Solution: ΔXYZ is the orthic triangle of ΔABC

AH is diameter of circumcircle of ΔAYZ,

$$AH = 2\text{R}\cos A = \frac{a}{\sin A}\cos A = a\cot A$$

Similarly the diameters of circumcircle of ΔBXZ and ΔCXY are $b\cot B$ and $c\cot C$.

Perimeter of $\Delta XYZ = YZ + ZX + XY$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= R(2\sin(A+B)\cos(A-B) + 2\sin C\cos C)$$

$$= R(2\sin(180° - C)\cos(A-B) + 2\sin C\cos(180° - A - B))$$

$$= 2R\sin C(\cos(A-B) - \cos(A+B))$$

$$= 4R\sin A\sin B\sin C$$

$$= 4R\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right) = \frac{abc}{2R^2} = \frac{2abc}{4R\cdot R}$$

Perimeter of $\Delta XYZ = \dfrac{2\Delta}{R} = \dfrac{2rs}{R}$

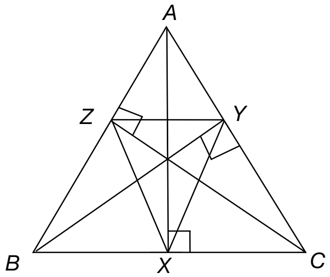
Perimeter of ΔXYZ : Perimeter of $\Delta ABC = \dfrac{r(2s)}{R} : 2s = r : R$

Example 142 *Find the area, circumradius and inradius of the orthic triangle of ΔABC.*

Solution:

Since area of Δ

$$= \frac{1}{2}(\text{Product of the sides}) \times \text{Sine of the included angle}$$



$$\therefore [XYZ] = \frac{1}{2} XY \cdot XZ \cdot \sin \angle YXZ$$

$$= \frac{1}{2} R \sin 2C \cdot R \sin 2B \sin(180° - 2A)$$

$$= \frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C$$

$$\text{Circumradius} = \frac{YZ}{2\sin \angle YXZ} = \frac{R\sin 2A}{2\sin(180° - 2A)} = \frac{R}{2}$$

That is, circumradius of orthic triangle is half the circumradius of ΔABC.

$$\text{The inradius of the orthic } \Delta XYZ = \frac{[XYZ]}{\text{Semi perimeter}}$$

$$= \frac{1}{2} \frac{R^2 \sin 2A \sin 2B \sin 2C}{2R \sin A \sin B \sin C}$$

$$= 2R \cos A \cos B \cos C$$

Thus for orthic triangle

$$\text{Area} = \frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C$$

$$\text{Circumradius} = \frac{R}{2}$$

In radius $= 2R \cos A \cos B \cos C$.

Example 143 *If x, y, z be the sides of the orthic triangle, prove that*

$$\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = \frac{a^2 + b^2 + c^2}{2abc}$$

Solution From example 140 on page 8.149, we have $x = a \cos A$, $y = b \cos B$ and $z = c \cos C$

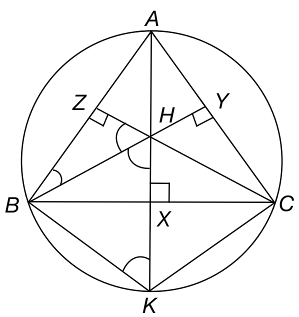
Hence,

$$\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$$

$$= \frac{a^2 + b^2 + c^2}{2abc}.$$

Example 144 *If H is the orthocentre of ΔABC and AH produced meets BC at X and the circumcircle of ΔABC at K then prove that $HX = XK$.*



Solution: In ΔBXH and ΔBYC

$\angle BXH = \angle BYC = 90°$

$\angle XBH = \angle YBC$

$\therefore$ By AA similarly

$\Delta BXH \sim \Delta BYC$

$\therefore \angle BHX = \angle BCY = \angle C$

Also $\angle ACB = \angle AKB = \angle C$

In ΔBXH and ΔBXK

$\angle BHX = \angle BKX = \angle C$

$BXH = \angle BXK = 90°$

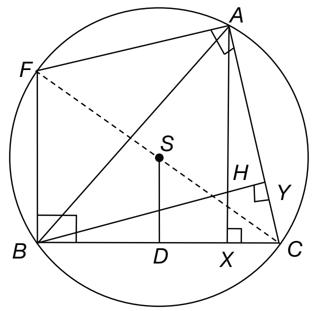
$BX = BX$ (Common)

$\therefore$ By AAS Congruence $\Delta BXH \cong \Delta BXK$

$\therefore HX = KX.$

Example 145 *If H is the orthocentre of ΔABC and S is the circumcentre and D is a mid-point of BC then prove that AH = 2SD.*

Solution: Join CS and produce it to cut the circumcircle at F. Join FB and FA.



Since CF is a diameter

$\therefore \angle FBC = \angle FAC = 90°$

Since $FB \perp BC$ and $AX \perp BC$

$\therefore FB \,||\, AX \,||\, AH$

also $FA \perp AC$, $BY \perp AC$

$\therefore FA \,||\, BY \,||\, BH$

$\therefore$ In quadrilateral $AFBH$

$AF \,||\, HB$ and $FB \,||\, AH$

$\therefore AFBH$ is a parallelogram

$\therefore AH = FB$

also in $\Delta\, CFB$, S and D are the mid-points of CF and CB respectively

$\therefore$ By mid-point theorem

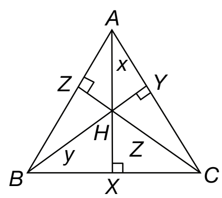
$SD \,||\, FB$ and $SD = \frac{1}{2} FB$

$\Rightarrow SD = \frac{1}{2} AH \; [\because AH = FB]$

$\Rightarrow AH = 2SD$

Example 146 *If x, y, z are the distances of the vertices of the ΔABC respectively from the orthocentre then prove that* $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$.

Solution:



$$[ABC] = [BHC] + [CHA] + [AHB]$$

$$\Delta = \frac{1}{2} yz \sin(\pi - A) + \frac{1}{2} zx \sin(\pi - B) + \frac{1}{2} xy \sin(\pi - C)$$

$$\frac{abc}{4R} = \frac{1}{2} yz \sin A + \frac{1}{2} zx \sin B + \frac{1}{2} xy \sin C$$

$$= \frac{1}{2} xyz \left[\frac{\sin A}{x} + \frac{\sin B}{y} + \frac{\sin C}{z}\right] = \frac{1}{2} xyz \left[\frac{a}{2Rx} + \frac{b}{2Ry} + \frac{c}{2Rz}\right]$$

$$\frac{abc}{4R} = \frac{xyz}{4R}\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)$$

$$\Rightarrow \frac{abc}{xyz} = \frac{a}{x} + \frac{b}{y} + \frac{c}{z}$$

Aliter: Since $A + B + C = \pi$

$\therefore A + B = \pi - C$

$\Rightarrow \tan(A + B) = \tan(\pi - C)$

$$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$

$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

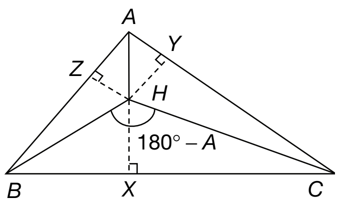
$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B + \tan C$

$$\Rightarrow \frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}+\frac{\sin C}{\cos C}=\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}\cdot\frac{\sin C}{\cos C}$$

$$\Rightarrow \frac{a}{2R\cos A}+\frac{b}{2R\cos B}+\frac{c}{2R\cos C}=\left(\frac{a}{2R\cos A}\right)\left(\frac{b}{2R\cos B}\right)\left(\frac{c}{2R\cos C}\right)$$

$$\Rightarrow \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}.$$ (As $x = 2R\cos A$ similarly others)

Example 147 *If H is the orthocentre of ΔABC. Prove that the radii of the circles circumscribing the triangles BHC, CHA, AHB, ABC are all equal.*



Solution: Since $\angle BHC = 180° - \angle A$

$\angle AHC = 180° - \angle B$

$\angle AHB = 180° - \angle C$

Let R_1 is the radius of the circumcircle of ΔBHC

$$\therefore R_1 = \frac{BC}{2\sin\angle BHC} = \frac{BC}{2\sin(180° - A)} = R$$

Similarly

$$R_2 = \frac{AC}{2\sin B} = R$$

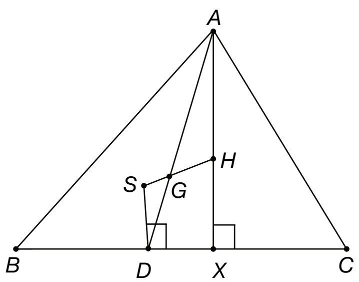
$$R_3 = \frac{AB}{2\sin C} = \mathrm{R}$$

$$\therefore R_1 = R_2 = R_3 = R$$

where R_1, R_2, R_3 and R are the circumradii of Δs BHC, AHC, AHB and ΔABC

8.13.10.5 Euler Line

The circumcentre S, the centroid G and the orthocentre H of a non-equilateral triangle are collinear and $HG = 2GS$. The line passing through H, G, S is called the Euler line.



Since $AX \perp BC$

$SD \perp BC$

$\therefore AX \,||\, SD$

Since $\dfrac{AH}{SD} = \dfrac{2}{1}$ and $\dfrac{AG}{GD} = \dfrac{2}{1}$

$$\Rightarrow \frac{AH}{SD} = \frac{AG}{GD}$$

also $\angle HAG = \angle SDG$

$\therefore$ By SAS similarity

$\Delta HAG \sim \Delta SDG$

$\Rightarrow \angle HGA = \angle SGD$

Since AD is a straight line, H, G, S are collinear

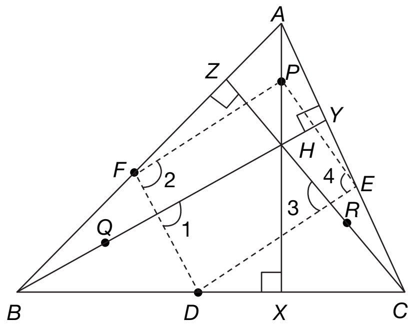
also $\dfrac{HG}{GS} = \dfrac{AH}{SD} = \dfrac{2}{1}$ or $HG = 2GS$

8.13.10.6 Nine Point Circle

The circle through the mid-points of the sides of a triangle also passes through the feet of the altitudes and the mid-points of the lines joining the orthocentre to the vertices. This circle is called the nine point circle of the triangle as there are nine fixed points on it, namely three mid-points of sides, three feet of altitudes, three mid-points of line segment joining the orthocentre and vertex.

Proof:

Given: In ΔABC,
$AX \perp BC$, $BY \perp AC$ and $CZ \perp AB$
H is the orthocentre.
D, E, F are the mid-points of BC, CA, AB respectively
P, Q and R, are the mid-points of AH, BH and CH respectively,



To prove: There is one circle passes through $D, E, F, X, Y, Z, P, Q, R$

In ΔABH, P, F are the mid-points of AH and AB respectively

$\therefore$ By mid-point theorem $PF \parallel BH$, *i.e.*, $PF \parallel BY$.

In ΔABC, F, D are the mid-points of AB, BC respectively

$\therefore$ By mid-point theorem $FD \parallel AC$

$\because \angle 1 = \angle CYB = 90°$ (Interior angles)

Also $\angle 2 = \angle 1 = 90°$ (Corresponding angles)

i.e., $\angle PFD = 90°$ (1)

also $\angle PXD = 90°$ (2)

Now In ΔAHC, P, E are the mid-points of AH, AC respectively

$\therefore$ By mid-point theorem

$PE \parallel HC$ *i.e.*, $PE \parallel ZC$

In ΔABC, E, D are the mid-points of AC, CB respectively

$\therefore$ By mid-point theorem $DE \parallel AB$

$\angle BZC = \angle 3 = 90°$ (Interior angles)

Also $\angle 4 = \angle 3 = 90°$ (Corresponding angles)

$\therefore \angle PED = 90°$ (3)

From Eqs. (1), (2) and (3)

Taking PD as a diameter if we draw a circle then it must passes through F, X and E

$\angle PFD = \angle PXD = \angle PED = 90°$.

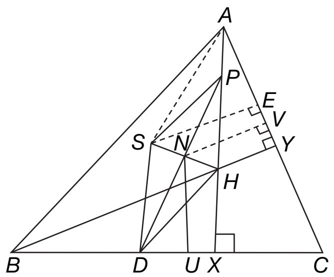
i.e., P, F, D, X, E are concyclic.

Similarly Q, D, E, Y, F are concyclic and R, E, Z, F, D are concyclic.

Since out of these, three point D, E, F are common and since from any three non-collinear points, there passes one and only one circle.

$\therefore P, Q, R, D, E, F, X, Y, Z$ are concyclic it is a nine point circle.

Theorem: The nine point centre of a triangle is collinear with the circumcentre and the orthocentre and bisects the segment joining them. Also radius of the nine point circle of a triangle is half the radius of the circumcircle of the triangle.



Proof: Let S be the circumcentre of ΔABC

$\because$ D and X lie on nine point circle.

$\therefore$ Its centre lie on the perpendicular bisector of DX. Let U be the mid-point DX. Let the perpendicular from U on BC meets SH at N. Since $SD \parallel NU \parallel HX$ and $DU = UX$

$\therefore$ $SN = NH$, *i.e.*, N is the mid-point of SH.

Now, to show that N is the centre of the nine point circle. Draw $NV \perp EY$

Since, $SE \perp AC$ and $HY \perp AC$

$\therefore$ $SE \parallel HY$

$\therefore$ $SEYH$ is a trapezium and N is a mid-pont of SH

Also $NV \parallel SE \parallel HY$

$\therefore$ V is the mid-point of EY, *i.e.*, NV is a $\perp$ bisector of EY.

That is, N is the point of intersection of perpendicular bisectors of DX and EY.

$\therefore$ N is the centre of nine point circle.

If follows that circumcentre, nine point centre and orthocentre are collinear

The nine point centre is the mid-point of the segment joining the circumcentre and orthocentre.

Now to show that the radius of the nine point circle is half the circumradius.

Since PD is a diameter of the nine point circle so N is the mid-point of PD

$\because$ SH and PD bisect each other at N

$\therefore$ $S, D, H\ P$ are the vertices of a parallelogram

$\Rightarrow SD = PH = AP$

Now, $SD \parallel AP$, $SD = AP$

$\therefore$ S, D, P, A are the vertices of a parallelogram

$\therefore$ $DP = SA = R$

$\Rightarrow 2PN = R$

$\Rightarrow 2r_N = R$

$\Rightarrow r_N = R/2$

where r_N is the radius of nine point circle.

Note: $\Delta PQR \cong \Delta DEF$ (by SSS congruence)

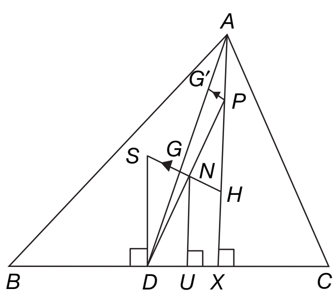
Where P, Q, R are the mid-point of AH, BH and CH and D, E, F are the mid-points of BC, CA, AB.

As $PQ = DE = (1/2)AB$

$QR = EF = (1/2)BC$

and $RP = DE = (1/2)\ CA$.

Theorem: In any triangle the circumcentre, the centroid, the nine point centre and the orthocentre are all collinear.



Proof: Through P draw $PG' \parallel HS$

So as to meet AD in G'

Let AD meets SH in G

We will show that

$AG' = G'G = GD$

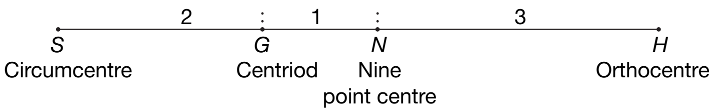
So as to conclude that

G divides AD in $2 : 1$

So consequently it is the centroid of ΔABC.
In ΔAGH, P is the mid-point of AH and $PG' \parallel HG$
$\therefore$ G' is the mid-point of AG
$\therefore$ $AG' = G'G$. (1)
In $\Delta PDG'$, N is the mid-point of PD and $NG \parallel PG'$
$\therefore$ by converse of mid-point theorem
G is the mid-point of DG'
i.e., $G'G = GD$ (2)
$\therefore$ From Eqs. (1) and (2)
$AG' = G'G = GD$
So $AG/GD = 2/1$
$\therefore$ $AG = (2/3)AD$
Thus G is the centroid of ΔABC which lies on the Euler line.

Aliter: Follow the proof given in Euler line.

Note: S, G, N, H are collinear with $SG/GH = 1/2$, $SN/NH = 1/1$, also $SG/GN = 2/1$.



8.13.10.7 Escribed Circles of a Triangle

The circle which touches the sides BC and two sides AB and AC produced of a triangle ABC and remains out of the triangle is called the escribed circle opposite to the angle A. Its radius is denoted by r_a (or r_1). Similarly r_b (or r_2) and r_c (or r_3) denote the radii of the escribed circles opposite to the angles B and C respectively. The centres of the escribed circles are called the ex-centres. The centre of escribed circle opposite to the angle A is the point of intersection of external bisector of angle B and C. The internal bisector also passes through the same point. This centre is generally denoted by I_a (or I_1) similarly others.

Standard results: In any ΔABC, we have

1. $r_a = \dfrac{\Delta}{s-a}, r_b = \dfrac{\Delta}{s-b}, r_c = \dfrac{\Delta}{S-C}$

2. $r_a = s\tan\dfrac{A}{2}, r_b = s\tan\dfrac{B}{2}, r_c = s\tan\dfrac{C}{2}$

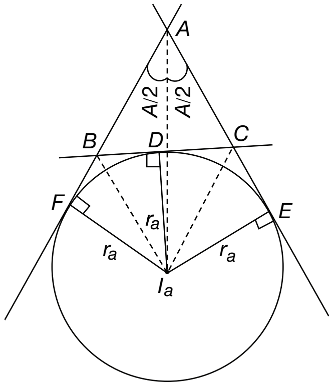
3. $r_a = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$;

$$r_b = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

$$r_c = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Proof:

1. Since each point on the angle bisector is equidistant from the arms of the angle.
   $\therefore$ $I_aD = I_aE = I_aF = r_a$



Now

$[ABC] = [ABI_a] + [ACI_a] - [BCI_a]$

$$\Delta = \frac{1}{2}AB \cdot I_aF + \frac{1}{2}AC \cdot I_aE - \frac{1}{2}BC \cdot I_aD$$

$$= \frac{1}{2}c \cdot r_a + \frac{1}{2}b \cdot r_a - \frac{1}{2}a \cdot r_a$$

$$= \frac{r_a}{2}(c + b - a) = \frac{r_a}{2}(a + b + c - 2a) = \frac{r_a}{2}(2s - 2a)$$

$$\Rightarrow r_a = \frac{\Delta}{s-a}$$

Similarly $r_b = \dfrac{\Delta}{s-b}$ and $r_c = \dfrac{\Delta}{s-c}$

2. Since the lengths of tangents to a circle from an external points are equal.

$\therefore AE = AF;\ BD = BF;\ CD = CE$

Now

$$\begin{aligned} AE + AF &= (AC + CE) + (AB + BF) \\ &= (AC + CD) + (AB + BD) \\ AF + AF &= AC + AB + (BD + CD) \\ &= AC + AB + BC \\ &= a + b + c = 2s \\ 2AF &= 2s \\ AF &= s = AE \end{aligned}$$

In ΔI_aAF, $\tan\dfrac{A}{2} = \dfrac{I_aF}{AF} = \dfrac{r_a}{s}$

$$\Rightarrow r_a = s\tan\frac{A}{2}$$

Similarly, $r_b = s\tan\dfrac{B}{2}$ and $r_c = s\tan\dfrac{C}{2}$

3. In ΔI_aBD, we have

$$\tan\left(\frac{\pi - B}{2}\right) = \frac{I_aD}{BD} = \frac{r_a}{BD}$$

or $\cot\dfrac{B}{2} = \dfrac{r_a}{BD}$

$$\Rightarrow BD = r_a\tan\frac{B}{2}$$

Similarly, $CD = r_a\tan\dfrac{C}{2}$

Now

$$a = BC = BD + DC = r_a\tan\frac{B}{2} + r_a\tan\frac{C}{2}$$

$$= r_a\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)$$

$$= r_a \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right)$$

$$= r_a \left(\frac{\sin \frac{B}{2} \cdot \cos \frac{C}{2} + \cos \frac{B}{2} \cdot \sin \frac{C}{2}}{\cos \frac{B}{2} \cdot \cos \frac{C}{2}} \right)$$

$$= r_a \frac{\sin \left(\frac{B+C}{2} \right)}{\cos \frac{B}{2} . \cos \frac{C}{2}} = \frac{r_a \sin \left(\frac{\pi - A}{2} \right)}{\cos \frac{B}{2} \cdot \cos \frac{C}{2}} = \frac{r_a \cos \frac{A}{2}}{\cos \frac{B}{2} \cdot \cos \frac{C}{2}}$$

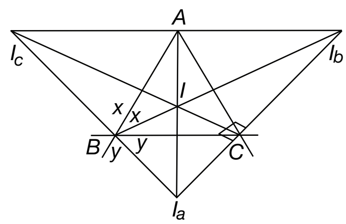
$$\Rightarrow r_a = \frac{a \cos \frac{B}{2} . \cos \frac{C}{2}}{\cos \frac{A}{2}}$$

$$r_a = \frac{2R \sin A \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}}{\cos \frac{A}{2}} = \frac{4R \sin \frac{A}{2} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$$

$$r_a = 4R \sin \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$$

Similarly, $r_b = 4R \cos \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \cos \frac{C}{2}$; $r_c = 4R \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \sin \frac{C}{2}$

8.13.10.8 Ex-central Triangle



Let ABC be a triangle and I be the centre of incircle. Let I_a, I_b, I_c be the centres of the escribed circles which are opposite to A, B, C respectively then $\Delta I_a I_b I_c$ is called the ex-central triangle of ΔABC.

Since IB bisects $\angle ABC$: $\angle ABI = \angle IBC = x$ (Say)

And I_aB bisects $\angle CBM$: $\angle CBI_a = \angle MBI_a = y$ (Say)

also $2x + 2y = 180°$

$\Rightarrow x + y = 90°$

$\angle I_aBI = 90°$

$I_aB \perp I_bB$

Thus $I_aI_c \perp IB$

Similarly, $I_aI_b \perp IC$

$I_bI_c \perp IA$

Hence ΔABC is the orthic triangle of its ex-central triangle $I_aI_bI_c$.

Sides and angles of the ex-central triangle:

In above figure $\angle BI_aC = \angle BI_aI + \angle CI_aI$

$= \angle BCI + \angle CBI$ (As IBI_aC is cyclic quadrilateral)

$= \frac{\angle C}{2} + \frac{\angle B}{2}$

$\angle BI_aC = \frac{180° - A}{2} = 90° - \frac{A}{2}$

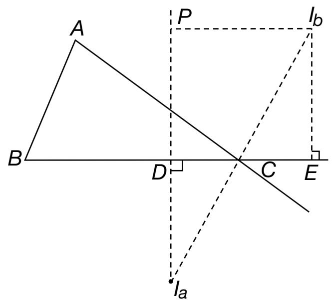
$\therefore \angle I_bI_aI_c = 90° - \frac{A}{2}$

Similarly

$\angle I_aI_bI_c = 90° - \frac{B}{2}$

$\angle I_bI_cI_a = 90° - \frac{C}{2}$

We already proved $BE = s \Rightarrow CE = s - a$.
Also $DC = s - b$
$\Rightarrow DE = s - a + s - b = c$
$\Rightarrow PI_b = DE = c$



In ΔI_aI_bP,

$I_aI_b = c \text{ cosec} \frac{C}{2}$

$= 2R \sin C \text{ cosec} \frac{C}{2}$

$I_aI_b = 4R \cos\left(\frac{C}{2}\right)$

Similarly, $I_bI_c = 4R \cos \frac{A}{2}$; $I_cI_a = 4R \cos \frac{B}{2}$

Area and circumradius of the ex-central triangle:

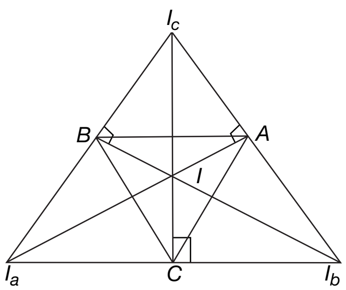
Area of $\Delta = \frac{1}{2}$ (Product of the sides) $\times$ (Sine of the included angle)

$$= \frac{1}{2}(I_aI_c)(I_aI_b) \sin(\angle I_bI_aI_c)$$

$$= \frac{1}{2} 4R\cos\left(\frac{B}{2}\right) \cdot 4R\cos\left(\frac{C}{2}\right)\sin\left(90° - \frac{A}{2}\right)$$

$$\Delta = 8R^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\text{Circumradius} = \frac{I_bI_c}{2\sin\angle I_bI_aI_c} = \frac{4R\cos\frac{A}{2}}{2\sin\left(90° - \frac{A}{2}\right)} = 2R.$$



Notes:

1. The excentres I_a, I_b, I_c of ΔABC form a triangle, whose sides pass through the vertices A, B, C. Since angle bisectors of an angle are at right angles. So the incentre I of ΔABC is the orthocentre of $\Delta I_aI_bI_c$.
2. A, B, C are the feets of the altitudes of $\Delta I_aI_bI_c$, if follows that the circumcircle of ΔABC is the nine point circle of $\Delta I_aI_bI_c$. Hence the circumcircle of ΔABC is bisector of lines I_bI_c, I_cI_a, I_aI_b and also the lines II_a, II_b and II_c.

Example 148 *In ΔABC, prove that* $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$

Solution:
$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} = \frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}$$
$$= \frac{3s-(a+b+c)}{\Delta}=\frac{3s-2s}{\Delta}$$
$$= \frac{s}{\Delta}=\frac{1}{r}$$

Example 149 *If $r_a = r_b + r_c + r$, then prove that angle A is a right angle.*

Solution. Since $r_a = r_b + r_c + r$

$$\Rightarrow r_a - r = r_b + r_c$$
$$\frac{\Delta}{s-a}-\frac{\Delta}{s}=\frac{\Delta}{s-b}+\frac{\Delta}{s-c}$$
$$\frac{s-(s-a)}{s(s-a)}=\frac{s-c+s-b}{(s-b)(s-c)}$$
$$\frac{a}{s(s-a)}=\frac{2s-(b+c)}{(s-b)(s-c)}$$
$$\frac{a}{s(s-a)}=\frac{a}{(s-b)(s-c)}$$
$$\Rightarrow \frac{(s-b)(s-c)}{s(s-a)}=1$$
$$\Rightarrow \tan^2\frac{A}{2}=1 \Rightarrow \frac{A}{2}=\frac{\pi}{4}$$
$$\Rightarrow A=\frac{\pi}{2}.$$

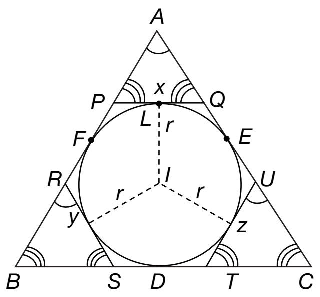
Example 150 *If A, B, C are the angles of a triangle, prove that*

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Solution: For any $\alpha, \beta, \gamma \in \mathbb{R}, \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)$

$$= 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$$
$$\Rightarrow \cos A + \cos B + \cos C + \cos(A+B+C) = 4\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{C+A}{2}\right)$$
$$\Rightarrow \cos A + \cos B + \cos C + \cos(180°) = 4\cos\left(90°-\frac{C}{2}\right)\cos\left(90°-\frac{A}{2}\right)\cos\left(90°-\frac{B}{2}\right)$$
$$\Rightarrow \cos A + \cos B + \cos C - 1 = 4\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)$$
$$\Rightarrow \cos A + \cos B + \cos C = 1 + 4\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right) = 1+\frac{r}{R}.$$

Example 151 *Tangents are parallel to the three sides are drawn to the incircle. If x, y, z are the lengths of the parts of the tangents with in the triangle then prove that* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$



Solution: Let $PQ = x$, $PQ \| BC$

$SR = y$, $SR \| CA$

$TU = z$, $TU \| AB$

We know that $AF = AE = s - a$

Also $AF = AP + PF = AP + PL$

And $AE = AQ + QE = AQ + QL$

$\Rightarrow$ Perimeter of $\Delta APQ = 2AE = 2(s - a)$

We can see that $\Delta APQ \sim \Delta ABC$

$$\frac{PQ}{BC}=\frac{\text{Perimeter of } \Delta APQ}{\text{Perimeter of } \Delta ABC}=\frac{2(s-a)}{2s}$$

$$\Rightarrow \frac{x}{a}=\frac{s-a}{s}$$

Similarly, $\frac{y}{b}=\frac{s-b}{s}$ and $\frac{z}{c}=\frac{s-c}{s}$

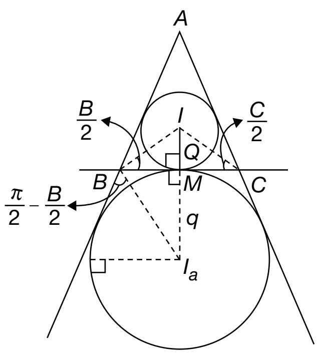
On adding, we get,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{3s-(a+b+c)}{s}=\frac{3s-2s}{s}=1$$

Example 152 *Let points $P_1, P_2, P_3, \ldots, P_{n-1}$ divides the side BC of a ΔABC into n parts. Let $r_1, r_2, r_3, \ldots, r_n$ be the radii of inscribed circles and let $q_1, q_2, \ldots, q_n$ be the radii of escribed circles corresponding to vertex A for triangle $ABP_1, AP_1P_2, \ldots, AP_{n-1}C$ and let r and q be the corresponding radii for the ΔABC. Show that*

$$\frac{r_1}{q_1}\cdot\frac{r_2}{q_2}\cdots\frac{r_n}{q_n}=\frac{r}{q}$$

Solution:



In ΔABC, we have

$$\frac{r}{q}=\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

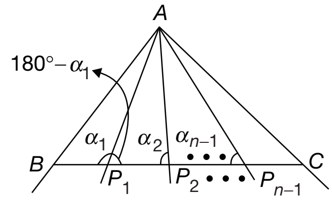
$$\Rightarrow \frac{r}{q}=\tan\frac{B}{2}\cdot\tan\frac{C}{2}$$

i.e., $\frac{r}{q}$ is product of tangents of half of base angles.

So, $\frac{r_1}{q_1}=\tan\frac{B}{2}.\tan\frac{\alpha_1}{2}$

And $\frac{r_2}{q_2}=\tan\left(\frac{180°-\alpha_1}{2}\right)\tan\frac{\alpha_2}{2}$

$$=\cot\alpha_1\tan\alpha_2$$



And so on $\dfrac{r_{n-1}}{q_{n-1}} = \tan\left(\dfrac{180° - \alpha_{n-2}}{2}\right)\tan\dfrac{\alpha_{n-1}}{2} = \cot\dfrac{\alpha_{n-2}}{2}\tan\dfrac{\alpha_{n-1}}{2}$

$$\frac{r_n}{q_n} = \tan\left(\frac{180° - \alpha_{n-1}}{2}\right)\tan\frac{C}{2} = \cot\frac{\alpha_{n-1}}{2}\tan\frac{C}{2}$$

Multiplying all, we get,

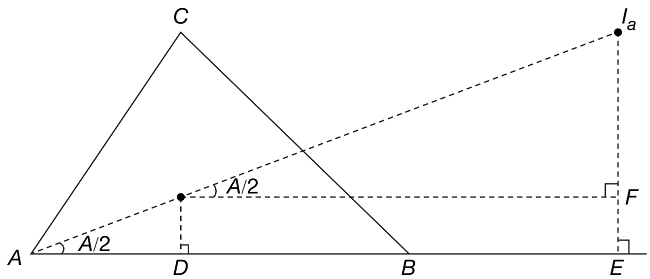
$$\frac{r_1}{q_1}\cdot\frac{r_2}{q_2}\cdots\frac{r_{n-1}}{q_{n-1}}\cdot\frac{r_n}{q_n} = \left(\tan\frac{B}{2}\tan\frac{\alpha_1}{2}\right)\left(\cot\frac{\alpha_1}{2}\tan\frac{\alpha_2}{2}\right)\cdots\left(\cot\frac{\alpha_{n-2}}{2}\tan\frac{\alpha_{n-1}}{2}\right)$$

$$\left(\cot\frac{\alpha_{n-1}}{2}\tan\frac{C}{2}\right).$$

$$= \tan\frac{B}{2}.\tan\frac{C}{2} = \frac{r}{q}$$

Example 153 *Find the distance between the incentre and ex-centres of ΔABC.*

Solution:



We know that $AE = s$, $AD = s - a \Rightarrow DE = s - (s - a) = a$

$\Rightarrow IF = a$

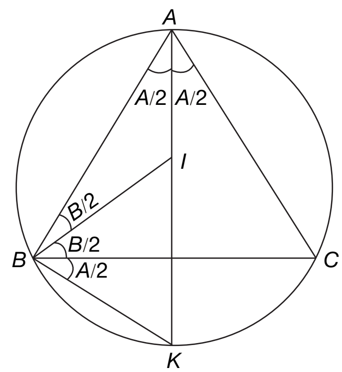
In ΔII_aF,

$$\sec\frac{A}{2} = \frac{II_a}{IF} = \frac{II_a}{a}$$

$$II_a = a\sec\frac{A}{2}$$

Example 154 *If I is the incentre of a ΔABC and if AI meets the circumcircle in K prove that $KI = KB$.*

Solution:



I is the incentre of ΔABC

$\angle IAB = (1/2)\angle A$

$\angle IBA = (1/2)\ \angle B$

$\angle KBC = \angle KAC = (1/2)\ \angle A$ (1)

$\therefore\ \angle IBK = \angle IBC + \angle CBK = \dfrac{1}{2}(\angle A + \angle B)$

Also In ΔABI by exterior $\angle$property

$\angle BIK = \angle IAB + \angle IBA = A/2 + B/2$ (2)

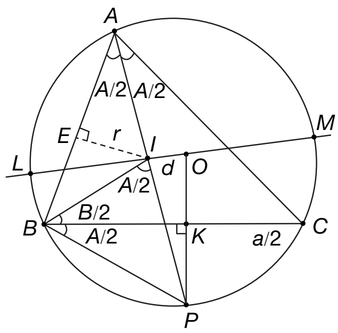
In ΔIBK, $\angle IBK = \angle BIK = 1/2(\angle A + \angle B)$ (From Eqs. (1) and (2))

$\therefore\ KI = KB$

Note: Angle bisector $\angle A$ and $\perp$ bisector of BC meet at the circumcircle. So K lies on the perpendicular bisector of BC $\therefore$ $KB = KC$

Also II_a is the diameter of circumcircle of triangle IBI_a, where I_a is the excentre as $\angle IBI_a = 90°$. Since mid-point of the hypotenuse is equidistant from the vertices, $KI = KB = I_aK = KC$.

Example 155 *Find the distance between the circumcentre and incentre of a triangle*



Solution: In ΔABC, let AP be the angle bisector, where P is a point on its circumcircle. I is the incentre and O is the circumcentre of ΔABC. Let $LIOM$ is a straight line, LM is a diameter.

$OL = R = OM$

Let $IO = d$,

$\therefore$ $LI = R - d$ and $IM = R + d$,

Let $IE \perp AB$ $\therefore$ $IE = r$

In ΔAEI

$$\sin\frac{A}{2} = \frac{r}{AI}$$

$$AI = \frac{r}{\sin A/2}$$

We know $BP = IP$ (From previous problem)

Also in a Δ, angle bisector of $\angle A$ and perpendicular bisector of BC meet at the circumcircle so OP is the perpendicular bisector of BC.

$\therefore$ $BK = a/2$

$$\text{In } \Delta BKP,\ \cos\frac{A}{2} = \frac{BK}{BP} = \frac{a}{2\cdot BP}$$

$$BP = \frac{a}{2\cos A/2} = \frac{2R\sin A}{2\cos A/2} = 2R\sin\frac{A}{2}$$

$$\therefore PI = BP = 2R\sin\frac{A}{2}$$

$\therefore$ Considering power of the point I with respect to circumcircle, we get,

$AI \cdot IP = LI \cdot IM = (R - d)(R + d)$

$$\Rightarrow \frac{r}{\sin\frac{A}{2}} 2R\sin\frac{A}{2} = R^2 - d^2$$

$$\Rightarrow d = \sqrt{R^2 - 2Rr}.$$

Example 156 *Find the distance between the circumcentre and excentre.*

Solution: Let O be the circumcentre and I be the incentre then AI produced passes through the excentre I_a.

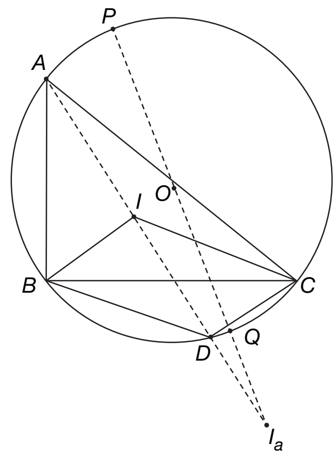
Let AI meets the circumcircle in D

Join CI, BI, CD, BD, CI_a, BI_a,

We know that $DB = DC = DI = DI_a$ (From previous problem)

Also D is the centre of the circle IBI_aC

In ΔBCI_a



Circumdiameter $II_a = \dfrac{BC}{\sin\angle BI_aC}$ (From sine rule)

$$II_a = \frac{a}{\sin\left(90° - \dfrac{A}{2}\right)}$$

$$2DI_a = \frac{2R\sin A}{\cos\dfrac{A}{2}} \quad (\text{As } II_a = 2DI_a)$$

$$\Rightarrow \quad DI_a = \frac{a}{2\cos\dfrac{A}{2}} = \frac{2R\sin A}{2\cos\dfrac{A}{2}}$$

$$\Rightarrow \quad DI_a = 2R\sin\frac{A}{2}$$

By writing power of the point I_a with respect to circumcircle of ΔABC, we get,

$I_aQ \cdot I_aP = I_aD \cdot I_aA$

$$\Rightarrow \quad (I_aO - R)\,(I_aO + R) = 2R\sin\frac{A}{2}\cdot r_a \cdot \operatorname{cosec}\frac{A}{2}$$

$$\Rightarrow \quad OI_a^{\,2} - R^2 = 2Rr_a$$

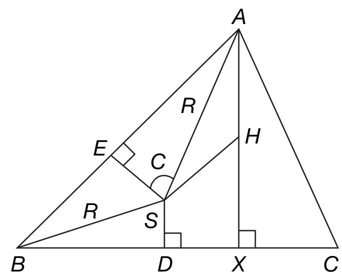
$$\Rightarrow \quad OI_a^{\,2} = R^2 + Rr_a$$

$$\Rightarrow \quad OI_a = \sqrt{R^2 + 2Rr_a}$$

Similarly, $OI_b = \sqrt{R^2 + 2Rr_b}$ and $OI_c = \sqrt{R^2 + 2Rr_c}$.

Example 157 *Find the distance between the circumcentre and the orthocentre of ΔABC.*

Solution: Let S and H be the circumcentre and orthocentre of ΔABC, and $\angle B < \angle C$.



$SE \perp AB$

$\angle ESA = \angle C$

$\therefore\ \angle EAS = 90° - \angle C$

Also $\angle HAB = 90° - \angle B$

$\angle SAH = \angle HAB - \angle BAS = 90° - \angle B - (90° - \angle C) = \angle C - \angle B$

Also $AH = 2SD = 2R\cos A$ and $SA = R$

For ΔSAH by using cosine formula

$$\begin{aligned} SH^2 &= SA^2 + AH^2 - 2SA\ AH\cos\angle SAH \\ &= R^2 + 4R^2\cos^2 A - 2R\cdot 2R\cos A\cdot\cos(C-B) \\ &= R^2 + 4R^2\cos A(\cos A - \cos(C-B)) \\ &= R^2 - 4R^2\cos A\,(\cos(C+B) + \cos(C-B)) \\ &= R^2\,(1 - 4\cos A\,(2\cos C\cdot\cos B)) \\ SH^2 &= R^2\,(1 - 8\cos A\cos B\cos C) \\ SH &= R\sqrt{1 - 8\cos A\cos B\sin C}. \end{aligned}$$

Build-up Your Understanding 17

1. Prove the following:

$$R = \frac{abc}{4rs} = \frac{1}{4}(r_a + r_b + r_c - r) = \frac{r + r_a + r_b - r_c}{4\cos C} = \frac{1}{4}(a+b)\sec\frac{(A-B)}{2}\sec\frac{C}{2}$$

$$= \frac{a^2 - b^2}{2c\sin(A-B)} = \frac{(r_a - r)(r_b - r)(r_c - r)}{4r^2} = \frac{r^2 s^2}{4}\left(\frac{1}{r} - \frac{1}{r_a}\right)\left(\frac{1}{r} - \frac{1}{r_b}\right)\left(\frac{1}{r} - \frac{1}{r_c}\right)$$

$$= \frac{\sum a\cos A}{4\prod \sin A} = \frac{\sum (b+c)\tan\frac{A}{2}}{4\sum \cos A} = \frac{r\sum \sin A}{2\prod \sin A}$$

2. Prove the following:

$$\Delta = \sqrt{r r_a r_b r_c} = r r_a \cot\frac{A}{2} = s(s-a)\tan\frac{A}{2} = \cos\frac{A}{2}\sqrt{bc(s-b)(s-c)}$$

$$= Rr(\sin A + \sin B + \sin C) = \frac{1}{4}(b^2 \sin 2C + c^2 \sin 2B) = \frac{\sin A \sin B}{2\sin(A-B)}(a^2 - b^2)$$

$$= \frac{a^2 + b^2 + c^2}{4(\cot A + \cot B + \cot C)} = \frac{(abc)^{2/3}}{2^{5/3}}(\sin 2A + \sin 2B + \sin 2C)^{1/3}$$

3. Show that the radii of the three escribed circles of a triangle are the roots of the equation, $x^3 - x^2(4R + r) + xs^2 - rs^2 = 0$.

4. If R_1, R_2 and R_3 be the diameter of the excircles of a ΔABC (opposite to the vertices A, B and C respectively), then prove that $\dfrac{a}{R_1} + \dfrac{b}{R_2} + \dfrac{c}{R_3} = \dfrac{R_1 + R_2 + R_3}{a + b + c}$.

5. Prove that $r^2 + r_a^2 + r_b^2 + r_c^2 = 16R^2 - (a^2 + b^2 + c^2)$.

6. In a triangle ABC, the incircle touches the sides BC, CA and AB at D, E, F respectively. If radius of incircle is 4 units and BD, CE and AF be consecutive natural numbers, find the sides of the triangle ABC.

7. D, E and F are the middle points of the sides of the triangle ABC; prove that the centroid of the triangle DEF is the same as that of ABC, and that its orthocentre is the circumcentre of ABC.

8. In a ΔABC, if $8R^2 = a^2 + b^2 + c^2$, show that the triangle is right angled.

9. In ΔABC, AD is the altitude through A; x, y, z are the inradii of ΔADC, ΔADB and ΔABC. Prove that $x^2 + y^2 = z^2$.

10. Let PQ be a diameter of the circumcircle of ΔABC whose centroid is G. Prove that PG bisects QH where H is the orthocentre of ΔABC.

11. PQ is a chord of a circle. Through the mid-point M of PQ chords AB and CD are drawn. AD and BC meet PQ at K and L. Then prove that M is the mid-point of KL.

12. Let the incircle touch the side BC of ΔABC at X. If A' is the mid-point of BC then prove that $A'I$ bisects AX.

13. If h, m, t are the altitude, the median and the internal bisector respectively from the same vertex of a triangle then prove that $4R^2h^2(t^2 - h^2) = t^4(m^2 - h^2)$ where R, is the circumradius of the triangle.

14. Prove that in any triangle ABC, $b^2 - c^2 = 2a\,A'D$ where D is the foot of the altitude from A and A' is the mid-point of AB.

15. If a circle be drawn touching the inscribed circle and circumscribed circles of a triangle and the side BC externally, prove that its radius is $\frac{\Delta}{a}\tan^2\frac{A}{2}$.

16. If each side of the triangle DEF is tangent to two of the three escribed circles of the triangle ABC such that all three escribed circles are circumscribed by ΔDEF, then prove that $\frac{EF}{a\cos A}=\frac{FD}{b\cos B}=\frac{DF}{c\cos C}$.

17. Let I and O be the incentre and circumcentre of ΔABC, respectively. Assume ΔABC is not equilateral (so $I \neq O$). Prove that $\angle AIO \le 90°$ if and only if $2BC \le AB + CA$.

18. In triangle ABC, the circle touches the sides BC, CA, AB respectively at D, E, F. if the radius of the incircle is 4 units and if BD, CE, AF are consecutive integers, find
(i) The perimeter of ΔABC (ii) The circumradius of ΔABC.

19. AD, BE, CF are the altitudes of ΔABC. Lines EF, FD, DE meet lines BC, CA, AB in points L, M, N, respectively. Show that L, M, N are collinear and the line through them is perpendicular to the line joining the orthocentre H and circumcentre O of ΔABC.

20. A triangle has sides of lengths 18, 24, and 30. Show that the area of this triangle, whose vertices are the incentre, the circumcentre and the centroid of the original triangle, has an integer measure.

21. Suppose the lengths of the three sides of ΔABC are integers and the in radius of the triangle is 1. Prove that the triangle is a right triangle.

22. If a, b, c are the lengths of the sides of ΔABC, prove that there exist positive real members x, y, z, such that $a = y + z$; $b = z + x$; $c = x + y$;
(i) Express the inradius 'r' and circumradius 'R' in terms of x, y, z; hence deduce the following:
(ii) (a) $\frac{R}{r} \ge \frac{b}{c}+\frac{c}{b}$ (b) $R \ge 2r$

23. In ΔABC, $AB = AC$, $\angle A = 100°$, the bisector of $\angle B$ meets AC in D. Prove that $BC = BD + AD$.

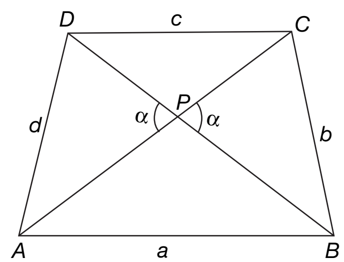
24. The centre of the circumcircle of ΔABC with $\angle C = 60°$ is O. Its radius is 2. Find the radius of the circle that touches AO, BO and the minor arc AB.

25. On the sides AB, AC of ΔABC, squares $AYXB$ arid $AQPC$ are constructed outside the Δ. Prove that CX, BP meets on the perpendicular from H to BC.

26. Prove that the straight line dividing the perimeter and area of a triangle in the same ratio passes through the incentre.

8.13.11 Area of a Quadrilaterals

8.13.11.1 Theorem I



Area of a quadrilateral is equal to half of the product of diagonals and sine of angle included between them.

Proof: $ABCD$ is any quadrilateral where $AB = a$, $BC = b$, $CD = c$ and $AD = d$ and $\angle DPA = \alpha$

Then

Area of ΔDAC = area ΔDPA + area ΔDPC

$$= \frac{1}{2} DP \cdot PA \sin\alpha + \frac{1}{2} DP \cdot PC \sin(\pi - \alpha)$$

$$= \frac{1}{2} DP(PA + PC) \sin\alpha$$

Area $\Delta DAC = \frac{1}{2} DP \cdot AC \sin\alpha$

Similarly area of $\Delta ABC = \frac{1}{2} AC \cdot PB \cdot \sin\alpha$

$\therefore$ Area $ABCD$ = Area of ΔADC + Area of $\Delta ABC = \frac{1}{2} DP \cdot AC \sin\alpha + \frac{1}{2} AC \cdot PB \sin\alpha$

$$= \frac{1}{2}(DP + PB)\, AC \sin\alpha$$

Area $ABCD = \frac{1}{2} BD \cdot AC \sin\alpha$

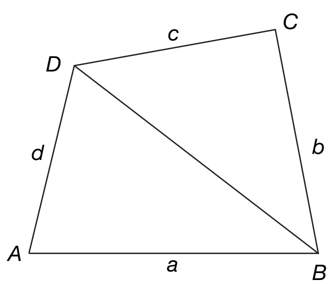
$$= \frac{1}{2} \text{(Product of the diagonals)} \times \text{(Sine of included angle)}.$$

8.13.11.2 Theorem 2

Let length of sides AB, BC, CD, DA of a quadrilateral $ABCD$ be a, b, c, d respectively and '2α' be the sum of a pair of opposite angles of it and 's' be the semi perimeter. Then area of the quadrilateral 'Δ' is given by

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$$

Proof: Consider ΔABD and ΔBCD



By cosine formula in both triangles, we get

$BD^2 = a^2 + d^2 - 2ad\cos A$

And $BD^2 = b^2 + c^2 - 2bc\cos C$

$\Rightarrow b^2 + c^2 - 2bc\cos C = a^2 + d^2 - 2ad\cos A$

$\Rightarrow b^2 + c^2 - a^2 - d^2 = 2(bc\cos C - ad\cos A)$ (1)

Also $[ABCD] = \Delta = [ABD] + [BCD] = \frac{1}{2} ad\sin A + \frac{1}{2} bc\sin C$

$\Rightarrow 4\Delta = 2\,(ad\sin A + bc\sin C)$ (2)

Squaring and adding Eqs. (1) and (2), we get,

$$(b^2 + c^2 - a^2 - d^2)^2 + 16\,\Delta^2 = 4(bc\cos C - ad\cos A)^2 + 4(ad\sin A + bc\sin C)^2$$

$$= 4[b^2c^2 + a^2d^2 - 2abcd\cos A\cos C + 2abcd\sin A\sin C]$$

$$= 4[b^2c^2 + a^2d^2 - 2abcd\,(\cos A\cos C - \sin A\sin C)]$$

$$= 4[b^2c^2 + a^2d^2 - 2abcd\cos(A + C)]$$

$$= 4[b^2c^2 + a^2d^2 - 2abcd\cos 2\alpha] \quad (\text{where } A + C = 2\alpha)$$

$$= 4[b^2c^2 + a^2d^2 - 2abcd\,(2\cos^2\alpha - 1)]$$

$$= 4[b^2c^2 + a^2d^2 + 2abcd - 4abcd\cos^2\alpha]$$

$$16\Delta^2 = 4(bc+ad)^2 - (b^2+c^2-a^2-d^2)^2 - 16abcd\cos^2\alpha$$
$$= [2(bc+ad)+b^2+c^2-a^2-d^2]\,[2(bc+ad)-b^2-c^2+a^2+d^2] - 16abcd\cos^2\alpha$$
$$= [(b+c)^2-(a-d)^2][(a+d)^2-(b-c)^2] - 16abcd\cos^2\alpha$$
$$= (b+c+a-d)(b+c-a+d)(a+d+b-c)(a+d-b+c) - 16abcd\cos^2\alpha$$
$$= (2s-2d)(2s-2a)(2s-2c)(2s-2b) - 16\,abcd\cos^2\alpha$$
(where $2s = a+b+c+d$)

$$\Rightarrow 16\Delta^2 = 16(s-a)(s-b)(s-c)(s-d) - 16\,abcd\cos^2\alpha$$
$$\Rightarrow \Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha$$
$$\Rightarrow \Delta = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$$

Notes:

1. For cyclic quadrilateral $A + C = \pi$, *i.e.*, $2\alpha = \pi \Rightarrow \alpha = \dfrac{\pi}{2} \Rightarrow \cos\alpha = 0$

   $\Rightarrow$ Area of the cyclic quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$,

   where $s = \dfrac{a+b+c+d}{2}$

   This formula is known as **Bramhagupta's** formula.
2. For tangential quadrilateral $a + c = b + d$

   $\therefore\ s = \dfrac{a+b+c+d}{2} = a+c = b+d$

   $\therefore\ s-a = c;\ s-c = a;\ s-b = d;\ s-d = b$

   $\therefore$ Area $= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$

   $= \sqrt{abcd - abcd\cos^2\alpha} = \sqrt{abcd(1-\cos^2\alpha)}$

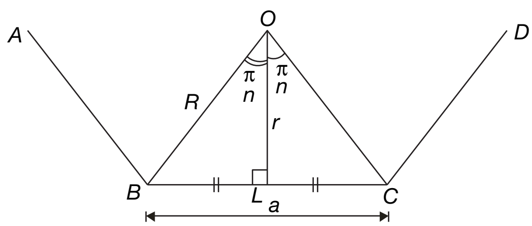
   $= \sqrt{abcd\sin^2\alpha}$

   $= \sqrt{abcd}\sin\alpha$ where $2\alpha = \angle A + \angle C$
3. For Cyclic as well as tangential quadrilateral area $= \sqrt{abcd}$ (As $\sin\alpha = 1$)

8.13.12 Regular Polygon

A regular polygon is a polygon which has all its sides as well as all its angles are equal.

If the polygon has n sides them sum of the internal angles is $(n-2)\pi$ and each angles is $\dfrac{(n-2)\pi}{n}$.



Let AB, BC and CD be three consecutive side of the regular polygon and let n be the number of its sides. Let O be the point of intersection of the bisectors of the angles $\angle ABC$ and $\angle BCD$. The point O is both the incentre and circumcentre of polygon and so $BL = LC = a/2$ where a is the side of the polygon.

Thus we have $OB = OC = R$ and $OL = r$, the circumradius and inradius respectively of n side regular polygon.

In ΔOLB $\sin\frac{\pi}{n} = \frac{BL}{OB} = \frac{a/2}{R} \Rightarrow R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

and $\tan\frac{\pi}{n} = \frac{BL}{OL} = \frac{a/2}{r} \Rightarrow r = \frac{a}{2}\cot\frac{\pi}{n}$

So circumference of circumcircle $= 2\pi R = \pi a \operatorname{cosec}\frac{\pi}{n}$

Circumference of incircle $= 2\pi r = \pi a \cot\frac{\pi}{n}$.

Also area of polygon (in terms of a) $= n \times [OBC] = n \times (1/2) \times BC \times OL$

$= \frac{n}{2} \times a \times \frac{a}{2}\cot\frac{\pi}{n} = \frac{na^2}{4}\cot\frac{\pi}{n}$

Area of polygon (in terms of 'r') $= n \times \frac{1}{2} \times BC \times OL = n\frac{a}{2} \times r = nr^2 \tan\frac{\pi}{n}$

Area of polygon (in terms of R) $= n \times \frac{1}{2} \times BC \times OL = \frac{n}{2} a \cdot r = n \times R\sin\frac{\pi}{n} \cdot R\cos\frac{\pi}{n}$

$= \frac{n}{2}R^2 \sin\frac{2\pi}{n}$.

Example 158 *Prove that difference of area of circumcircle and incircle of a regular polygon is area of a circle taking any one side of the polygon as diameter.*

Solution: Area of circumcircle $= \pi R^2 = \frac{\pi a^2}{4}\operatorname{cosec}^2\frac{\pi}{n}$

Area of incircle $\pi r^2 = \frac{\pi a^2}{4}\cot^2\frac{\pi}{n}$

So area of circumcircle-area of incircle $= \frac{\pi a^2}{4}\left(\operatorname{cosec}^2\frac{\pi}{n} - \cot^2\frac{\pi}{n}\right) = \pi\left(\frac{a}{2}\right)^2$

= area of a circle taking any one side of the regular polygon as a diameter.

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1. If $2a$ be the side of a regular polygon of n sides, R and r be the circumradius and inradius, prove that $R + r = a\cot\frac{\pi}{2n}$.
2. With reference to a given circle, A_1 and B_1 are the areas of the inscribed and circumscribed regular polygons of n sides, A_2 and B_2 are corresponding quantities for regular polygons of $2n$ sides. Prove that A_2 is a geometric mean between A_1 and B_1. Also prove that B_2 is a harmonic mean between A_2 and B_1.
3. Obtain a relation between the shortest diagonal, longest diagonal and a side of a regular nonagon.

4. Two regular polygons of n and $2n$ sides have the same perimeter, show that their areas are in the ratio $2\cos\frac{\pi}{n}:1+\cos\frac{\pi}{n}$.
5. If a, b, c, d are the sides of a quadrilateral described about a circle then prove that $ad\sin^2\frac{A}{2}=bc\sin^2\frac{C}{2}$.
6. Show that if a convex quadrilateral with side lengths a, b, c, d and area $\sqrt{abcd}$ has an inscribed circle, then it is a cyclic quadrilateral **[Putnam, 1970]**
7. The circumference of the unit circle is divided into eight equal arcs by points A, B, C, D, E, F, G, H. Chords connecting point A, to each of the other points, are joined. Find the product of the lengths of all these chords. Generalize your result.
8. Consider $A_1A_2A_3, \ldots A_n$, a regular polygon inscribed in a unit circle. Evaluate the following:
   (i) $|A_1A_2|^2+|A_1A_3|^2+\cdots+|A_1A_n|^2$
   (ii) $|A_1A_2|\,|A_1A_3|\cdots|A_1A_n|$
   (iii) $\prod\limits_{1\le i<j\le n}|A_iA_j|$
9. If A_1, A_2, A_3, …, A_n be the vertices of a n-sided regular polygon, such that $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$, find the value of n. **[INMO, 1992]**
10. Among all quadrilaterals with given lengths side of $AB=a$, $BC=b$, $CD=c$, $DA=d$, find the one with the greatest area.
11. A regular octagon with 1 unit-long sides is inscribed in a circle. Find the radius of the circle. Also find the radius of its in-circle.

8.14 Construction of Triangles

It is well-known that a triangle can be constructed in each of the following cases:

1. When all the three sides are given.
2. When one side and two angles are given.
3. When two sides and the included angle are given.

Beside the above three cases, there are many other cases when it is possible to construct the triangle. We are going to describe some of them. But before we do so we shall discuss one case, which is of the special interest.

What can we do if two sides and one angle (other than the included angle) are given?

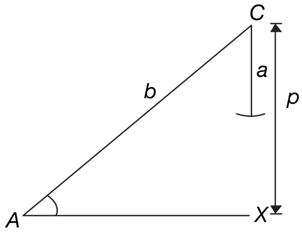
To construct a ΔABC when a, b, A are given:

There may exist no triangles, one triangle, or two triangles depending on the relation between the given parts as we shall see below. Because of the possibility of having two triangles, this case is called the Ambiguous case.

To discuss the existence and uniqueness of the solution we shall procced geometrically first.

We construct angle A and cut of $AC=b$. This fixes the vertex C. with C as centre and 'a' as radius we draw an arc in order to locate (if possible)B.

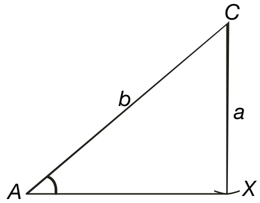
For the sake of convenience let us consider the cases $A<90°$, $A>90°$ separately.



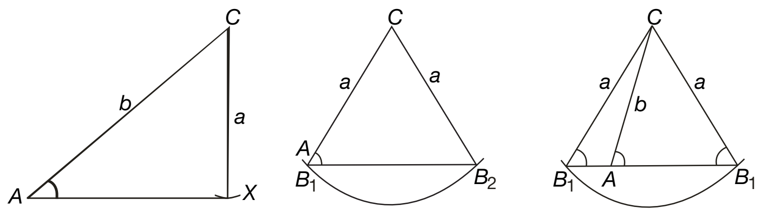
Case 1: $A < 90°$

Several possibilities arise:

1. In $a < p$ (where $p = b \sin A$ is the length of the perpendiculars from C on AX), then the arc does not cut AX and no triangle is possible.
2. If $a = p$, then arc touches AX. Therefore, one triangle is possible and it is right angled.



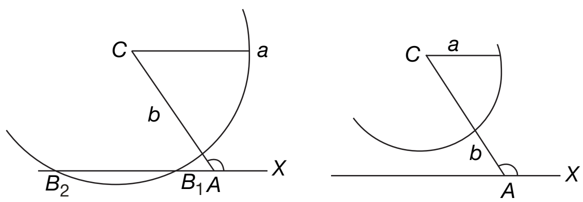
3. If $a > p$, then the arc cut AX at two points; both these point lie to the right of A if $a < b$; one of them lies to the right of A and the other coincides with A if $a = b$; and one of them lies to the right of A and the other to the left of A if $a > b$. Thus two triangles are possible if $a < b$ and only one triangle is possible if $a \geq b$, because of the possibility of two triangles, the case $b \sin A < a < b$, A acute is called the ***ambiguous case***.

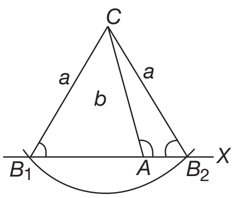


Case 2: $A > 90°$:

The following possibilities arise:

1. If $a \leq b$, the arc does not cut AX at any point to the right of A and no triangle is possible.





2. If $a > b$, then the arc cuts AX at two points, only one of which lies to the right of A and therefore, only one triangle is possible. This completes the discussion.

Let us discuss above construction algebraically also

To discuss the case, when a, b, A are given Algebraically, we shall use the sine formula. For the sake of convenience we shall discuss the case $A < 90°$, and $A > 90°$ separately.

Case 1: $A < 90°$:

The following possibilities arise:

1. If $a < b \sin A$, then form the formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} \tag{1}$$

$\sin B > 1$ and consequently no solution is possible.

2. If $a = b \sin A$, then form (1), $\sin B = 1$, so that $B = 90°$. Therefore, there is one solution and the triangle is right angled.
3. If $a > b \sin A$, then (1) gives two values of B, one of which is acute and the other obtuse. If $a \geq b$, then $A \geq B$, so that only the acute value of B is permissible and consequently there is only one solution.
   If $a < b$ then $A < B$, so that both the values of B are possible and consequently there may be two solutions.

Case 2: $A > 90°$:
The following possibilities arise:

1. If $a \leq b$, then $A \leq B$, so that B must also been obtuse angle which is not possible. Hence no solution is possible.
2. If $a > b$, then only the acute value of B is permissible and therefore, only one triangle is possible.
   Having determined B (wherever there exists a permissible value of B), we determine C by the formula $C = 180° - (A + B)$. The remaining side c is then found as in the SAS case. In the ambiguous case the values of C and c corresponding to the values of B have to be found separately.

Remark: We can discuss the ambiguous case by using the cosine formula also.
If a, b, A are given, then the cosine formula for a gives

$$a^2 = b^2 + c^2 - 2\,bc \cos A$$

$$c^2 - 2\,bc \cos A + b^2 - a^2 = 0 \tag{1}$$

Solving Eq. (1) as a quadratic in c, we have

$$c = \frac{2b \cos A \pm \sqrt{4b^2 \cos^2 A - 4(b^2 - a^2)}}{2}$$

$$= b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A} \tag{2}$$

Since c is the length of a side of a triangle, therefore, it must be positive. We have therefore to determine as to how many of the values of c given by Eq. (2) are positive for any given set of values of a, b and A.

Two different possibilities arise:

1. $A < 90°$; If $A < 90°$, $\cos A$ is positive.
   Three sub-cases arise:
   (i) If $a < b \sin A$, then $a^2 < b^2 \sin^2 A$, so that $a^2 - b^2 \sin^2 A < 0$. The two values of c are imaginary and no triangle is possible.
   (ii) If $a = b \sin A$, then $a^2 = b^2 \sin^2 A$, so that $a^2 - b^2 \sin^2 A = 0$. There is only one value of c $(= b \cos A)$ from Eq. (2) which is positive. Therefore, only one triangle is possible.
   (iii) If $a > b \sin A$, then $a^2 > b^2 \sin^2 A$, so that $a^2 - b^2 \sin^2 A > 0$. In this case Eq. (2) gives two real and distinct values of c. One of these values, namely

$$b \cos A + \sqrt{(a^2 - b^2 \sin^2 A)}$$

   is surely positive; the other value

$$b \cos A - \sqrt{(a^2 - b^2 \sin^2 A)}$$

   is positive if $b \cos A > \sqrt{(a^2 - b^2 \sin^2 A)}$

   $\Rightarrow b^2 \cos^2 A > a^2 - b^2 \sin^2 A$

   $\Rightarrow b^2 > a^2$

   $\Rightarrow b > a$.

   Therefore two triangles are possible if $b > a$ and only one triangle is possible if $b \leq a$.

2. $A > 90°$: If $A > 90°$, cos A is negative so that b cos A is negative

The value of $b\cos A - \sqrt{(a^2 - b^2 \sin^2 A)}$ is surely negative

The value of $b\cos A + \sqrt{(a^2 - b^2 \sin^2 A)}$ is positive if

$\sqrt{(a^2 - b^2 \sin^2 A)} > -b\cos A$

$a^2 - b^2 \sin^2 A > b^2 \cos^2 A$,

i.e., if $a^2 > b^2$

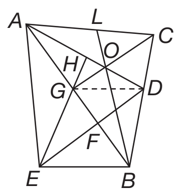
i.e., if $a > b$

Thus we find that when A > 90°, no triangle is possible $a \le b$ and only one triangle is possible when $a > b$.

8.14.1 Summary of the Various Possibilities

1. $A < 90°$
   - (i) $a < b \sin A$ No triangle
   - (ii) $a = b \sin A$ One triangle
   - (iii) $a > b \sin A$ Two triangles if $a < b$; one triangle if $a \ge b$
2. $A > 90°$
   - (i) $a \le b$ No triangle
   - (ii) $a > b$ One triangle.

Example 159 *Construct a triangle whose median lengths are given as m_a, m_b, m_c.*



Solution: Construct a ΔADE whose side are of length m_a, m_b, m_c. Draw median AF and EH of this triangle, meeting each other at G. Produce AF to B so that $GF = FB$.

Join BD and produce it to C, so that $DC = BD$, Join AC.

ΔABC is the desired triangle.

We shall show that medians of ΔABC are of lengths equal to the sides of ΔADE.

Join GC to meet AD in O, Join BO and produce it to meet AC in L

Since $BD = DC$, by construction, therefore AD is a median of ΔABC.

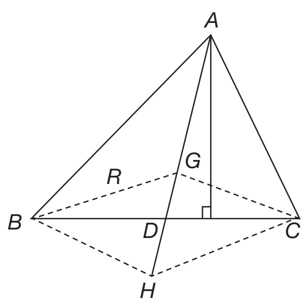
Since G is the centroid of triangle AED.

Therefore, $AG = 2GF = GF + FB = GB$

So that G is the mid-point of AB. Consequently CG is the median of ΔABC. Also since O is the point of intersection of the medians AD and CG, therefore it is the centroid of ΔABC, and consequently BL is also a median of ΔABC.

Since GB and ED bisect each other at F, therefore, E, B, D, G are the vertices of a parallelogram. Since G and D are mid-points of AB, BC respectively, therefore $GD \parallel AC$ and $GD = (1/2)AC = AL$. Now $EB \parallel GD$, and $EB = GD$, and $GD \parallel AC$, $GD = AL$, therefore, $EB \parallel AL$ and $EB = AL$. Therefore E, B, L, A are the vertices of a parallelogram. Consequently $BL = EA$.

Since $EG \parallel DC$ and $EG = DC$, therefore E, D, C, G are the vertices of a parallelogram. Consequently $CG = DE$. Since the medians AD, BL, CG of ΔABC are respectively equal to the sides AD, AE, ED of ΔADE, the proof is complete.



Example 160 *Given the lengths m_a, m_b of two medians and the length h_a of a altitude, show how to construct the triangle.*

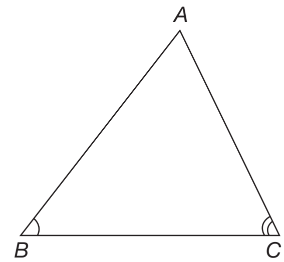
Solution: Construct a right angled ΔALD such that $AL = h_a$, $AD = m_a$, and $\angle ALD = 90°$

Produce AD to H so that $DH = 1/3\ AD$. With H as centre draw an arc equal to $2/3\ m_b$ cutting DL produced in C. With D as centre and DC as radius draw an arc cutting CD produced in B. Join AB, AC. Then ABC is the desired triangle.

Proof: It is obvious that altitude $AL = h_a$. Also since D is the mid-point of BC, therefore AD is a median. If G be the centroid, then $GD = DH$, and $BD = DC$. Therefore B, H, C, G are the vertices of parallelogram.

Therefore $CH = BG$. But $CH = \frac{2}{3}\ m_b$. Therefore $BG = \frac{2}{3}\ m_b$, showing that the length of the median from B is m_b.

Example 161 *Given the altitude h_a, h_b, h_c of a triangle. Show how to construct the triangle.*



Solution: If a, b, c be the lengths of the sides of the triangle, Δ be the area of the triangle then

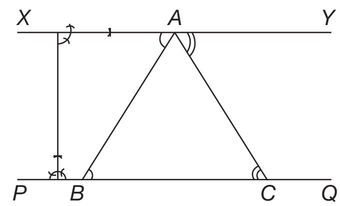
$$ah_a = bh_b = ch_c = 2\Delta$$

showing that, a, b, c are proportional to $\frac{1}{h_a}, \frac{1}{h_b}, \frac{1}{h_c}$ respectively.

By the construction for third proportional, we can construct $\frac{1}{h_a}, \frac{1}{h_b}, \frac{1}{h_c}$.

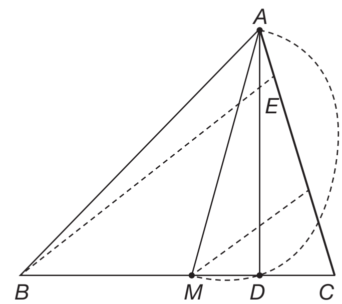
The $\Delta A'B'C'$ having these lengths as sides will be equiangular to the desired triangle(ABC, say)

$$B'C' = \frac{1}{h_a}, C'A' = \frac{1}{h_b} A'B' = \frac{1}{h_c}$$



Draw two parallel lines XY and PQ distant h_a from each other
Take a point A in XY and draw
$\angle XAB = \angle A'B'C'$. $\angle YAC = \angle A'C'B'$.
ΔABC is the desired triangle.

Example 162 *Explain the construction of the ΔABC, with necessary proof, when its altitudes AD and BE and the median AM are given* **[RMO, 1993]**



Solution:

Step 1: Construct a right angled triangle ADM, having its hypotenuse equal to the median AM and one of the sides equal to the altitude AD, a convenient way of doing this is to draw a line AM equal to the given median and then draw a semicircle having AM as a diameter, with A as centre and radius AD, draw an arc cutting the semicircle at D. Join AD.

Step 2: With M as centre and radius 1/2 BE draw an arc cutting the semicircle at H.

Step 3: Join AH and produce it to meet MD produced at C.

Step 4: With M as centre and radius equal to MC draw an arc meeting CM produced at B.
ΔABC is the desired triangle.

Justification: Since $MB = MC$(by construction), therefore M is the mid-point of BC. Therefore AM is a median. Also AD is an altitude by construction)

It only remains to be seen that the perpendicular from B to AC is equal to $2MH$. This is an immediate consequence of the fact that in right angled ΔBEC, M is the mid-point of BC and $MH \,||\, BE$($\angle BEC$ and MHC both are right angles). Therefore

$MH = \frac{1}{2}\ BE$
$BE = 2MH$.

Build-up Your Understanding 19

1. Copy a segment. In other words, mark off a segment that exactly matches the length of a given segment on a different straight line.
2. Copy an angle. Given an angle, make another angle of exactly the same size somewhere else.
3. Bisect a segment.
4. Bisect an angle. Given an angle, find a line through the vertex that divides it in half.
5. Construct a line perpendicular to a given line through a point on the given line.
6. Construct a line perpendicular to a given line and passing through a point not on the given line.
7. Given a line L and a point P not on L, construct a new line that passes through P and is parallel to L.
8. Construct an angle whose size is the sum or difference of two given angles.
9. Given three segments, construct a triangle whose sides have the same lengths as the segments.
10. Construct the perpendicular bisector of a line segment.
11. Given three points, construct the circle that passes through all of them.
12. Given a circle, find its centre.
13. Given a triangle T, construct the inscribed and circumscribed circles. The inscribed circle is a circle that fits inside the triangle and touches all three edges; the circumscribed circle is outside the triangle except that it touches all three of the vertices of the triangle.
14. Construct angles of 90°, 45°, 30°, 60°, 72°.
15. Construct a regular pentagon. (A regular pentagon is a five sided figure all of whose sides and angles are equal.)
16. Given a point P on a circle C, construct a line through P and tangent to C.
17. Given a circle C and a point P not on C, construct a line through P and tangent to C.
18. Given two circles C_1 and C_2, find lines internally and externally tangent to both.
19. Given segments of lengths A and B, construct a segment of length $A + B$ or $A - B$.
20. Given segments of lengths A, B, and 1, construct a segment of length AB.
21. Given segments of lengths A and 1, construct a segment of length $1/A$. Also construct B/A length where segment of length B is provided.
22. Given segments of lengths A and B, construct a segment whose length is $\sqrt{AB}$. Also draw $\sqrt{A}$ where segments of length l is given.
23. Given a rectangle, construct a square with exactly the same area.
25. Given a semicircle centred at a point C with diameter AB, find points I and J on AB, and points H and G on the semicircle such that the quadrilateral $GHIJ$ is a square.
26. Given a quadrant of a circle (two radii that make an angle of 90° and the included arc), construct a new circle that is inscribed in the quadrant (in other words, the new circle is tangent to both rays and to the quarter arc of the quadrant).
27. Given a point A, a line L that does not pass through A, and a point B on L, construct a circle passing through A that is tangent to L at the point B.
28. Given two points A and B that both lie on the same side of a line L, find a point C on L such that AC and BC make the same angle with L.
29. Given two non-parallel lines L_1 and L_2 and a radius r, construct a circle of radius r that is tangent to both L_1 and L_2.
30. (i) Construct an angle of $22\frac{1}{2}°$ using a ruler and compass only.

    (ii) Use this construction to solve the following problem:

If $v = \dfrac{1}{\tan 22\frac{1}{2}^\circ}$ and $u = \dfrac{1}{\sin 22\frac{1}{2}^\circ}$ then, v satisfies a quartic equation and u satisfies a quadratic equation with rational coefficients.

31. (a) Construct a regular hexagon, inscribed in a circle.
(b) Use this construction to draw two other circles to cut each other orthogonally.
(c) Justify your construction.

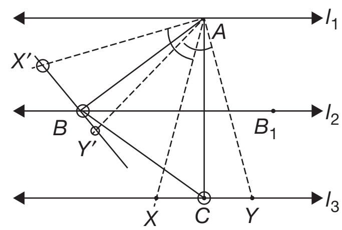
32. Construct a right angled triangle, with hypotenuse 'c' such that, the median drawn to the hypotenuse, is the GM of the two legs of the triangle. Justify.

33. Given an angle $\angle QBP$ and a point L, outside the angle $\angle QBP$. Draw a straight line through L, meeting BQ in A and BP in C, such that, ΔABC has a given perimeter. Justify your construction.

34. Given the vertex A, the orthocentre H and the centroid G, construct the triangle. Justify your construction.

35. Using a ruler and compass only, show how to bisect a triangle, by a straight line, perpendicular to the base, Justify your construction.

36. Given a triangle ABC, explain how you will find
(i) Points P, Q, R on the sides AB, BC, CA, such that $APQR$ is a rhombus
(ii) Show that the area of this rhombus cannot exceed one half of the area of ΔABC.
(iii) When does the equality hold?



37. Given are three parallel lines. You need to construct an equilateral triangle with each parallel line containing one of the vertices of the triangle.

38. Given any two rectangles anywhere in a plane, how can you draw a single line which will separate each rectangular region into two regions of equal area?

39. Describe the method (with proof) of constructing the triangle when two of its sides are given along with the median to the third side.

40. Describe the method (with proof) of constructing the triangle ABC, given the side BC and the medians BE and CF.

41. Using only compasses construct segment 2, 3, 4, … and in general n times as great as a given segment AA_1 (n is any natural number).

Solved Problems

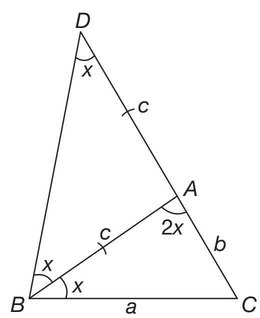
Problem 1 *In a ΔABC, $\angle A = 2\angle B$, if and only if, $a^2 = .\ b(b + c)$.* **[INMO, 1992]**

Solution: It is given that

$$\angle A = 2\angle B$$

Let, $\angle B = x$.
Then, $\angle A = 2x$
Produce CA to D, such that
$AD = AB$
$\therefore \quad \angle ABD = \angle ADB$ and $\angle ABD + \angle ADB = \angle BAC = 2x$
$\therefore \quad \angle ABD = \angle ADB = x$.
$\Delta ABC \sim \Delta BDC$ (AAA similarity)

$$\frac{BC}{DC} = \frac{AC}{BC}$$



i.e., $$\frac{a}{b+c} = \frac{b}{a} \Rightarrow b(b+c) = a^2$$

Converse: Given $a^2 = b\,(b+c)$

To prove that: $\angle A = 2\angle B$

Proof: We have the same construction as before and hence, use the same figure.

Now, $\angle ABD = \angle ADB = \frac{1}{2}\angle BAC$

Note: Here, we need to prove $\angle BAC = 2\angle ABC$. We cannot take it for granted

In ΔACB and ΔBCD,

since $a^2 = b(b+c)$

We have $\frac{a}{b+c} = \frac{b}{a}$

$\Rightarrow \frac{CB}{CD} = \frac{AC}{BC}$

and $\angle C$ is common

So, $\Delta BCD \sim \Delta ACB$

$\angle CBA = \angle CDB = \angle B$ (1)

And also, $\angle ADB = \angle ABD$

(since, $AB = AD$)

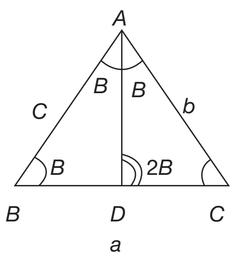
But, $\angle BAC$ = sum of the exterior angles

$= \angle ADB + \angle ABD$ (2)

$= 2\angle ADB = 2\angle CDB$

$= 2\angle CBA = 2\angle B$

which was to be proved.



Aliter 1: Draw AD, the bisector of $\angle A$, so that $\angle BAD = \angle DAC = \angle B$, as $\angle A = 2\angle B$.

Now $\angle ADC = 2\angle B$ (Exterior angle)

Also,

$\Delta ABC \sim \Delta DAC$ (AAA)

$\therefore \frac{AC}{CD} = \frac{AB}{AD} = \frac{BC}{AC}$

Thus, $\frac{BC}{AC} = \frac{a}{b} = \frac{AB+AC}{AD+CD} = \frac{b+c}{BD+CD}$ (as $AD = BD$, isosceles triangle)

i.e., $\frac{a}{b} = \frac{b+c}{a}$

Thus, $a^2 = b(b+c)$.

Converse: If $a^2 = b(b+c)$, then $\angle A = 2\angle B$.

$a^2 = b^2 + bc \Rightarrow a^2 - b^2 = bc$

$\therefore (2R\sin A)^2 - (2R\sin B)^2 = (2R\sin B)(2R\sin C)$

$\Rightarrow \sin^2 A - \sin^2 B = \sin B \sin C$

(ii) Now, *i.e.*, $\sin(A+B)\cdot\sin(A-B) = \sin B\sin C$

i.e., $\sin(A-B) = \sin B$ [as $\sin(A+B) = \sin C$]

Thus, $A - B = B$ (as A, B, C are acute)

$\therefore \angle A = 2\angle B$.

Aliter 2:

Given $\angle A = 2\angle B$, prove $a^2 = b(b+c)$.

$A = 2B \Rightarrow A - B = B \Rightarrow \sin(A-B) = \sin B$

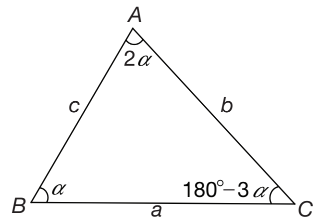
$\Rightarrow \sin(A-B)\cdot\sin(A+B) = \sin B\cdot\sin(A+B)$

$\Rightarrow \sin^2 A - \sin^2 B = \sin B \sin C$

$\Rightarrow (2R\sin A)^2 - (2R\sin B)^2 = (2R\sin B)(2R\sin C)$

$\Rightarrow a^2 - b^2 = bc \Rightarrow a^2 = b(b+c).$

Aliter 3:



Given: In a ΔABC, $\angle A = 2\angle B = 2\alpha$

To prove: $a^2 = b(b+c)$

Proof: Using sine rule in ΔABC, we get $\dfrac{a}{\sin 2\alpha} = \dfrac{b}{\sin\alpha} \Rightarrow a = 2b\cos\alpha$ (1)

And by using cosine rule in ΔABC, we get $\cos B = \cos\alpha = \dfrac{a^2+c^2-b^2}{2ac}$

$\Rightarrow \dfrac{a}{2b} = \dfrac{a^2+c^2-b^2}{2ac}$ (from Eq. (1))

$\Rightarrow a^2c = a^2b + c^2b - b^3 \Rightarrow a^2(c-b) = b(c-b)(c+b)$

$\Rightarrow a^2 = b(b+c)$ (for $c \neq b$)

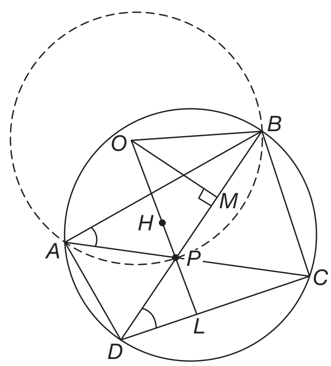
In case of $b = c$, we will have $\alpha = \pi - 3\alpha \Rightarrow \alpha = \dfrac{\pi}{4} \Rightarrow \angle A = 2\alpha = \dfrac{\pi}{2}$

$\Rightarrow a^2 = b^2 + c^2 = b^2 + bc = b(b+c)$

$\Rightarrow a^2 = b(b+c)$

Hence proved.

Problem 2 *Suppose, ABCD is a cyclic quadrilateral. The diagonals AC and BD intersect at P. Let, O be the circumcentre of ΔAPB and H, the orthocentre of ΔCPD. Show that O, P, H are collinear.*



Solution: Given, *ABCD* is a cyclic quadrilateral.

'*O*' is the circumcentre of ΔAPB.

To explain, if *M* is the mid-point of *PB*, then *OM* is perpendicular to *PB* in the in Fig. 3.12, *H* is the orthocentre of ΔCPD.

Let, *OP* produced meet *DC* in *L*.

To prove: *O*, *P* and *H*, are collinear.

To prove that *H* lies on *OP* or *OP* produced.

Or, in other words, *OP* produced is perpendicular to *DC*.

Proof: Since quadrilateral *ABCD* is cyclic,

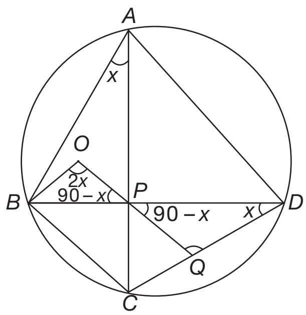
$\angle CDB = \angle CAB = \angle PAB = \dfrac{1}{2}\angle POB$ (Since, *O* is the circumcentre of ΔPAB) = $\angle POM$ ($= \angle BOM$) as *OM* is the perpendicular bisector of *PB*.

In ΔLDP and *MOP*,

$\angle LDP = \angle POM$

$\angle DPL = \angle OPM$ (Vertically opp. Z^s)

$\therefore$ $\angle PLD = \angle PMO = 90°$ and hence the result.



Aliter:

Join OP and produce it to meet CD at Q. If O, P, H are collinear, we need to prove that H lies on the line OP (produced).

$\Rightarrow PQ$ is an altitude in ΔPCD as H is the orthocentre of ΔPCD.

Let, $\angle PDC = 90°$; this shows that $\angle BAC = x$, so that $\angle BOP = 2x$ (angle at the centre).

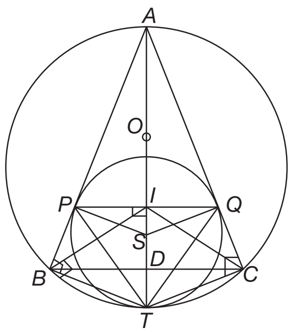
$$\therefore \angle OBP = \angle OPB = \frac{180 - 2x}{2} = 90 - x$$

Thus, $\angle QPD = 90 - x$ (Vertical opposite angle)

Now, $\angle PQD = 180 - (\angle QPD + \angle PDQ) = 180 - (90 - x + x) = 90°$

$\therefore PQ$ is the altitude $\Rightarrow O, P, H$ are collinear.

Problem 3 *In a ΔABC, $AB = AC$. A circle is internally drawn touching the circum-circle of ΔABC, and also touching the sides AB and AC at P and Q, respectively. Prove that the mid-point of PQ is the incentre of ΔABC.*



Solution:

Let, $\angle ABC = \angle ACB = \beta°$.

AT is the angle bisector of $\angle A$. I is the mid-point of PQ. Now, $AP = AQ$ as the smaller circle touches AB and AC at P and Q, respectively. The centre of the circle PQT lies on the angle bisector of $\angle A$, namely, AT, since PQ is the chord of contact of the circle PQT. $PQ \perp AT$ and the mid-point I of PQ lies on AT.

Now, to prove that I is the incentre of ΔABC, it is enough to prove that BI is the angle bisector of $\angle B$ and CI is the angle bisector of $\angle C$, respectively. By symmetry, $\angle PTI = \angle QTI = \alpha$

Now, $\angle ABT - 90°$ ($\because$ AT is diameter of $\odot ABC$)

$\therefore \quad \angle PBT = 90°$

Also, $\quad \angle PIT = 90°$

$\therefore \quad PBTI$ is cyclic.

$\therefore \angle PBI = \angle PTI = \alpha$ (Angle in the same segment)

$\angle IBD = \angle ABD - \angle ABI = \beta - \alpha$

$\angle TBC = \angle TAC = 90° - \beta$

$\therefore \angle IBT = \angle IBD + \angle DBT$

$= \beta - \alpha + 90° - \beta = 90° - \alpha$

Since, $PBTI$ is cyclic,

$\angle IPT = \angle IBT = 90° - \alpha$ (1)

$\angle BPT = 180° - \angle TPA = 180° - \angle API - \angle IPT$

$= 180° - \beta - 90° + \alpha$

$= 90° + \alpha - \beta$ (2)

But, APT is a tangent to circle PQT.

$\therefore \quad \angle BPT = \angle PQT - \angle IQT$

From Eqs. (1) and (2), we get

$90° + \alpha - \beta = 90° - \alpha$

$2\alpha = \beta$

$\therefore \quad \angle IBD = \beta - \angle PBI = 2\alpha - \alpha = \alpha$

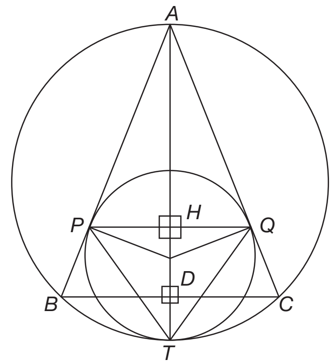
$\therefore \quad \angle IBD = \angle PBI$

$\therefore \quad BI$ is the angle bisector of $\angle B$.

Hence, the result.

Aliter:

Two circles touches internally then the line joining their centres passes through the point of contact.



AE is a diameter of a circle and since $AB = AC$

$\therefore$ AE bisects $\angle A$

$\therefore$ AE is a perpendicular bisector of BC

i.e., $BC \perp AD$

Also $AP = AQ$ [as length of the tangents drawn from an external point to the circle are equal]

Also $AB = AC$

$\therefore \dfrac{AP}{AB} = \dfrac{AQ}{AC} \Rightarrow PQ \parallel BC$

Since $AE \perp BC$

$\Rightarrow AE \perp PQ$ at H

H is a mid-point of PQ as AE lies along the diameter of smaller circle as well.

In ΔPHE and ΔQHE

$PH = QH$

$\angle PHE = \angle QHE = 90°$

$HE = HE$

$\therefore$ By SAS congruence, $\Delta PHE \cong \Delta QHE$

$\Rightarrow \angle PEH = \angle QEH = \theta$

Since $\angle APQ = \angle PEQ = \angle AQP = 2\theta$

Since $PQ \parallel BC$

$\therefore \angle ABC = \angle ACB = \angle APQ = 2\theta$

Since AE is a diameters, $\angle ABC = 90°$

Also $\angle PHE = 90°$

$\therefore$ $PBEH$ is a cyclic quadrilateral

$\therefore \angle PEH = \angle PBH = \theta$

$\therefore \angle HBC = 2\theta - \theta = \theta$

HB bisects $\angle PBC$ and HA bisects $\angle BAC$

$\therefore$ H is the incentre of ΔABC and H is the mid-point of PQ.

Problem 4 *Prove that if the two angle bisectors of a triangle are equal, then the triangle is isosceles.* (*This theorem is known as* ***Steiner Lehmus Problem***)

Solution: ABC is a triangle with AD and BE the bisectors of $\angle A$ and $\angle B$, respectively. They intersect at K. Given, $AD = BE$.

Draw $\angle BEF$ and $\angle EBF$ equal to $\angle BAD$ and $\angle ADB$, respectively. Draw AH and FG perpendicular to AC and FB (produced if necessary).

(i) $\Delta ADB \equiv \Delta EBF$ (ASA)

$AB = EF$ ($\because$ $AD = BE$, $\angle DAB = \angle FEB$,

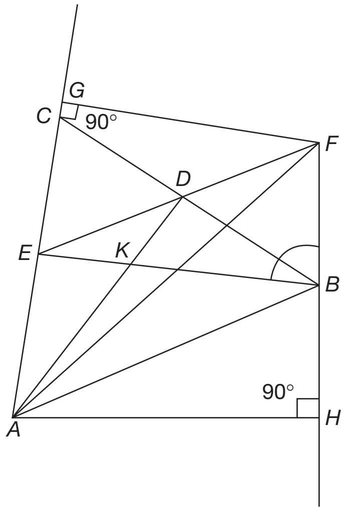
$DB = BF$ $\angle ADB = \angle EBF$)

Jakob Steiner

18 Mar 1796–1 Apr 1863

Nationality: Swiss

Problem was proposed by Daniel Christian Ludolph Lehmus (Jul 3 1780–Jan 18 1863) a German mathematician and it was solved by Steiner.



(ii) $\angle AEF = \angle AEK + \angle KEF = \angle AEK + \angle EAK$

$= \angle AKB$

$= \angle KDB + \angle KBD$

$= \angle EBF + \angle EBA$

$= \angle ABF = \angle FEG = \angle ABH$

(iii) $\Delta ABH \equiv A\ FEG$ (By steps (i), (ii) and construction and AAS)

$\therefore \quad AH = FG$

$BH = EG$

(iv) $\Delta AFG \equiv \Delta FAH$ (Hypotenuse and length)

$\therefore \quad AG = FH$

(v) $AE = AG - GE$

$= FH - HB$

$= FB$

$= FD$ (by step (i))

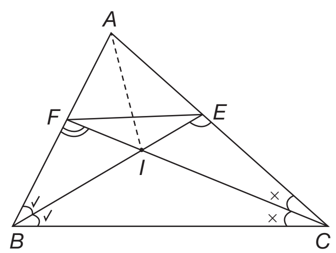
(vi) $\Delta ABE = \Delta BAD$ (SSS)

$\therefore \quad \angle EAB = \angle DBA$

$\Rightarrow \quad \angle A = \angle B \quad \Rightarrow \quad CB = CA.$

Aliter:

Let ABC be a triangle, in which, angle bisectors of B and C are equal, *i.e.*, $BE = CF$.



Then $\dfrac{CF}{\sin B} = \dfrac{BC}{\sin BFC} = \dfrac{a}{\sin\left(B + \dfrac{C}{2}\right)}$ from ΔBFC

and $\dfrac{BE}{\sin C} = \dfrac{BC}{\sin BEC} = \dfrac{a}{\sin\left(C + \dfrac{B}{2}\right)}$ from ΔBEC

As $CF = BE$,

$$\frac{\sin\left(B + \frac{C}{2}\right)}{\sin B} = \frac{\sin\left(C + \frac{B}{2}\right)}{\sin C}$$

or $$\frac{\sin B}{\sin C} = \frac{\sin\left(B + \frac{C}{2}\right)}{\sin\left(C + \frac{B}{2}\right)}$$

or $$\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}} = \frac{2\sin\frac{C}{2}\cdot\sin\left(B + \frac{C}{2}\right)}{2\sin\frac{B}{2}\cdot\sin\left(C + \frac{B}{2}\right)}$$ (Expand $\sin B$ and $\sin C$)

i.e., $$\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}} = \frac{\cos B - \cos(B + C)}{\cos C - \cos(B + C)} = \frac{\cos B + \cos A}{\cos C + \cos A}$$

$$i.e., \quad \frac{\cos\frac{B}{2}-\cos\frac{C}{2}}{\cos\frac{C}{2}} = \frac{\cos B-\cos C}{\cos C+\cos A} = \frac{2\cos^2\frac{B}{2}-1-2\cos^2\frac{C}{2}-1}{\cos C+\cos A}$$

$$i.e., \quad \frac{\cos\frac{B}{2}-\cos\frac{C}{2}}{\cos\frac{C}{2}} = \frac{2\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)\left(\cos\frac{B}{2}+\cos\frac{C}{2}\right)}{\cos C+\cos A}$$

$$i.e., \quad \cos\frac{B}{2}-\cos\frac{C}{2}=0 \quad \text{or} \quad \angle B=\angle C$$

or, $\cos\frac{B}{2}+\cos\frac{C}{2}=0$, which is impossible as $\frac{A}{2}, \frac{B}{2}, \frac{C}{2}$ are all acute and hence positive.

This proves the claim.

Problem 5 *Let, r be the radius of the inscribed circle of a right-angled ΔABC. Show that r is less than half of either leg and less than one fourth of the hypotense.*

Solution: Draw the diameters through the points of contact of circle with the sides of the triangle.

$$GG' < CD < AC$$

where CD is the altitude of the right ΔABC, with $\angle C = 90°$.

$$\Rightarrow \quad 2r < AC$$

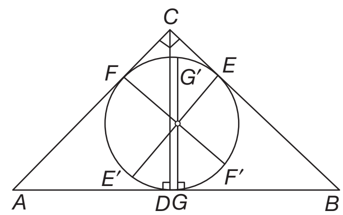
$$\Rightarrow \quad r < \frac{AC}{2}$$

Again, $GG' < CD < CB$

$$\Rightarrow \quad 2r < CB$$

$$\Rightarrow \quad r < \frac{CB}{2}$$

Now, CD is less than half of a chord of the circumcircle of the right ΔACB ($\angle C = 90°$).



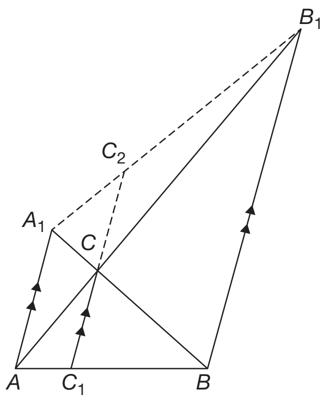
$$\therefore \quad CD \le \frac{AB}{2}$$

$$\Rightarrow \quad GG' < CD \le \frac{AB}{2}$$

$$\text{or} \quad 2r < \frac{AB}{2} \qquad i.e., r < \frac{AB}{4}.$$

Problem 6 *Let C_1, be any point on side AB of a ΔABC. Draw C_1C meeting AB at C_1. The lines through A and B parallel to CC_1 meet BC produced and AC produced at A_1 and B_1, respectively. Prove that*

$$\frac{1}{AA_1} + \frac{1}{BB_1} = \frac{1}{CC_1}$$



Solution: AA_1, BB_1 and CC_1 are parallel line segments and hence,

$$\frac{CC_1}{A_1A} = \frac{C_1B}{AB} \tag{1}$$

Also
$$\frac{CC_1}{B_1B} = \frac{AC_1}{AB} \tag{2}$$

Adding Eqs. (1) and (2), we have

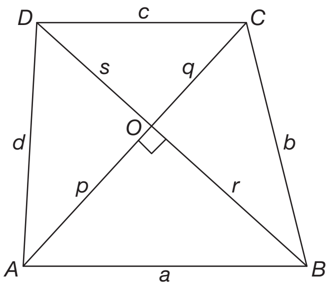
$$\frac{CC_1}{A_1A} + \frac{CC_1}{B_1B} = \frac{C_1B + AC_1}{AB} = \frac{AB}{AB} = 1 \tag{3}$$

Dividing Eq. (3) by CC_1, we get

$$\frac{1}{A_1A} + \frac{1}{B_1B} = \frac{1}{CC_1}$$

Note: That ABB_1A_1 is a trapezium and C_1C_2 is the harmonic mean of the parallel sides AA_1BB_1, and C_1C_2 is parallel to the parallel sides.

Problem 7 *Prove that the diagonals of a (convex) quadrilateral are perpendicular, if and only if the sum of the squares of one pair of opposite sides equals that of the other.*



Solution: Let a, b, c and d be the measures of the sides AB, BC, CD, and DA of the quadrilateral. The diagonals intersect at O. Let $OA = p$, $OB = r$, $OC = q$ and $OD = s$.

If AC is not perpendicular to BD, let $\angle AOB$ be obtuse.

Then, by the extension of the Pythagoras theorem,

$a^2 > p^2 + r^2$; $b^2 < r^2 + q^2$

$c^2 > s^2 + q^2$; $d^2 < p^2 + s^2$

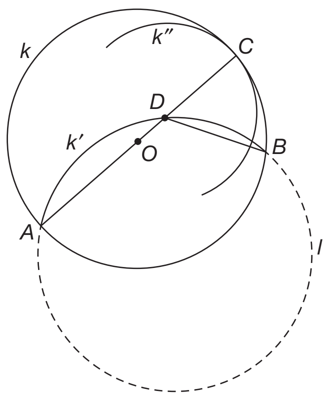
$a^2 + c^2 > p^2 + r^2 + s^2 + q^2 > b^2 + d^2$

Thus, $a^2 + c^2 > b^2 + d^2$

which is a contradiction as it is given that $a^2 + c^2 = b^2 + d^2$ and $\angle AOB \not> 90°$

If AC is perpendicular to BD, then

$$a^2 = p^2 + r^2$$
$$c^2 = s^2 + q^2$$
$$a^2 + c^2 = p^2 + q^2 + r^2 + s^2 = (p^2 + s^2) + (q^2 + r^2)$$
$$= d^2 + b^2.$$



Problem 8 *Let A and B be the points on a circle k. Suppose that an arc k′ of another circle l connects A with B and divides the area inside the circle k into two equal parts. Prove that arc k′ is longer than the diameter of k.*

Solution: As arc k' bisects the area of the circle k, so k' cannot entirely lie on one side of any diameter of circle k.

Hence, every diameter of k intersects k'. Let, AC be one such diameter and k' intersects AC at D, Now the centre O of the circle k lies inside the circle l, hence the radius AO of circle k lies inside l and now, D lies on the radius OC.

Length or arc $ADB > AD + DB$.

As we have to prove that arc $ADB > AC = AD + DC$, we should show that $DB > DC$.

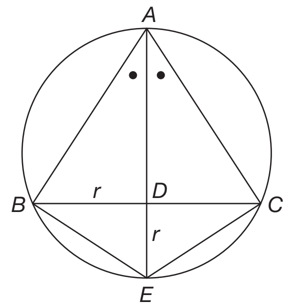
Now the circle k'' with centre D and radius DC is a circle touching k, internally, and B lies outside this circle k'', so the radius of k'' is less than DB, *i.e.*, $DC < DB$ or $DB > DC$.

$\Rightarrow$ arc $ADB > AD + DB > AD + DC = AC$

$\Rightarrow$ arc $ADB >$ the diameter of k.

Note: O lies inside circle k' as every diameter of k meets the circle k' (*i.e.*, arc AB) as k' bisects area in k.

Problem 9 *ABC is a triangle, the bisector of ∠A, meets BC in D. Show that AD is less than the geometric mean of AB and AC.*



Solution: Draw the circumcircle of ΔABC and let the bisector AD of $\angle A$ meet the circumcircle again at E.

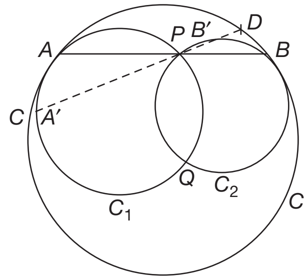
ΔABD is similar to ΔAEC (AA similarity)

$$\therefore \frac{AD}{AC} = \frac{AB}{AE}$$

$$\Rightarrow AB \times AC = AD \cdot AE > AD^2 \quad (\because AE > AD)$$

$\Rightarrow$ $AD < \sqrt{AB \times AC}$ which was to be proved.

Problem 10 *Two given circles intersect in two points P and Q. Show that how to construct a segment AB passing through P and terminating on the two circles such that AP · PB is a maximum.*



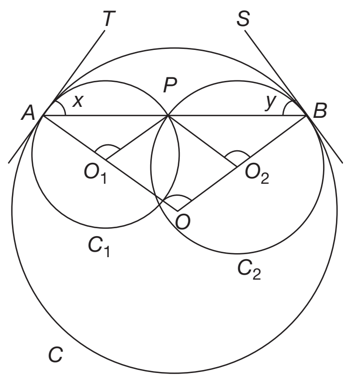
Solution: Let, C_1, C_2 be two circles. We first show that if APB is a straight line such that there is a circle C touching C_1 at A and C_2 at B, then AB is the segment giving the required maximum.
Let, $A'P$ and PB' be any other chords so that $A'PB'$ may be collinear and the extension of these chords meet the circle C at C and D.

$$CP \cdot PD = AP \cdot PB > A'P \times PB'$$

$\therefore$ $AP \cdot PB$ is maximum.

Now, we need to construct a chord APB. For this, we need to construct a circle C touching C_1 and C_2 at points A and B so that APB are collinear. Let us find the properties of the points A and B.

Let, O be the centre of circle C and O_1, and O_2 be the centre of circles C_1, and C_2. Now, C and C_1 touches at A.



$\therefore$ AO_1O are collinear. Similarly, BO_2O are collinear. Let, AT, BS be the common tangents to circles C and C_1, and C and C_2 respectively.

Let, $\angle PAT = x$ and $\angle PBS = y$ since AT is tangent to circle C.

$$\angle PAT = x = \frac{1}{2}\angle AOB \qquad \text{(Angle in the alternate segment theorem)}$$

Since, BS is tangent to circle C.

$$\angle PBS = y = \frac{1}{2}\angle AOB$$

$\therefore$ $x = y$. Since, AT is tangent to circle C_1, we get

$$\angle PAT = x = \frac{1}{2}\angle AO_1P$$

Similarly, since BS is tangent to circle C_2, we get

$$\angle PBS = y = \frac{1}{2}\angle BO_2P$$

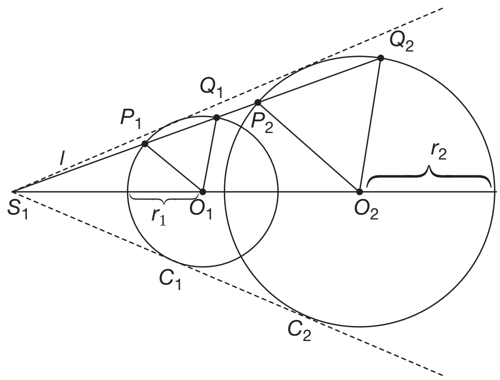
$$\therefore \quad \angle AO_1P = \angle AOB = \angle BO_2P$$

$$\therefore \quad \Delta AO_1P \sim \Delta PO_2B$$

$$\therefore \quad \frac{AP}{PB} = \frac{AO_1}{PO_2} = \frac{r_1}{r_2}.$$

Therefore, the line segment AB must be such that P divides AB internally in the ratio $r_1{:}r_2$. Further, $PO_2 \| OO_1$ and $PO_1 \| OO_2$.

So, join PO_1 and PO_2. Through O_1 draw a line parallel to PO_2 to meet circle C_1 in A. Through O_2 draw a line parallel to PO_1 to meet the circle C_2 in B. Now, these two parallel lines drawn meet at O. If we draw a circle with O as centre and radius $OA = OB$, then the circle touches C_1 at A and C_2 at B. By retracing the arguments, we can prove that APB is collinear and AB is the required chord.

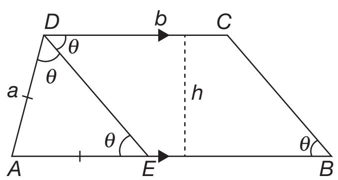


Note: In the previous problem, the line AB and O_1O_2 meets in a point S_1. Point S_1 divides O_1O_2 externally in the ratio $r_1{:}r_2$. The point S_1 is called the external centre of similitude of circles C_1 and C_2. If we draw a line l through S_1 meeting C_1 in P_1, Q_1, and C_2 in P_2, Q_2, then $O_1P_1 \| O_2P_2$ and $O_1Q_1 \| O_2Q_2$.

Moreover, the direct common tangents to the two circles C_1 and C_2 meet at S_r.

Problem 11 *In a trapezium ABCD, AB $\|$ CD, m$\angle D$ = 2m$\angle B$. If AD = a, CD = b, and the distance between AB and CD is h, give an expression for the area of the trapezium.*

Solution: Let the bisector of $\angle D$ meet AB at E. Since,



$CD \| AB$, $\angle EDC = \angle DEA$ (Alternate interior angles) (1)

As $\angle D = 2\angle B$

$$\Rightarrow \quad \angle DEA = \frac{1}{2}\angle ADC = \angle B \qquad (2)$$

$\Rightarrow$ $DE \| BC$ (Corresponding angles are equal)

Hence, $EBCD$ is a parallelogram and hence, $EB = b$ units.

By Eq. (1), in ΔADE, $\angle D = \angle E = \theta$ and hence,

$AE = AD = a$

So, $AB = AE + EB = (a + b)$ units

So, the area of the trapezium is $= \frac{1}{2}h(a+2b)$ sq. units

$$= \frac{1}{2}h(a+2b) \text{ sq.units.}$$

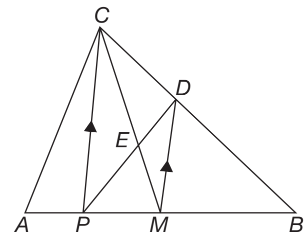
Problem 12 *Let, M be the mid-point of the side AB of ΔABC. Let, P be a point on AB between A and M and let MD be drawn parallel to PC, intersecting BC at D. If the ratio of the area of ΔBPD to that of ΔABC is denoted by r, then examine which of the following is true?*

(i) $\frac{1}{2} < r < 1$ *depending upon the position of P.*

(ii) $r = \frac{1}{2}$

(iii) $\frac{1}{3} < r < \frac{2}{3}$ *depending upon the position of P.*

Solution: Join PD and MC and let them intersect at E.



Area of ΔBPD = Area of ΔBMD + Area of ΔMDP

= Area of ΔBMD + Area of ΔMDC

($\Delta MDP = \Delta MDC$ as both the triangles lie on the same base MD and between the same parallels PC and MD)

= Area of ΔCMB

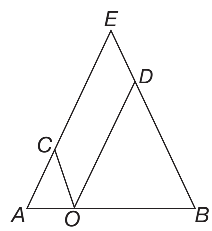
$= \frac{1}{2}$ Area of ΔABC (as M is the mid-point of AB)

Thus, $\frac{\text{Area of } \Delta BPD}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2}\text{Area of } \Delta ABC}{\text{Area of } \Delta ABC} = \frac{1}{2}$

Thus, $r = \frac{1}{2}$ (independent of P)

Problem 13 *Let O be an arbitrary point situated in the segment AB. Construct equilateral ΔAOC and ΔBOD. Let, E be the point of intersection of AC and BD. Show that CODE is a parallelogram. When will it be a rhombus?*

Solution: In the figure ΔAOC and ΔBOD being equilateral $\angle COD = 180° - (\angle COA + \angle BOD) = 180° - (60° + 60°) = 60°$.



The exterior $\angle ODE$ of $\Delta OBD = 60° + 60° = 120°$. Again, the exterior $\angle OCE$ of $\Delta OCA = 60° + 60° = 120°$.

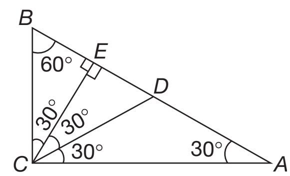
Therefore, the remaining

$$\angle CED = 360° - (120° + 120° + 60°) = 60°$$

In quadrilateral $OCED$, opposite angles are equal, implying that the opposite sides are parallel. Thus, it is a parallelogram. In this parallelogram, if the adjacent sides $OC = OD$ (*i.e.*, all sides are equal), then it becomes a rhombus. For this, we should have $AO = OC = OD = OB$, *i.e.*, $AO = OB$ or O should be the mid-point of the segment AB (Also note that ΔAEB is also equilateral).

Problem 14 *ABC is a triangle, $\angle A = 30°$, $\angle B = 60°$ and AB = 10 cm. Find the length of the shorter trisector of $\angle C$.*

Solution: In Fig. 3.33, CE and CD are the trisectors of $\angle C$. $\angle CED = 90°$ and hence, $CE < CD$. (In ΔCED, CD is the hypotenuse.) Thus, it is required to find the length of CE.



$AB = 10$ cm, $\angle B = 60°$, $\angle A = 30°$

$\Rightarrow \quad BC = 5$ cm (and $AC = 5\sqrt{3}\ CM$)

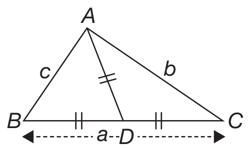
Again, in ΔBCE,

$\angle CEB = 90°$, $BC = 5$ cm, $CE = 5\cos 30°$

$$\Rightarrow \quad CE = \frac{5}{2}\sqrt{3} \text{ cm} \qquad \left(BE = \frac{5}{2}\right)$$

[CD can also be calculated from the right-angled ΔECD: $\angle CED = 90°$, $\angle ECD = 30°$, $CE = \frac{5}{2}\sqrt{3}$, $CD = 5$ cm $= DA$. Thus, $BE = ED = \frac{5}{2}$ cm., DA = 5 cm, $CE = \frac{5}{2}\sqrt{3}$ cm, $CD = 5$ cm and clearly, $\frac{5}{2}\sqrt{3} < 5$ and the shorter trisector has a length of $\frac{5}{2}\sqrt{3}$ cm.]

Problem 15 *In ΔABC, in the usual notation, the area is $\frac{1}{2}bc$ sq.units. AD is the median to BC. Prove that $\angle ABC = \frac{1}{2}\angle ADC$.*



Solution:

$$\Delta = \frac{1}{2}bc \sin A = \frac{1}{2}bc$$

$\Rightarrow \quad \sin A = 1$

$\Rightarrow \quad \angle A = 90°$.

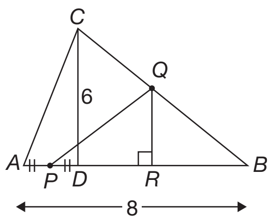
Since AD is the median and $\angle A = 90°$, D, the mid-point of BC is the centre of the circumcircle of ΔABC.

So, $AD = BD = DC$

$$\angle ABC = \frac{1}{2}\angle ADC$$

(Angle subtended by AC at the circumference $= \frac{1}{2}$ angle subtended by AC at the centre.)

Problem 16 *Let, ABC be an acute angled triangle and CD be the altitude through C. If $AB = 8$ units and $CD = 6$ units, find the distance between the mid-points of AD and BC.*



Solution: Let P, be the mid-point of AD and Q be the mid-point of BC.

Draw QR perpendicular to AB.

In ΔCDB and ΔQRB, CD and QR are both perpendicular to AB and hence, parallel.

Since, Q is the mid-point of CB, R is the mid-point of DB.

(by the basic proportionality theorem, $\Delta CDB = \Delta\, QRB$)

$$\therefore \quad QR = \frac{1}{2}CD = \frac{1}{2}\times 6 = 3 \text{ units}$$

$$\therefore \quad PR = PD + DR = \frac{1}{2}(AD + DB) = \frac{1}{2}\times 8 = 4 \text{ units}$$

So, in the right-angled ΔPQR

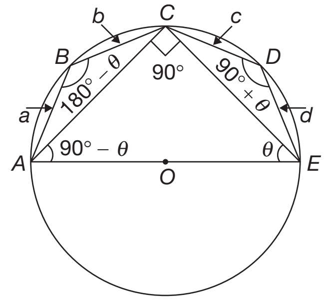
$$PQ = \sqrt{4^2 + 3^2} = 5.$$

Problem 17 *ABCDE is a convex pentagon inscribed in a circle of radius 1 units with AE as diameter. It $AB = a$, $BC = b$, $CD = c$, $DE = d$, then prove that*

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$

Solution: Since, AE is the diameter $\angle ACE = 90°$ and $AC^2 + CE^2 = AE^2 = 2^2 = 4$. By cosine formula (for ΔABC)

$$AC^2 = a^2 + b^2 - 2ab\cos(180° - \theta)$$
$$= a^2 + b^2 + 2ab\cos\theta$$



Similarly, in ΔCED

$CE^2 = c^2 + d^2 - 2cd\cos(90° + \theta)$

$\quad = c^2 + d^2 + 2cd\sin\theta$

$\therefore\ AC^2 + CE^2 = a^2 + b^2 + c^2 + d^2\ 2ab\cos\theta + 2cd\sin\theta$

In ΔACE, $\dfrac{AC}{AE} = \sin\theta$

$\Rightarrow\quad AC = 2\sin\theta > b \qquad (AE = 2) \qquad (1)$

and $\dfrac{CE}{AE} = \cos\theta \qquad (AE = 2)$

$\Rightarrow\quad CE = 2\cos\theta > c \qquad (2)$

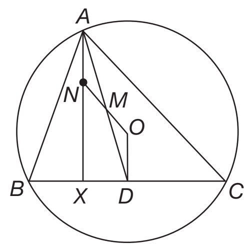
(Because, in ΔABC and ΔCDE, $\angle B$ and $\angle D$ are obtuse angles. Here, AC is the greatest side of ΔABC, and CE is the greatest side of ΔCDE)

$AC^2 + CE^2 = a^2 + b^2 + c^2 + d^2 + 2ab\cos\theta + 2cd\cos\theta = 4$

$\Rightarrow\quad a^2 + b^2 + c^2 + d^2 + ab\cdot 2\cos\theta + cd\cdot 2\sin\theta = 4$

$\Rightarrow\quad a^2 + b^2 + c^2 + d^2 + abc + bcd < 4 \qquad$ (by Eqs. (1) and (2))

Problem 18 *O is the circumcentre of ΔABC and M is the mid-point of the median through A. Join OM and produce it to N so that $OM = MN$. Show that N lies on the altitude through A.*



Solution: Let AD be the median through A, and M be the mid-point of AD. Join OD.

Since, D is the mid-point of BC and O is the circumcentre, OD is perpendicular to BC.

In ΔDMO and ΔAMN,

$DM = AM \qquad$ (M is the mid-point of AD)

$OM = NM \qquad$ (Given)

$\angle DMO = \angle AMN \qquad$ (Vertically opposite angles)

So, the triangles are congruent.

$\angle MDO = \angle MAN \qquad$ (Corresponding angles of congruent triangles)

So, $AN \parallel OD \qquad$ ($\angle MDO$ and $\angle MAN$ are alternate interior angles and are equal)

But, OD is perpendicular to BC and hence, AN produced is perpendicular to BC, *i.e.*, N lies on the perpendicular through A to BC, *i.e.*, N lies on the altitude through A).

Problem 19 *Prove in ΔABC, if one angle is equal to $120°$, the triangle formed by the feet of the angle bisectors is right-angled.*

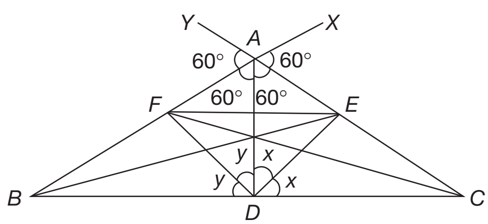
Solution: Produce

$\overrightarrow{BA}$ to X;

$\angle CAX = 180° - 120° = 60°$

Now, AC bisects $\angle DAX$.

So, in $\Delta ABD, \overrightarrow{BE}$ is



the internal bisector of $\angle ABD$ and $\overrightarrow{AE}$ is the bisector of the exterior $\angle DAX$ of ΔBAD and so, E is the centre of excribed circle of ΔABD, opposite to the vertex B.

So, DE is the bisector of the exterior $\angle ADC$ of ΔABD

$$\angle ADE = \angle CDE$$

Similarly, $\overrightarrow{AB}$ is the bisector of the external $\angle DAY$ of ΔADC and $\overrightarrow{CF}$ is the internal bisector of $\angle C$. So, F is the centre of the excribed circle of ΔADC, opposite to vertex C.

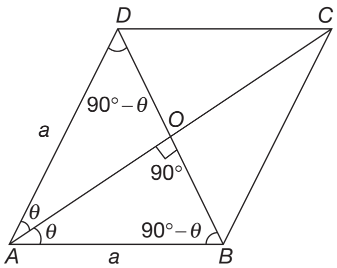
So, DF is the bisector of the exterior $\angle ADB$ of ΔADC

So, $\angle ADF = \angle FDB$

$$\therefore\ \angle FDE = x + y = \frac{1}{2}(2x + 2y) = \frac{1}{2}\times 180° = 90°$$

So, ΔFDE is a right-angled triangle at D.

Problem 20 *A rhombus has half the area of the square with the same side length. Find the ratio of the longer diagonal to that of the shorter one.*



Solution: If a is the side of the rhombus, then area of the rhombus is $\frac{1}{2}a^2 \sin 2\theta \times 2$.

But, by hypothesis, this area is equal to $\frac{1}{2}a^2$,

i.e., $\frac{1}{2}a^2 = a^2 \sin 2\theta$

$\Rightarrow\ 2\theta = 30°$ or $150°$

$\Rightarrow\ \theta = 15°$ or $75°$.

[If the acute angle of the rhombus is 30°, the other angle which is obtuse is 150°.]

By sine formula, $\dfrac{BD}{\sin 2\theta} = \dfrac{AB}{\sin(90 - \theta)}$ (In ΔABD)

$$\Rightarrow\ BD = \frac{a \times 2\sin\theta\cos\theta}{\cos\theta} = 2a\sin\theta$$

Again, $\dfrac{AC}{\sin(180 - 2\theta)} = \dfrac{a}{\sin\theta}$ (In ΔABC)

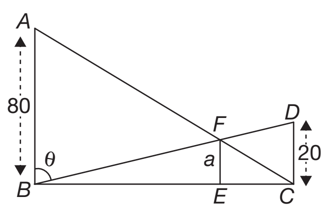
$AC : BD = \cos\theta : \sin\theta$

[If $\theta = 15°$, then $AC > BD$ and if $\theta = 75°$, $BD > AC$]

$AC : BD = \cos 15° : \sin 15° = \cot 15°$.

Problem 21 *Two vertical poles* 20 *m and* 80 *m high stand apart on a horizontal plane. The height of the point of intersection of the lines joining the top of each pole to the foot of the other is in metres. Find a.*

ΔABF and ΔCDF are similar



Solution: $\angle AFB = \angle DFC$ (Vertically opposite angles)

$\angle FAB = \angle FCD$ (Alternate interior angles)

by angle-angle similarity triangles are similar.

$$\therefore\ \frac{AF}{CF} = \frac{AB}{CD} = \frac{80}{20} = \frac{4}{1}$$

$$\Rightarrow\ \frac{AF}{CF} = \frac{4}{1}$$

$$\Rightarrow \quad \frac{AF+FC}{FC} = \frac{4+1}{1} = \frac{5}{1}$$

$$\Rightarrow \quad \frac{AC}{FC} = \frac{5}{1}$$

ΔABC and ΔFEC are similar $\quad (\because \; AB$ and FE are $\|)$

$$\frac{AB}{FE} = \frac{AC}{FC}$$

$$\Rightarrow \quad \frac{80}{FE} = \frac{5}{1}$$

or $5FE = 80$

$\Rightarrow \quad FE = 16$

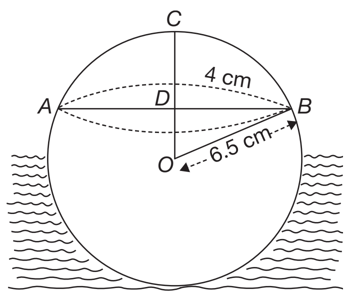
Thus, $a = 16$ metres.

Aliter: Using result of Problem 6, we get

$$\frac{1}{a} = \frac{1}{20} + \frac{1}{80} \Rightarrow a = 16 \text{ metres.}$$

Problem 22 *A ball of diameter* 13 *cm is floating so that the top of the ball is* 4 *cm above the smooth surface of the pond. What is the circumference in centimetres of the circle formed by the contact of the water surface with the ball.*

Solution: We should find the circumference of the circle on AB as diameter.



$CD = 4$ cm

$$OC = OB = \frac{13}{2} = 6.5\,\text{cm}$$

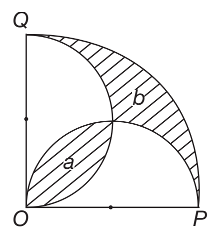
So, $OD = 6.5$ cm $-$ 4 cm $= 2.5$ cm

$$DB = \sqrt{(6.5)^2 - (2.5)^2} = 6\,\text{cm}$$

So, the circumference of the circle is $2\pi \times 6$ cm $= 12\pi$ cm.

Problem 23 *OPQ is a quadrant of a circle, and semicircles are drawn on OP and OQ. Show that the shaded areas a and b are equal.*

Solution: Area of the quadrant = areas of the two semicircles + $b - a$ [Since the sum of the areas of the two semicircles include the area shaded 'a' twice)



$$\text{The area of quadrant } = \frac{1}{4}\pi r^2$$

$$i.e., \; \frac{1}{4}\pi r^2 = \frac{1}{2}\pi\left(\frac{r}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{r}{2}\right)^2 + b - a$$

$$\Rightarrow \quad \frac{1}{4}\pi r^2 = \frac{1}{4}\pi r^2 + b + a$$

$\Rightarrow \quad b - a = 0$

$\Rightarrow \quad a = b.$

Problem 24 *ABC is a right-angled triangle with $\angle B = 90°$. M is the mid-point of AC and $BM = \sqrt{117}$ cm. The sum of the lengths of sides AB and BC is 30 cm. Find the area of the triangle.*

Solution: M is the centre of the circum-circle of the right angled ΔABC and hence,

$$AM = CM = BM = \sqrt{117}\text{ cm.}$$

$$AC^2 = a^2 + c^2 = (a+c)^2 - 2ac$$

$= 900 - 2ac$

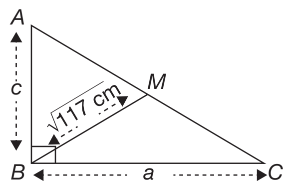
But, $AC = 2\sqrt{117}$

($\because$ $AC = 2AM = 2MC = 2BM$)

So, $AC^2 = 4 \times 117 = 900 - 2ac$

$\Rightarrow$ $2ac = 900 - 4 \times 117 = 900 - 468 = 432$ sq. cm.

$\Rightarrow$ $\dfrac{1}{2}ac = \dfrac{432}{4} = 108\,\text{cm}^2$



Problem 25 *In a ΔABC, the incircle touches the sides BC, CA and AB at D, E and F respectively. If the radius of the incircle is 4 units and, if BD, CE and AF are consecutive integers, find the lengths of the sides of the triangle.*

Solution: The inradius of the triangle is given by the formula

$$r = \frac{\Delta}{s}$$

where Δ is the area of the triangle, s is the semi-perimeter.

Lets take BD, CE and AF are $n - 1$, $n + 1$, n, so that the sides BC, CA and AB may be $2n$, $(2n + 1)$ and $(2n - 1)$

Area of the triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{3n(n)(n-1)(n+1)}$

And hence, $\dfrac{\Delta}{s} = \dfrac{\sqrt{3n^2(n^2-1)}}{3n} = 4$

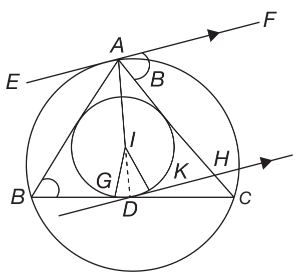
$\Rightarrow$ $144n^2 = 3n^2(n^2 - 1)$

$\Rightarrow$ $(n^2 - 1) = 48$

$\Rightarrow$ $n = 7$ (because -7 is not applicable.)

Therefore, the sides of the triangle are $(2 \times 7 - 1)$, (2×7) and $(2 \times 7 + 1)$ or 13 cm, 14 cm, 15 cm or 15 cm, 14 cm and 13 cm

Problem 26 *AD is the internal bisector of $\angle A$ in ΔABC. Show that the line through D, drawn parallel to the tangent to the circumcircle at A, touches the inscribed circle.*



Solution: Let, EF be the tangent to the circumcircle through A. AD is the bisector of $\angle A$ and DH is parallel to EF meeting AC at H.

Let the incircle touch the side BC at G.

$\angle ADH = 180° - \angle DAF$

$= 180° - \dfrac{A}{2} - B$

$= C + \dfrac{A}{2}$

(Since, $\angle HAF = \angle ABC$, being angles in alternate segments.)

If the incircle touches BC at G, then

$\angle ADG = \angle DAC + \angle ACD$

(Exterior angle = Sum of the remote interior angles)

$= \frac{A}{2} + C$

i.e., $\angle IDG = \angle IDH$ (1)

Let the tangents through D to the incircle meet it at G and K. Where G and K lies on the opposite sides of ID (Since, the incircle touches the side BC at G, here GD is one tangent from D, the other is DK).

So, $\angle IDG = \angle IDK$

But, $\angle IDG = \angle IDH$ (from Eq. (1))

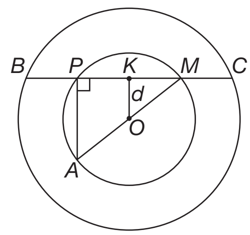
Therefore, $\angle IDK = \angle IDH$

But, both K and H are on the same side of ID and hence, K is a point of DH or DH is a tangent to the through D.

Problem 27 *Given two concentric circles of radii R and r. From a point P on the smaller circle, a straight line is drawn to intersect the larger circle at B and C. The perpendicular to BC at P intersects the smaller circle at A. Show that*

$$PA^2 + PB^2 + PC^2 = 2(R^2 + r^2).$$

Solution: Let, BC meet the smaller circle at P and M.



Through P, draw PA *perpendicular* to BC meeting the smaller circle at A.

Since, $\angle APM = 90°$,

AM is the diameter of the smaller circle,

or, $AM = 2r$

Let OK be the perpendicular from O to BC.

$OK = d$ units; $BK = KC$; $PK = KM$ (1)

Now, $PA^2 + PB^2 + PC^2$

$$= PA^2\ (PC - PB)^2 + 2PC \cdot PB$$

$$= PA^2 + (PC - MC)^2 + 2PC \cdot PB \quad \text{(by Eq. (1))}$$

$$= PA^2 + PM^2 + 2PC \cdot PB$$

$$= AM^2 + 2PC \cdot PB$$

$$= 4r^2 + 2PC \cdot PB$$

Now, $R^2 = OB^2 = OK^2 + BK^2 = d^2 + \frac{1}{4}BC^2$

$$r^2 = OM^2 = OK^2 + KM^2 = d^2 + \frac{1}{4}PM^2$$

$$\therefore\ R^2 - r^2 = \frac{1}{4}(BC^2 - PM^2) = \frac{1}{4}(BC + PM)(BC - PM)$$

$$= \frac{1}{4}(2BK + 2PK)(2BK - 2PK)$$

$= (BK + PK)(BK - PK)$

$= (CK + PK)(BP)$

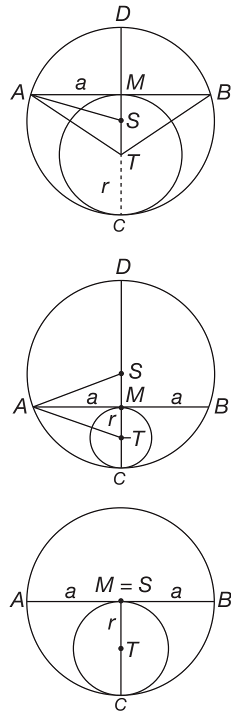
$= PC \cdot BP$

Or $2(R^2 - r^2) = 2PC \cdot PB$

$\therefore\ PA^2 + PB^2 + PC^2 = 4r^2 + 2PC \cdot PB$

$= 4r^2 + 2(R^2 - r^2)$

$= 2R^2 + 2r^2 = 2(R^2 + r^2)$.



Problem 28 *A circle of radius r touches a straight line at a point M. Two points A and B are chosen on this line on opposite sides of M, such that MA = MB = a. Find the radius of the circle passing through A and B and touching the given circles, respectively.*

Solution: Let, T and S be the centres of the smaller and the larger circles, respectively.

TS = distance between the centres of the two circles

$= SC - TC$

$= (R - r)$

In the first figure,

$SM = TM - TS$

$= r - (R - r)$

$= (2r - R)$

In the second figure,

$SM = SC - MC = (R - 2r)$

The radius of the larger circle $SA = R$, and in the right $\angle d\Delta SAM$

$R^2 = SA^2 = a^2 + SM^2 = a^2 + (R - 2r)^2 = a^2 + R^2 + 4r^2 - 4Rr$

$\Rightarrow \quad 4Rr = a^2 + 4r^2$

$$\Rightarrow \quad R = \frac{a^2 + 4r^2}{4r}$$

Note that $SM^2 = (R - 2r)^2 = (2r - R)^2$ and hence, we get the same value for R.

In the third figure, there is yet another possibility. The larger circle may have AB as diameter still touching the smaller circle. In this special case, $R = a = 2r$. Since, M is the centre of the larger circle.

Problem 29 *A tangent at P to a circle with centre O, cuts two other parallel tangents AC and BD at A and B. The parallel tangents touch the circle at C and D. Show that AC · BD is a constant.*

Solution: $AC \parallel BD$.

OC and OD are the radii through the point of contact of the tangents. If OQ is a radius parallel to AC and BD,

$\therefore$ C, O and D are collinear.

Join AO and BO.

In ΔACO and ΔAPO are congruent hypotenuse and leg congruence in right-angled triangles.

$\therefore \angle AOP = \angle AOC$

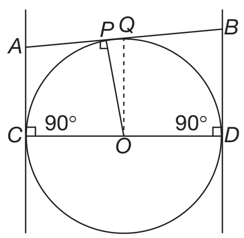
Similarly, $\angle BOD = \angle POB$.

But, COD is a straight line.

Thus, $\angle AOB = \angle AOP + \angle POB$

$$= \frac{1}{2}(\angle COA + \angle AOP + \angle POB + \angle BOD)$$

$$= \frac{1}{2} \times 180° = 90°$$



Again, OP is the radius through P, the point of contact of the tangent AB and hence, OP is perpendicular to AB.

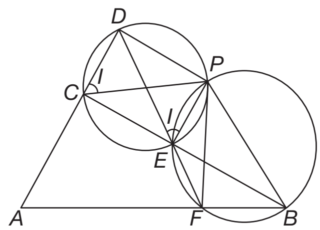
Thus, $AP \cdot PB = OP^2 = r^2$

But, $AP = AC, \quad PB = BD$

$\therefore AC \cdot BD = AP \cdot PB = r^2$

which is a constant for any given circle.

Problem 30 *AB, BC, AD, and DF are four straight lines as shown in the figure and their intersections A, B, C, D, E, F form four triangles, ΔADF, ΔCDE, ΔEBF and ΔABC. Show that the circumcircle of these four triangles intersect at the same point.*



Solution: Without loss of generality let us take that the circumcircles of ΔDCE and ΔEFB meet at P.

We should show that the circumcircles of ΔADF and ΔABC, pass through P (*i.e.*, $ADPF$ and $ABPC$ are cyclic quadrilaterals, and $DCEP$ and $FBPE$ are cyclic).

$\angle DCP = \angle DEP$

(In the circle through $DCEP$, angles fall on the same segment.) $= \angle FBP$ ($FBPE$ is a cyclic quadrilateral and exterior angle = interior opposite angle).

This implies, in the quadrilateral $ABPC$, exterior $\angle DCP$ = interior opp. $\angle ABP$.

So, $ABPC$ is a cyclic quadrilateral or the circumcircle of ABC passes through P.

Again, considering quadrilateral $ADPF$

$\angle ADP = \angle CDP = \angle PEB$ ($CDPE$ is a cyclic quadrilaterial and interior $\angle$ = exterior opposite angle) $= \angle PFB$ (in the circle through $PEFB$, PB subtends equal angles at E and F or angles on the same segment).

Thus, one interior $\angle$ of the quadrilateral $ADPF$ = exterior opposite angle of the same quadrilateral.

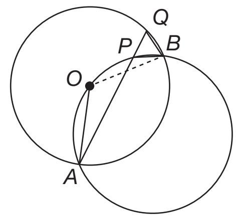
So, $ADPE$ is a cyclic quadrilateral and hence, the result.

Note: If you take any two triangles and consider their circumcircle, you will get the same result.

Problem 31 *A circle AOB, passing through the centre O of another circle, cuts the latter circle at A and B. A straight line APQ is drawn meeting the circle AOB in P and the other circle in Q.*

Prove that PB = PQ.

Solution:



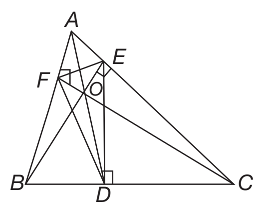
$$\begin{aligned}\angle PQB &= \angle AQB \\ &= \frac{1}{2}\angle AOB = \frac{1}{2}\angle APB \\ &= \frac{1}{2}(\angle PQB + \angle PBQ)\end{aligned}$$

$$\Rightarrow \quad \angle PQB - \frac{1}{2}\angle PQB = \frac{1}{2}\angle PBQ$$

$$\Rightarrow \quad \angle PQB = \angle PBQ$$

$$\Rightarrow \quad PQ = PB$$

Problem 32 *ABC is a triangle. AD, BE, and CF are the altitudes from the vertices A, B, and C, respectively. Show that the ΔDEC, ΔDBF, and ΔAEF are similar.*



Solution: ΔDEF is called the *pedal triangle.*

'O' is the orthocentre of the Δ.

Quadrilaterals $OECD$, $ODBF$, $OFAE$, $BCEF$, $ACDF$ and $ABDE$ are cyclic.

$\angle FCA = \angle OCE = \angle ODE = \angle ADE$ (From cyclic quadrilateral $OECD$)

($\because$ $\angle ADC = 90°$)

But, $\angle FCA = 90° - A$

$\therefore$ $\angle ADE = 90° - A$

$\therefore\ \angle EDC = \angle A \quad (\because\ \angle ADC = 90°)$

$\angle DCE = \angle C$ of ΔABC

$\therefore\ \angle DEC = \angle B$

Similarly, in ΔBFD,

$\angle FBD = \angle B$

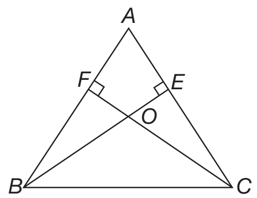
$\angle BFD = \angle C$ and $\angle FDB = \angle A$

and in ΔAFE

$\angle FAE = \angle A$, $\angle AFE = \angle C$, and hence, $\angle AEF = \angle B$.

Thus, ΔAFE, ΔBFD, and ΔCED are equiangular and hence, each being similar to ΔABC.

Problem 33 *Given the base and vertical angle of a triangle, find the locus of its orthocentre and incentre.*



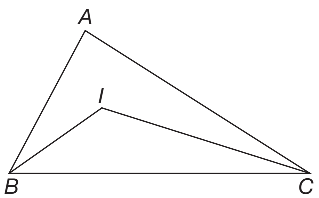
Solution: Let, ABC be a triangle on the given base BC having its vertical angle (a given angle).

Let, BE and CF be the altitudes from B and C meeting at O which is the orthocentre.

$\angle FOE = 180° - \angle A$

(As O, E, A, and F are concyclic.)

So, the locus of O is the circular arc on BC which contains an angle whose measure is $180° - A$.



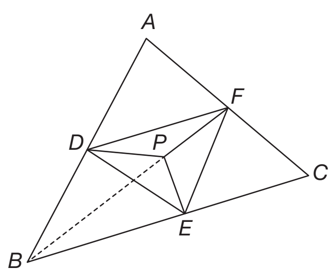
To find the locus of the incentre, let the bisectors of $\angle B$ and $\angle C$ meet at I.

$$\angle BIC = 180° - \frac{1}{2}(B + C)$$

$$= 180° - \frac{1}{2}(180° - A) = 90° + \frac{A}{2}.$$

So, the locus is the arc of the circle on BC containing an angle whose measure is $90° + \dfrac{A}{2}$.

Problem 34 *Let, ABC be an arbitrary acute-angled triangle. For any point, P, lying within this triangle, let D, E, F denote the feet of the perpendiculars from P onto the sides AB, BC, and CA, respectively. Determine the set of all possible positions of the point P for which the ΔDEF is isosceles. For which positions of P will the ΔDEF become equilateral?*



Solution: Suppose, $DE = DF$. Since, ΔPDB and ΔPEB are right, angled. P, D, B, E are concyclic, and PB is the diameter of the circle through these points.

$$\therefore\ \frac{DE}{\sin B} = PB$$

$$\left(\text{in any triangle, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \text{ by sine formula}\right)$$

$\Rightarrow\quad DE = PB \sin B$

Similarly, $DF = PA \sin A$

Since, $DE = DF$

$\Rightarrow\quad PB \sin B = PA \sin A$

$$\Rightarrow\quad \frac{PA}{PB} = \frac{\sin B}{\sin A} = \frac{b}{a}\ \left(i.e.,\ \frac{AC}{BC}\right)$$

This implies that P must lie on a circle, called Appolonius circle

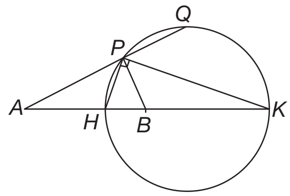
The Apollonius circle corresponding to the points A, B and the constant $\frac{b}{a}$.

Since ΔDEF is isosceles, whenever any two of the three sides are equal, the locus of P is the set of three Apollonius circles $\left(A, B, \frac{b}{a}\right)$. The ΔDEF is equilateral, if and only if, the point P lies on any two of these circles, *i.e.*, it will be the set of points common to the above circles taken two by two.

Notes:

1. The locus is only that portion of the Apollonius circles that lie inside A as it is expected that the point to be inside the Δ.
2. All the three circles are concurrent. The common point of concurrence lies inside ΔABC. Therefore, Only one point P exists, such that ΔDEF is equilateral.

Apollonius circle theorem:



A, B are two fixed points and P is a moving point, such that $\frac{PA}{PB}$ is a constant.

Then the locus of P is a circle. (*Prove*)

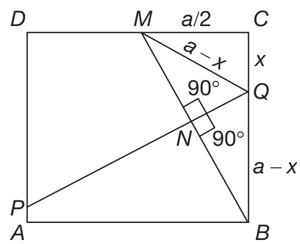
Proof: Produce AP to Q.

Divide AB, internally and externally in the ratio $\frac{PA}{PB} = \lambda$ at H and K, respectively.

$$\frac{AH}{HB} = \lambda \quad \Rightarrow \quad \frac{PA}{PB} = \frac{AK}{BK}.$$

So, PH and PK are the internal and external bisectors of $\angle APB$ and hence, $\angle HPK = 90°$. So, P lies on a circle on HK as diameter.

Problem 35 *A square sheet of paper ABCD is so folded that B falls on the mid-point, M, of CD. Prove that the crease will divide BC in the ratio* 5 : 3.



Solution: When the square paper is folded, the vertex B touches the mid-point M of DC, the crease PQ, so formed, is the perpendicular bisector of MB.

Thus, $MQ = BQ$.

If $QC = x$ units and the side of the square is 'a' is units, then the right ΔMCQ,

$$MQ = QB = a - x, \; MC = \frac{a}{2}, CQ = x$$

$$\Rightarrow \quad (a-x)^2 = \frac{1}{4}a^2 + x^2$$

$$\Rightarrow \quad 2ax = \frac{3}{4}a^2$$

$$\Rightarrow \quad x = \frac{3}{8}a \text{ as } a \neq 0$$

Thus, $CQ : QB :: \frac{3}{8}a : \left(a - \frac{3}{8}a\right)$

$$= \frac{3}{8}a : \frac{5}{8}a$$

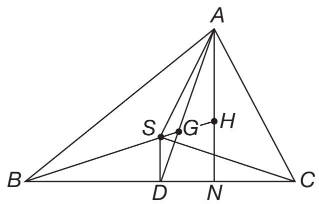
$$= 3 : 5.$$

$\therefore BQ : QC = 5 : 3.$

Problem 36 *Given are three non-collinear points A, H and G. Construct a triangle with A as vertex, H as orthocentre and G as the centroid.*

Solution:

1. Join AG, and produce it to D, such that $GD = \frac{1}{2} AG$
2. Produce AH and draw DN *perpendicular* to AH produced to meet it at N. Extend DN on both the sides.



Draw DS perpendicular to DN to meet HG produced at S.

(H, G, S are colinear points. The line joining these points is called *the Euler line*. In a Δ, the circumcentre, the centroid, the orthocentre and nine-point centre lie on a line. This line is called the Euler line). On DN extended cut-off SB and SC, equal to SA on the opposite sides of D. Now, ABC is the required Δ with the given data (or draw a circle with centre S and radius SA, to cut DN extended at B and C).

Problem 37 *If* $\angle A + \angle B + \angle C = \pi$, *then show that* $\cot A + \dfrac{\sin A}{\sin B \cdot \sin C}$ *retains the same value if any two of the angles A, B and C be interchanged.*

Solution: $\cot A + \dfrac{\sin A}{\sin B \cdot \sin C}$

$$= \cot A + \frac{\sin[\pi - (B + C)]}{\sin B \cdot \sin C}$$

$$= \cot A + \frac{\sin(B + C)}{\sin B \cdot \sin C}$$

$$= \cot A + \frac{\sin B \cos C + \cos B \sin C}{\sin B \cdot \sin C}$$

$$= \cot A + \cot C + \cot B$$

Thus, even when two of the three angles are interchanged, the value of the given expression remains the same.

Problem 38 *Show that* $\sin 55° - \sin 19° + \sin 53° - \sin 17° = \cos 1°$

Solution: $\sin 55° - \sin 19° + \sin 53° - \sin 17°$

$$= (\sin 55° + \sin 53°) - (\sin 19° + \sin 17°)$$

$$= 2\sin\frac{108°}{2}\cos\frac{2°}{2} - 2\sin\frac{36°}{2}\cdot\cos\frac{2°}{2}$$

$$= 2\cos 1°[\sin 54° - \sin 18°]$$

$$= 2\cos 1°\left[\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right]$$

$$= 2\cos 1° \times \frac{1}{2} = \cos 1°$$

Problem 39 *Find x, y, z ∈ R satisfying* $\dfrac{4\sqrt{x^2+1}}{x} = \dfrac{5\sqrt{y^2+1}}{y} = \dfrac{6\sqrt{z^2+1}}{z}$ *and* $xyz = x + y + z$.

Solution: Let, $x = \tan\alpha$, $y = \tan\beta$, $z = \tan\gamma$, $\dfrac{-\pi}{2} < \alpha, \beta, \gamma < \dfrac{+\pi}{2}$

$$\frac{4\sqrt{(\tan^2\alpha+1)}}{\tan\alpha}=\frac{5\sqrt{(\tan^2\beta+1)}}{\tan\beta}=\frac{6\sqrt{(\tan^2\gamma+1)}}{\tan\gamma}$$

$$\Rightarrow \quad \frac{4}{\sin\alpha}=\frac{5}{\sin\beta}=\frac{6}{\sin\gamma}.$$

Again, $\tan\alpha\tan\beta\tan\gamma = \tan\alpha + \tan\beta + \tan\gamma$

$\Rightarrow \quad \tan\alpha(\tan\beta\tan\gamma - 1) = (\tan\beta + \tan\gamma)$

$$\Rightarrow \quad -\tan\alpha=\frac{(\tan\beta+\tan\gamma)}{1-\tan\beta\tan\gamma}=\tan(\beta+\gamma)$$

$\Rightarrow \quad \tan(k\pi - \alpha) = \tan(\beta + \gamma)$

$\Rightarrow \quad \alpha + \beta + \gamma = k\pi$

Taking $k = 1$, we get $\alpha + \beta + \gamma = \pi$ which implies that there exists a triangle whose angles are α, β, and γ and whose sides opposite to these angles are proportional to 4, 5 and 6.

Let the sides of such Δ be $4k$, $5k$ and $6k$.

$$s = \text{semi-perimeter of the triangle} = \frac{15k}{2}$$

$$\tan\frac{\alpha}{2}=\sqrt{\frac{(s-5k)(s-6k)}{s(s-4k)}}=\sqrt{\frac{\frac{5k}{2}\times\frac{3k}{2}}{\frac{15}{2}k\times\frac{7}{2}k}}=\sqrt{\frac{1}{7}}$$

$$x=\tan\alpha=\frac{2t}{1-t^2}=\frac{2\sqrt{\frac{1}{7}}}{1-\frac{1}{7}}=\frac{\sqrt{7}}{3}$$

Similarly, $y = \tan\beta = \dfrac{5\sqrt{7}}{9}$, and $z = \tan\gamma = 3\sqrt{7}$

$$\left[\tan\frac{\beta}{2}=\sqrt{\frac{(s-4k)(s-6k)}{s(s-5k)}}\ \text{and}\ \tan\frac{\gamma}{2}=\sqrt{\frac{(s-4k)(s-6k)}{s(s-6k)}}\right]$$

where α, β, and γ are measures of the angles A, B, and C of ΔABC.

Problem 40 *If $a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x = 0$ for all $x \in R$, show that $a_0 = a_1 = a_2 = a_3 = 0$.*

Solution: Let, $f(x) = a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x$

$$f(0) = a_0 + a_1 + a_2 + a_3 = 0 \tag{1}$$

$$f\left(\frac{\pi}{2}\right) = a_0 - a_2 = 0 \quad \Rightarrow \quad a_0 = a_2 \tag{2}$$

$$f\left(\frac{\pi}{3}\right) = a_0 + \frac{1}{2}a_1 - \frac{1}{2}a_2 - a_3 = 0$$

$$\Rightarrow \quad \frac{1}{2}a_2 + \frac{1}{2}a_1 - a_3 = 0$$

$$\Rightarrow \quad a_3 = \frac{1}{2}(a_2 + a_1) \tag{3}$$

$$f\left(\frac{\pi}{4}\right) = a_0 + \frac{a_1}{\sqrt{2}} - \frac{a_3}{\sqrt{2}} = 0$$

$$\Rightarrow \quad a_2 + \frac{(a_1 - a_3)}{\sqrt{2}} = 0$$

$$\text{or } a_2 = \frac{(a_3 - a_1)}{\sqrt{2}} \tag{4}$$

Substituting in Eq. (1) the values obtained shown in Eqs. (2) and (3)

$$2a_2 + a_1 + \frac{1}{2}(a_1 + a_2) = 0$$

$$\Rightarrow \quad 5a_2 + 3a_1 = 0$$

$$\text{or } a_2 = \frac{-3}{5}a_1 \tag{5}$$

From Eqs. (4) and (5), we get:

$$\left(\frac{1}{\sqrt{2}} - \frac{3}{5}\right)a_1 = \frac{1}{\sqrt{2}}a_3 \tag{6}$$

Again, from Eqs. (3), (5), and (6), we get:

$$\left(\frac{1}{\sqrt{2}} - \frac{3}{5}\right)a_1 = \frac{1}{2\sqrt{2}}\left[\left(a_1 - \frac{3}{5}a_1\right)\right]$$

$$= -\frac{1}{2\sqrt{2}} \times \frac{2}{5}a_1 = \frac{1}{5\sqrt{2}}a_1$$

$$\Rightarrow \quad \left(\frac{1}{\sqrt{2}} - \frac{3}{5} - \frac{1}{\sqrt{2.5}}\right)a_1 = 0$$

$$\Rightarrow \quad \frac{(5 - 3\sqrt{2} - 1)}{5\sqrt{2}}a_1 = 0$$

$$\Rightarrow \quad \frac{(4 - 3\sqrt{2})}{5\sqrt{2}}a_1 = 0, \text{ but } \frac{4 - 3\sqrt{2}}{5\sqrt{2}} \neq 0$$

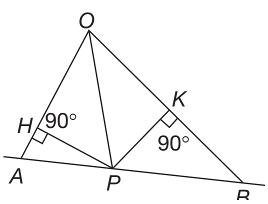
$$\therefore a_1 = 0$$

$$\therefore a_3 = 0 \left[\text{as } a_3 = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{3}{5}\right)a_1\right]$$

$$\therefore \; a_2 = \frac{-3}{5}a_1 = 0$$

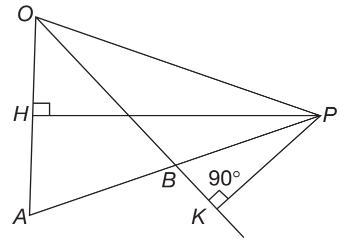
$$a_0 = a_2 = 0$$

Thus, $a_0 = a_1 = a_2 = a_3 = 0$.



Problem 41 *If any straight line is drawn cutting three concurrent lines OA, OB, OP at A, B, P, then*

$$\frac{AP}{PB} = \frac{AO \sin AOP}{BO \sin POB}$$



Solution:

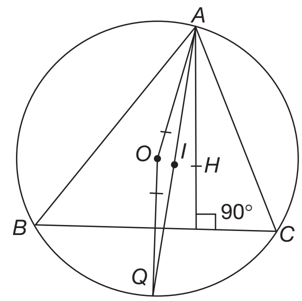
$$\frac{AP}{PB}=\frac{\Delta AOP}{\Delta POB}=\frac{\frac{1}{2}AO\cdot PO\cdot \sin AOP}{\frac{1}{2}BO\cdot PO\cdot \sin BOP}$$

$$=\frac{AO\sin AOP}{BO\sin BOP}$$

or $$\frac{AP}{PB}=\frac{\Delta AOP}{\Delta POK}=\frac{\frac{1}{2}OA\cdot PH}{\frac{1}{2}BO\cdot PK}=\frac{\frac{1}{2}OA\cdot OP\cdot \sin HOP}{\frac{1}{2}OA\cdot OP\cdot \sin POK}$$

$$=\frac{OP\sin AOP}{OP\sin POB}.$$

Problem 42 *ABC is a triangle. O, I and H are its circumcentre, in-centre and ortho-centre. Show that $\angle OAI = \angle HAI$.*



Solution: Let, AI meet the circumcircle at Q.

$OA = OQ$

(radii of the circum circle)

$\angle OAI = \angle OQI$

O is the circumcentre and AQ bisects $\angle BAC$

$\therefore$ arc BQ = arc QC

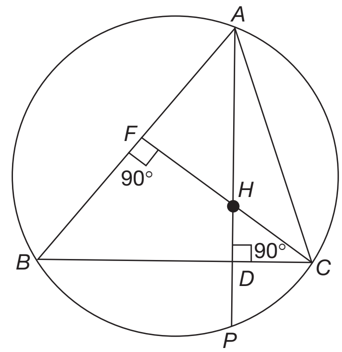
$\therefore$ OQ is perpendicular to chord of arc BC

$\therefore$ $OQ \parallel AH$ (both being perpendicular to the same line BC).

$\therefore \angle HAI = \angle HAQ = \angle AQO = \angle OAQ = \angle OAI$

$\therefore$ AI bisect $\angle HAO$.

Problem 43 *If the altitude AD meets the circumcircle of the ΔABC at P and, if H is the orthocentre, show that $HD = PD$.*



Solution:

$\angle CPD = \angle CPA$

$= \angle CBA = \angle CBF$

$= 90° - \angle FCB$

$= 90° - \angle HCD$

$= \angle DHC = \angle CHD$

$\therefore CP = CH$

$\therefore CD$ is the perpendicular bisector of PH ($\because \angle CDH = 90°$)

$\therefore DH = DP$ or $HD = PD$.

Problem 44 *ABC is a triangle. The altitudes from A, B, C meet the opposite sides BC, CA, AB at D, E, F. Here, H is the orthocentre of ΔABC. Show that the bisectors of the angles of ΔDEF are concurrent at H.*

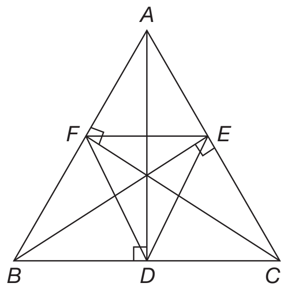
Solution: $FHDB$, $EHDC$ and $AFHE$ are cyclic quadrilaterals.

$\therefore$ In the cyclic quadrilateral $FHDB$

$\angle HDF = \angle FBH$ (angles in the same segment)

$= \angle ABE$

$= 90° - A$ (1)



In the cyclic quadrilateral $EHDC$.

$\angle EDH = \angle ECH$ (angles in the same segment)

$= \angle ACF$

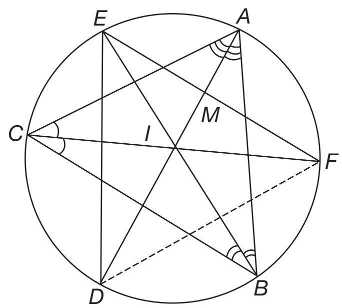
$= 90° - A$ (2)

From Eqs. (1) and (2), we get $\angle HDF = \angle HDE$

i.e., HD bisects $\angle FDE$.

Similarly, we can prove that FH and EH bisect angles $\angle DFE$ and $\angle DEF$, which implies that the bisectors of $\angle D$, $\angle E$, and $\angle F$ of ΔDEF pass through H, the orthocentre of ΔABC (*i.e.*, H is the in-centre of the pedal ΔDEF).

Problem 45 *ABC is a triangle that is inscribed in a circle. The angle bisectors of A, B, C meet the circle at D, E, F. Show that AD is perpendicular to EF.*



Solution: Let AD intersect EF at M.

Consider the ΔIMF

$\angle MFI = \angle EFC$

$= \angle EBC$ (Angles in the same segment)

$= \dfrac{B}{2}$

$\angle MIF = 180° - \angle MIC$

$$= 180° - \left[180° - \frac{A}{2} - \frac{C}{2}\right] \quad (\text{In } \Delta AIC)$$

$$= \frac{A}{2} + \frac{C}{2}$$

$$= \frac{1}{2}(180° - B)$$

$$= 90° - \frac{B}{2}$$

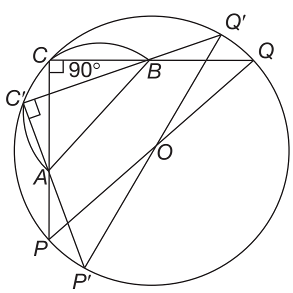
$\therefore \angle IMF = 180° - [\angle MFI + \angle MIF]$

$$= 180° - \left(\frac{B}{2} + 90° - \frac{B}{2}\right) = 90°$$

i.e., AD is perpendicular to EF.

Similarly, we can prove that BE and CF are perpendiculars to FD and ED.

Problem 46 *Given a circle and two points A and B inside the circle. If possible, construct a right-angled triangle inscribed in the circle, such that one leg of the right-angled triangle contains A and another leg contains B.*

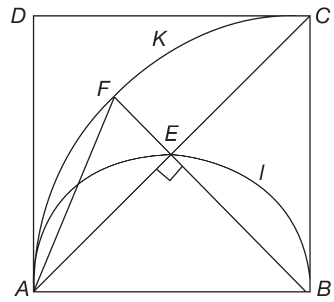


Solution: On AB as diameter, draw a semi-circle to cut the given circle at, say, C and C'. Join CA and CB. Extend them to meet the circle at P, Q.

Then, ΔPCQ is the required triangle. Since, $\angle ACB = \angle PCQ = 90°$, PQ will be the diameter. Similarly, if the other point C' is joined to A and B and extended to meet the given circle at P', Q', then $\Delta P'C'Q'$ is the Δ satisfying the given condition.

The semi-circle on AB, as diameter, may cut the circle at two points or touch the circle, or the full circle itself may be in the interior of the given circle. Accordingly, there are two two right angled triangles, or one right angled triangle triangle, or no right angled triangle satisfying the hypothesis.

Problem 47 *Let, ABCD be a square, and k be the circle with centre B passing through A and C. Let, I be the semi-circle inside the square with diameter AB. Let, E be a point on I, and the extension of BE meet the circle k at F. Prove that ∠DAF = ∠EAF.*



Solution:

(i) $BA = BF$ (Radius of the circle k.)

(ii) $\angle AEB = 90°$ (Angle in the semi-circle.)

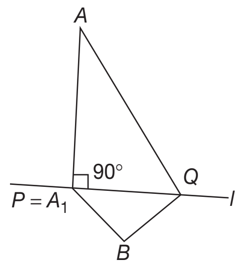
(iii) $\angle EAF = 90° - \angle AFE = 90° - \angle AFB$

$= 90° - \angle BAF$ ($BA = BF$ by Step (i))

$= \angle BAD - \angle BAF$

$= \angle FAD$ or $\angle DAF$.

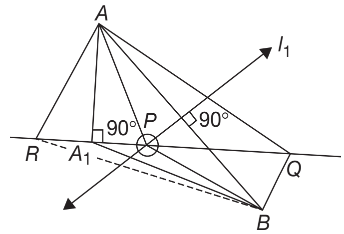
Problem 48 *Let l be a given line. A and B are the given points on the plane. Choose a point P on l, so that the longer of the segments, AP or BP, is as short as possible. (If AP = BP, either segment may be taken as the longer segment).*



Solution: If A is further away from l than B, *i.e.*, B is nearer to l than A is, draw AA_1 perpendicular to l (first figure).

(i) If $AA_1 > BA_1$, then $A_1 = P$. For any other point, Q on l, $BQ < AQ$ and $AQ > AA_1$, as AQ is the hypotenuse of the right angled ΔAA_1Q.

(ii) If $AA_1 < BA_1$ draw the perpendicular bisector l_1 of AB meeting l at P (second figure).



Now, $AP = BP$.

If Q is a point on l, such that B and Q are on the same half-plane determined by l_1, then $AQ > BQ$. But, then $AQ > AP$, so the longer segment is not the least.

Again, if R is a point on l, so that A and R lies on the same half-plane determined by l, then $AR < BR$.

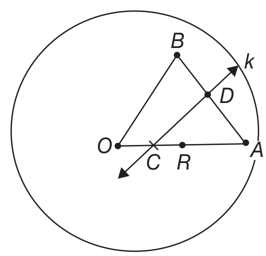
But, BR is not the shortest as $\angle BPR > 90°$ and hence, $BR > BP$. Thus, the point on l with the required property is P.

Problem 49 *Let, A and B be two points inside a given circle k. Prove that there exist infinitely many circles through A and B which lies entirely in k.*

Solution: Join A and B to the centre (O) of the circle k.

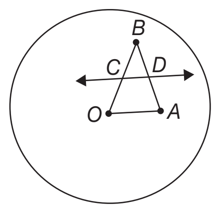
If P is a point on OA, any circle with centre P and radius PA lies entirely inside k, since A is an interior point of k.

Similarly, if Q is a point on OB and the circle with its centre Q and radius QB lies entirely inside k.



Since, OA is less than the radius of the circle k, and the circle with O as centre and radius OA lies inside circle k.

(It is the concentric circle with k) and circle with centre P and radius PA is a circle touching the concentric circle of k with radius OA internally, and hence, this circle lies entirely inside k. Similarly, for the point Q on OB, the following explanation can be given.



Let the perpendicular bisector of AB meet OA at C (or, this perpendicular bisector may meet OB).

Now, the set of centres of the set of circles passing through A and B are the points on this perpendicular bisector.

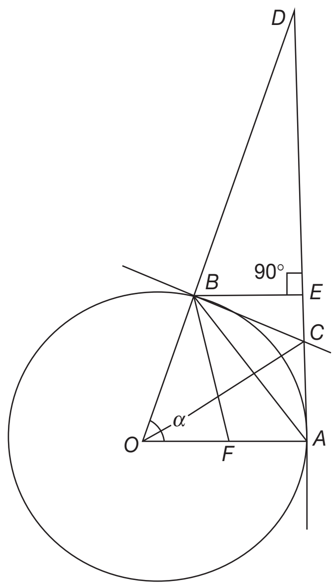
Taking any point P on line segment DC as centre and radius $PA = PB$, an infinite number of circles can be constructed. All those would lie entirely on k. This is because there are infinite number of points as P on line segment DC.

Problem 50 *Show that the radian measure of an acute angle is less than the harmonic mean of its sine and its tangent.*

Solution: Let the acute angle in the problem be α. The harmonic mean of sin α and tan α is

$$\frac{2}{\dfrac{1}{\sin\alpha}+\dfrac{1}{\tan\alpha}}=\frac{2\sin\alpha}{1+\cos\alpha}=\frac{4\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}{2\cos^2\dfrac{\alpha}{2}}=2\tan\frac{\alpha}{2}$$

So, we should prove $\alpha < 2\tan\dfrac{\alpha}{2}$.



In Fig. 3.79, $m\angle AOB = \alpha$ radians and the radius of the circle with centre O is 1 unit. *i.e.*, $OA = OB = 1$.

Arc $AB = d < \dfrac{\pi}{2}$ sq. units

Let the tangents at A and B intersect at C.

Let OB produced meet the tangent at A at the point D and BE perpendicular to AD.

(i) Area of the sector OAB

$$=\frac{1}{2}\times\alpha\times 1=\frac{\alpha}{2} \text{ sq. units}$$

But the sector OAB is contained in the quadrilateral $OACB$.

(ii) $\therefore$ Area of the sector < Area of the quadrilateral.

$\Rightarrow$ Area of the sector < 2 area of ΔOAC $\qquad (\because \Delta OAC = \Delta OBC)$

Area of $\Delta OAC = \dfrac{1}{2}OA\times AC = \dfrac{1}{2}\times 1\times\tan\dfrac{\alpha}{2}$ sq. units

$$\therefore \frac{\alpha}{2} < 2\times\frac{1}{2}\tan\frac{\alpha}{2}$$

$$\alpha < 2\tan\frac{\alpha}{2} \text{ as required}$$

Problem 51 *Show that if* α, β *and* γ *are angles of an arbitrary triangle, then* $\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2} < \dfrac{1}{4}$.

Solution: $\alpha+\beta+\gamma = 180°$ and hence, $\dfrac{\alpha}{2}, \dfrac{\beta}{2}$ and $\dfrac{\gamma}{2} < 90°$.

Since, $\dfrac{1}{2}(\alpha+\beta+\gamma) = 90°$

$$\Rightarrow \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\gamma = 90°$$

$$\frac{\alpha}{2} = 90° - \frac{1}{2}(\beta+\gamma) < 90 - \frac{1}{2}\beta < 90°$$

$$\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2} < \sin\left(90° - \frac{1}{2}\beta\right)\cdot\sin\frac{\beta}{2} = \cos\frac{\beta}{2}\sin\frac{\beta}{2}$$

$$=\frac{1}{2}\sin\beta < \frac{1}{2}$$

(i) Suppose, γ is the smallest of the 3 angles, then $\gamma \le \frac{180°}{3} = 60°$ and $\frac{\gamma}{2} \le 30°$.

(ii) So, $\sin\frac{\gamma}{2} \le \sin 30° = \frac{1}{2}$.

From Steps (i) and (ii), we have $\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2}\cdot\sin\frac{\gamma}{2} < \frac{1}{2}\times\frac{1}{2}$

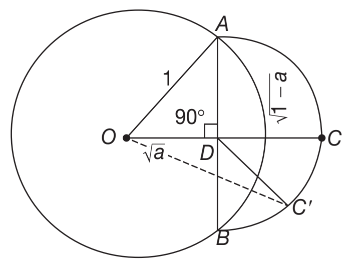
$\Rightarrow \quad \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cdot\sin\frac{\gamma}{2} < \frac{1}{4}$.

Problem 52 *A semi-circle is drawn outwardly on chord AB of the circle with centre O and unit radius. The perpendicular from O to AB meets the semi-circle on AB at C.*

(i) *Show that if C′ is any other point on the semi-circle, then $OC > OC'$.*
(ii) *Determine AB, so that OC has maximum length.*

Solution:

(i) $OC' < OD + DC' = OD + DC = OC$
(ii) Let, $OD = \sqrt{a}$ units



So that

$AD = \sqrt{1-a}$ units

$\therefore AD = BD = DC$

$= \sqrt{1-a}$ units

$\therefore OC^2 = (OD + DC)^2$

$= (\sqrt{a} + \sqrt{1-a})^2$

$= 1 + 2\sqrt{a(1-a)}$

If OC is to be a maximum, then OC^2 should also be a maximum.

For this, $1 + 2\sqrt{a(1-a)}$ should be maximum.

i.e., $a(1-a)$ should be maximum.

$$a(1-a) = a - a^2 = \frac{1}{4} - \left(a - \frac{1}{2}\right)^2$$

So, $a - a^2$ is a maximum, when $a - \frac{1}{2} = 0$, *i.e.*, $a = \frac{1}{2}$.

This implies that $OD = \sqrt{a} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

$\therefore$ In ΔAOB, $OA = OB = 1$

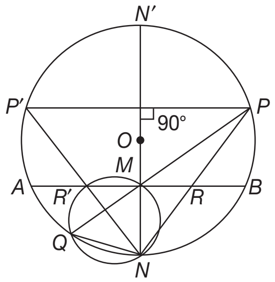
$OD = \frac{\sqrt{2}}{2}$

$\therefore AD = \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

$\therefore AB = 2AD = \sqrt{2}$

Thus, the sides of the ΔABO are in the ratio $1:1:\sqrt{2}$. So $\angle AOB = 90°$.
Thus, to determine AB, draw two radii OA, OB, inclined at an angle of 90° at O.

Problem 53 *AB is a chord of a circle with centre O, and ON is a radius perpendicular to AB, meeting AB at M. P is any point on the major segment. Join PM and extend it to meet the circle at Q. Join PN and let it intersect AB at R. Prove that RN > MQ.*



Solution: Draw the diameter NON'. Let, P' be the reflection of P in the diameter NON'. N is its own image under this reflection (Since N lies on the axis of reflection NON').

Since, AB is perpendicular to NON', R is reflected to the point R', which is the intersection of $P'N$ and AB. [$PN \to P'N$ and since, $R \in PN$ and AB, $R' \in P'N$ and AB, as AB is reflected to AB, (but not point-wise) as AB is perpendicular to $N'N$].

$\therefore$ $RN = R'N$

PP' and AB are parallel as both are perpendicular to NN'.

$\therefore$ $\angle NR'M = \angle NP'P = \angle NQP$

(NP subtends equal angles at P' and Q on the circle)

$= \angle NQM$

i.e., NM subtends equal angles at R' and Q.

$\therefore$ Points N, Q, R', M are concyclic

$\angle R'MN = 90°$ ($\because$ $R'M \parallel P'P$ and NM perpendicular to AB and $P'P$)

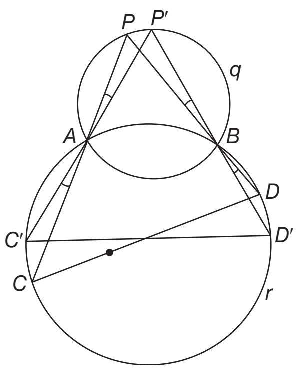
$\therefore$ $R'N$ is the diameter of the circle through the points Q, R', M, N and QM is a chord.

$\therefore$ $R'N > QM$

($\angle QNM = \angle QNN' < 90°$ as NN' is a diameter of the given circle.

$\therefore$ QM cannot be the diameter of the circle through $QNMR'$).

Problem 54 *Suppose, two circles q and r intersect at A and B. P is a point on the arc of q which lies out side r. PA and PB are joined and produced to meet the second circle at C and D. Show that for all positions of P on the circle q, the length of CD is a constant.*



Solution: Let, P' be any other point on the arc of the circle q lying outside the circle r.

Let, $P'A$ and $P'B$ meet the circle, again, at C' and D'.

We are required to show that $CD = CD'$.

$\angle PAP' = \angle P'BP$ (Angle in the same segment)

Now, $\angle C'AC =$ Vertically opposite $\angle PAP'$

$= \angle P'BP$

$=$ Vertically opposite $\angle D'BD$

In the circle r, $\angle C'AC = \angle D'BD$

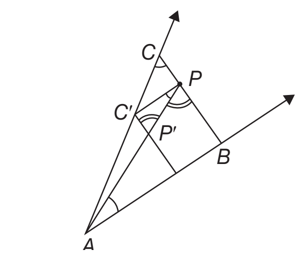
$\therefore$ arc $C'C =$ arc $D'D$

$\therefore$ arc $C'C +$ arc $C'D' =$ arc $D'D +$ arc $D'C$

$\Rightarrow$ arc $C'D' =$ arc CD

$\therefore$ Chord $C'D' =$ Chord CD.

Problem 55 *Show how to construct a chord BPC in a given angle A, through a given point P, such that* $\dfrac{1}{BP} + \dfrac{1}{PC}$ *is maximum, where P is in the interior of* $\angle A$.



Solution: Draw $PC' \parallel AB$ and $P'C' \parallel BC$ as shown in the figure.

$\Delta AP'C'$ is similar to ΔAPC.

[$\because$ $\angle P'AC' = \angle PAC$, $\angle ACP = \angle AC'P'$] and $\Delta PC'P'$ is similar to ΔABP.

[$\because$ $\angle C'P'P = \angle BPA$; $\angle C'PP' = \angle BAP$]

$$\therefore \quad \frac{P'C'}{PC} = \frac{AP'}{AP} \qquad (1)$$

and $$\frac{P'C'}{PB} = \frac{P'P}{PA} \qquad (2)$$

Adding Eqs. (1) and (2), we get:

$$\frac{P'C'}{PC} + \frac{P'C'}{PB} = \frac{AP' + P'P}{PA}$$

$$\Rightarrow \quad P'C'\left(\frac{1}{PC} + \frac{1}{PB}\right) = 1$$

or $$\frac{1}{PB} + \frac{1}{PC} = \frac{1}{P'C'}$$

If the quantity $\frac{1}{PB} + \frac{1}{PC}$ is a maximum, then $P'C'$ should be minimum.

But, $C'P'$ is minimum if $C'P'$ is $\perp r$ to AP. But, $P'C'$ is parallel to BC and $P'C'$ perpendicular to AP implies BC should be perpendicular to AP. So, join the vertex A of the given angle to the given point P and draw perpendicular to AP through P, terminated by the arms of the given angle A at C and B. Now, we have the chord BPC satisfying the hypothesis.

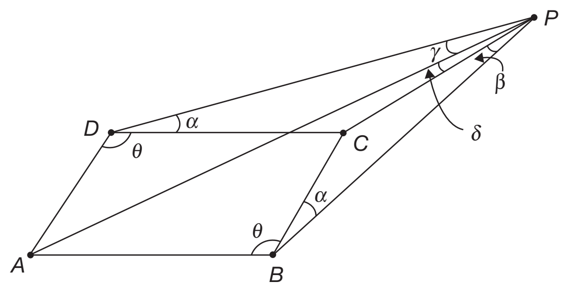
Problem 56 *If lines PB and PD outside a parallelogram ABCD make equal angles with the sides BC and DC respectively, then prove that* $\angle CPB = \angle DPA$.

Solution: Let, $\angle PBC = \angle PDC = \alpha$

$\angle CPB = \beta$

$\angle DPC = \gamma$

$\angle APC = \delta$



and, $\angle ADC = \angle ABC = \theta$

In ΔCDP, $$\frac{CD}{\sin(\gamma + \delta)} = \frac{PC}{\sin\alpha} \qquad (1)$$

In ΔBCP, $$\frac{BC}{\sin\beta} = \frac{PC}{\sin\alpha} \qquad (2)$$

From Eqs. (1) and (2), we get:

$$\frac{CD}{BC} = \frac{\sin(\gamma + \delta)}{\sin\beta} \qquad (3)$$

In ΔAPD $$\frac{AD}{\sin\gamma} = \frac{AP}{\sin(\theta + \alpha)} \qquad (4)$$

In ΔAPB $\dfrac{AB}{\sin(\beta+\delta)}=\dfrac{AP}{\sin(\theta+\alpha)}$ (5)

From Eqs. (4) and (5), we get:

$$\frac{AB}{AD}=\frac{\sin(\beta+\delta)}{\sin\gamma} \quad (6)$$

But, $CD = AB$ and

$AD = BC$.

$\therefore$ From Eqs. (4) and (6), we get:

$$\frac{\sin(\gamma+\delta)}{\sin\beta}=\frac{\sin(\beta+\delta)}{\sin\gamma}$$

$\therefore$ $\sin\gamma \sin(\gamma+\delta) = \sin\beta \sin(\beta+\delta)$

$\cos\delta - \cos(2\gamma+\delta) = \cos\delta - \cos(2\beta+\delta)$

$\therefore$ $\cos(2\beta+\delta) - \cos(2\gamma+\delta) = 0$

$\therefore$ $2\sin(\gamma+\beta+\delta)\sin(\gamma-\beta) = 0$

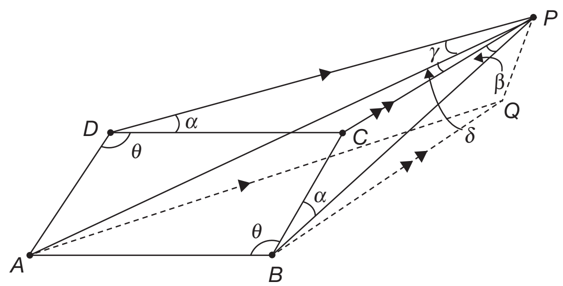
But, $\beta+\gamma+\delta \neq 0$ and it cannot be $=\pi$

$\therefore$ $\sin(\gamma-\beta) = 0$.

$\therefore$ $\beta = \gamma$.

Hence, the required result.

Aliter: Choose a point Q, such that both $BCPQ$ and $ADPQ$ are parallelograms (Q can be chosen to satisfy this condition as $AD \parallel BC$ and $AD = BC$).



Now, $\angle BPQ = \angle BAQ = \alpha$

$\{PD \parallel AQ$ and $CD \parallel AB$ $\therefore$ $\angle PDC = \angle QAB = \alpha$

$\angle CBP = \angle BPQ$ alternate angles for the parallel lines BC and $QP\}$

$\therefore$ $BQPA$ is concyclic.

$\therefore$ $\angle APB = \beta+\delta = \angle AQB = \angle DPC$

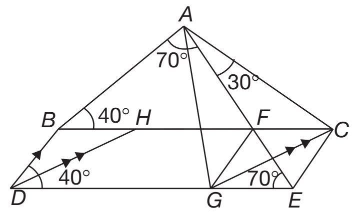
$\because$ $DP \parallel AQ$ and $CP \parallel BQ = \gamma+\delta$

$\therefore$ $\beta = \gamma$,

hence, the result.

Problem 57 *Given an isosceles ΔABC with base angle $40°$. Extend AB to D, such that $AD = BC$. Join DC. Find $\angle DCB$.*

Solution: Through D, draw a line DE parallel to BC to meet the line through C parallel to AB at E. Join AE to meet BC in F. Through F draw a line parallel to BD to meet DE in G. Join CG and AG. Through D draw a line parallel to CG to meet BC in H. $DE = BC = AD = a$. ΔADE is isosceles. ΔABF is also isosceles.



$\therefore\ \angle DAE = \angle DEA = 70°$

Now, $AB = BF = c$

$\therefore\ CE = BD = AD - AB = a - c$

Also, $CF = BC - BF = a - c$

$\therefore\ CE = CF$

$\therefore\ GECF$ is a rhombus.

$\therefore\ CG$ bisect $\angle ECF$

$\therefore\ \angle GCB = 20°$.

Now, $DGCH$ is a prallelogram with $DG = BF = CH = BA = c$

$\Delta DBH = \Delta GEC$

$\{\because\ \ DH = CG, BD = CE$ and $\angle GCE = \angle BDH\}$

In $\Delta ACG\ \angle ACG = \angle ACB + \angle BCG = 40° + 20° = 60°$

Since, $GECF$ is a rhombus, $FE \perp GC$,

$\therefore\ AE \perp CG$

Also, $EG = EC$

$\therefore$ By symmetry,

$\Delta AEG = \Delta AEC$

$\therefore\ \angle GAC = \angle GAE + \angle EAC = 2\angle EAC$ (by congruence)

$= 2[\angle BAC - \angle BAE] = 2[100° - 70°] = 60°$

$\therefore\ \angle ACG$ is an equilateral A

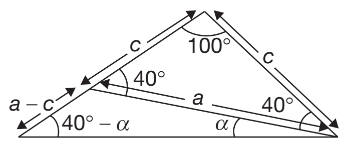
$AC = CG = AG = b = c$ ($\because\ \ \Delta ABC$ is isosceles)

$\therefore\ DGCH$ is a rhombus.

$\therefore\ DC$ bisects $\angle FCG$

$\therefore\ \angle DCB = 10°$.

Aliter 1:



$$\frac{a}{c} = \frac{\sin 100°}{\sin 40°} = \frac{\sin 80°}{\sin 40°}$$

$$\frac{a}{a-c} = \frac{\sin(40° - \alpha)}{\sin\alpha} = \frac{\sin 80°}{\sin 80° - \sin 40°}$$

$$= \frac{\cos 10°}{2\cos 60° \cdot \sin 20°}$$

$$= \frac{\cos 10°}{2\frac{1}{2}\cdot 2\sin 10° \cos 10°} = \frac{1}{2\sin 10°}$$

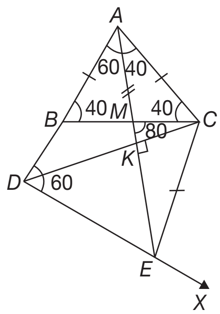
$$\therefore\quad \sin 40° \cot\alpha - \cos 40° = \frac{1}{2\sin 10°}$$

$$\cot\alpha = \frac{2\cos 40° \sin 10° + 1}{2\sin 10° \sin 40°} = \frac{\sin 50° - \sin 30° + 1}{2\sin 10° \sin 40°}$$

$$= \frac{\cos 40° + \frac{1}{2}}{2\sin 10° \sin 40°} = \frac{2\cos 40° + 1}{2(\cos 30° - \cos 50°)} = \frac{2\cos 40°}{\sqrt{3} - 2\sin 40°}$$

$$= \frac{\cos 40° + \cos 60°}{\sin 60° - \sin 40°} = \frac{2\cos 50° \cos 10°}{2\cos° \sin 10°} = \cot 10$$

$\therefore\ \alpha = 10°$.



Aliter 2:

Solution: Since $AB = AC$

$\angle ABC = \angle ACB = 40°$

$\therefore\ \angle BAC = 100°$

Construct an $\angle ADX = 60°$

Draw an arc DE on DX such that $AD = DE$.

Join AE which cuts BC at M and DC at K.

Now in ΔADE, $\angle ADE = 60°$ and $AD = DE$

$\therefore\ \Delta ADE$ is an equilateral triangle

$AD = DE = AE$ (1)

So $\angle DAE = 60°$; $\angle EAC = 40°$

In ΔAMC,

$\angle MAC = \angle MCA = 40°$

$\Rightarrow AM = MC$

Since $BC = AD = AE$

$MC = AM$

$\Rightarrow\quad BC - MC = AE - AM$

$\Rightarrow\quad BM = ME$ (2)

In ΔAMB and ΔCME

$AM = CM$

$\angle AMB = \angle CME = 80°$ (VOA)

$MB = ME$ (Proved above)

By SAS Congruence

$\Delta AMB \cong \Delta CME$

$\Rightarrow AB = CE$

But $AB = AC \Rightarrow AC = CE$

Also $AD = DE$

$\therefore\ ACED$ is a kite

$AE \perp CD$

$\therefore\ \Delta MKC$, by ASP of triangle

$90° + 80° + \angle MCK = 180°$

$\Rightarrow \angle MCK = 10°$

$\Rightarrow \angle DCB = \angle MCK = 10°$

Problem 58 *Let, ABC be a triangle of area* Δ *and* $A'B'C'$ *be the triangle formed by the altitudes* h_a, h_b, h_c *of* ΔABC *as its sides with area* Δ' *and* $A''B''C''$ *be the triangle formed by the altitudes of* $\Delta A'B'C'$ *as its sides with area* Δ''. *If* $\Delta' = 30$ *and* $\Delta'' = 20$, *find* Δ.

Solution: Let a, b, c be the sides of ΔABC.

Let, h_a, h_b, h_c be the sides of $\Delta A'B'C$. They are also the altitudes of ΔABC.

Let, h'_a, h'_b, h'_c be the sides of $\Delta A''B''C''$. They are also the altitudes of $\Delta A'B'C$.

$$\frac{1}{2}ah_a = \frac{1}{2}ah_b = \frac{1}{2}ah_c = \Delta \quad \therefore \quad h_a = \frac{2\Delta}{a}$$

$$\frac{1}{2}h_a \cdot h'_a = \frac{1}{2}h_b \cdot h'_b = \frac{1}{2}h_c \cdot h'_c = \Delta'$$

$$h'_a = \frac{2\Delta'}{h_a} = \frac{2\Delta'}{\frac{2\Delta}{a}} = \frac{a\Delta'}{\Delta}$$

$$\Delta''^2 = \frac{h'_a + h'_b + h'_c}{2} \cdot \frac{h'_a + h'_b - h'_c}{2} \cdot \frac{h'_a - h'_b + h'_c}{2} \cdot \frac{h'_b + h'_c - h'_a}{2}$$

$$= \frac{1}{2^4}\left[\frac{a\Delta'}{\Delta} + \frac{b\Delta'}{\Delta} + \frac{c\Delta'}{\Delta}\right]\left[\frac{a\Delta'}{\Delta} + \frac{b\Delta'}{\Delta} - \frac{c\Delta'}{\Delta}\right] \times \left[\frac{a\Delta'}{\Delta} - \frac{b\Delta'}{\Delta} + \frac{c\Delta'}{\Delta}\right]\left[\frac{b\Delta'}{\Delta} + \frac{c\Delta'}{\Delta} - \frac{a\Delta'}{\Delta}\right]$$

$$= \frac{\Delta'^4}{2^4\Delta^4}(a+b+c)(a+b-c)(a-b+c)(b+c-a) = \frac{\Delta'^4}{\Delta^4}\cdot\Delta^2 = \frac{\Delta'^4}{\Delta^2}$$

$\Delta' = 30$, $\Delta'' = 20$

$$\therefore\ \Delta^2 = \frac{\Delta'^4}{\Delta''^2} = \frac{30^4}{20^2} = \frac{3^4 \times 10^4}{2^2 \times 10^2} \qquad \Delta = \frac{3^2 \times 10}{2} = 45.$$

Check Your Understanding

1. Prove that, in ΔABC, whose sides AB, BC, CA have measures 4 cm, 3 cm and $\sqrt{5}$ cm respectively, the medians AK and CL are mutually perpendicular.
2. Let D be an arbitrary point on side AB of a given triangle ABC and let E be the intersection point where CD intersects the external common tangent to the incircles of triangles ACD and BCD. As D assumes all positions between A and B, prove that the point E traces the arc of the circle.
3. In ΔABC, M is the mid-point of BC. P is any point on AM; PE, PF are perpendiculars to AB, AC respectively. If $EF \parallel BC$, prove the triangle is either right-angled or isosceles.

4. Let C_1 and C_2 be circles whose centres are 10 units apart, and whose radii are 1 and 3. Find the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the mid-point of the line segment XY. **[Putnam, 1996]**
5. Prove that the quadrilateral formed by the angle bisectors of a cyclic quadrilateral, is also cyclic.
6. AD, BE, CF are the altitudes of ΔABC. If P, Q, R are the mid-points of DE, EF, FD, respectively, then show that the perpendicular from P, Q, R to AB, BC, CA, respectively, are concurrent.
7. The larger base of an isosceles trapezoid equals a diagonal and the smaller base equals the altitude of the trapezoid. Find the ratio of the smaller base to the larger base of this trapezoid.
8. Suppose the angle formed by the two rays OX and OY, is the acute angle α and A is a given point on the ray OX. Consider all circles touching OX at A and intersecting OY at B, C. Show that, the centres of all triangles ABC lie on the same straight line.
9. Let I be the incentre of ΔABC. Let the incircle of ΔABC touch the sides BC, CA, AB at K, L, M respectively. The line through B parallel to MK meets the lines LM and LK at R and S respectively. Prove that $\angle RIS$ is acute.
10. In a rhombus $ABCD$, $\angle A = 60°$. Let K be a point on the diagonal AC; choose points L, M on AB, AC respectively, such that, $KLBM$ is a parallelogram. Show that the triangle LMD is equilateral.
11. Construct a triangle, given its perimeter, the angle opposite the base and the altitude to the base. Justify.
12. Given ΔABC. Let line EF bisects $\angle BAC$ and $AE \cdot AF = AB \cdot AC$.
    Find the locus of the intersection P of lines BE and CF.

13. The diameter AB of a circle is divided into four equal parts at P, Q, R in that order. CD is a chord of the circle through P, such that, $2PD = 3AP$. Find the ratio of the area of quadrilateral $ACBD$ to that of triangle CAP.

14. In ΔABC, $\angle A = 75°$; $\angle B = 60°$; CF and AD are the altitudes from C and A respectively. H is the orthocentre and O is the circumcentre. Prove that O is the incentre of ΔCHD.

15. In ΔABC, D is a point on BC, such that, AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$. Also suppose $CD = AB$. Prove that $\angle A = 72°$.

16. ABC is a scalene triangle. Equilateral triangles ABC, BCA, CAB, are drawn outside the triangle ABC. Prove that AA_1, BB_1, CC_1 concur, say at a point K. Prove further that $AA_1 = KA + KB + KC$.

17. Let $ABCD$ be a cyclic quadrilateral. Prove that the incentres of the triangles ABC, BCD, CDA and DAB form a rectangle.

18. A circle cuts the sides of ΔABC internally as follows: BC at D, D'; CA at E, E'; AB at F', F. If AD, BE, CF are concurrent, prove that, AD', BE', CF' are also concurrent.

19. The incircle of ΔABC has centre I and touches the side BC at D. Let the mid-points of AD and BC be M and N respectively. Prove that, M, I, N are collinear.

20. D, E, F are the feet of the altitudes of ΔABC and G, H, I are the points of contact of the incircle of ΔDEF with the sides of ΔABC. Prove that, ΔABC and ΔGHI have the same Euler's line (*i.e.*, the line through the circumcentre and centroid).

21. Perpendiculars from a point P on the circumcircle of ΔABC are drawn to lines AB, BC with feet at D, E, respectively. Find the locus of the circumcentre of ΔPDE as P moves around the circle.

22. The sum of two adjacent angles of a trapezium is 90°. The lengths of two parallel sides are 'a' and 'b' respectively. Show that the length of the line segment joining the mid-points of the two parallel sides is $\frac{1}{2}|a-b|$.

23. Let ABC be an acute angled triangle and let D, E, F be the feet of the perpendiculars from A, B, C respectively to BC, CA, AB. Let the perpendiculars from F to CB, CA, AD, BE meet them at P, Q, R and N respectively. Prove that the points P, Q, M, and N are collinear.

24. Circles S_1 and S_2 with centres O_1, O_2 respectively intersect each other at points A and B. Ray O_1B intersects S_2 at point F and ray O_2B intersects S_1 at point E. The line parallel to EF and passing through B intersects S_1 and S_2 at points M and N, respectively. Prove that B is the incentre of ΔEAF and $MN = AE + AF$.

[Russian MO, 1995]

25. On the circumcircle of ΔABC, let A' be the mid-point of arc. (Not containing A). Let I be the incentre of ΔABC. Prove the following results:

(i) A, I, A' are collinear.

(ii) A' is the circumcentre of ΔBIC.

26. Given the base and the vertical angle of ΔABC, prove that the area and perimeter of ΔABC are maximum when the triangle is isosceles.

27. Triangle ABC has a right angle at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q respectively. The points M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN. **[British MO, 1995]**

28. Let I be the incentre of ΔABC and let X, Y, Z be the feet of the perpendiculars from I on the sides BC, CA, AB respectively. If IX meets YZ as N, then prove that A, N and the mid-point A' of BC are collinear.

29. ΔABC has incentre I and the incircle touches BC, CA at D and E respectively. Let BI meets DE at G. Prove that, $\overline{AG}$ is perpendicular to $\overline{BG}$.

30. $ABCD$ is a cyclic quadrilateral; points C_1, A_1 are marked on the rays BA, DC respectively, so that, $DA = DA_1$ and $CB = C_1B$. Prove that the diagonal BD intersects the segment $A_1 C_1$ at its mid-point.

31. In an acute angled triangle ABC, $\angle A$ is 30°, 'H' is the orthocentre and 'M' is the mid-point of BC. On the line HM, take a point T, such that $HM = MT$. Prove that $AT = 2BC$.

32. Given any acute angled triangle ABC, let points x, y, z be located as follows: X is the point, where the altitude from A on BC meets the outward facing semicircle on BC as diameter. Points Y and Z are defined similarly. Prove the result: $[BCX]^2 + [CAY]^2 + [ABZ]^2 = [ABC]^2$, where the notation $[PQR]$ denotes the area of ΔPQR.

33. $ABCD$ is a square. E is a point inside the square, such that $\angle EBA = \angle EAB = 15°$. Prove that ΔCED is equilateral.

34. In ΔABC, suppose $AB > AC$. Let P and Q be the feet of the perpendiculars from B and C to the angle bisector of $\angle BAC$, respectively. Let D be on line BC such that $DA \perp AP$. Prove that lines BQ, PC and AD are concurrent.

35. Through a point on the hypotenuse of a right angled triangle, lines are drawn parallel to the other two sides, so that the triangle is divided into a square and two triangles. If the area of one of the two small right triangles is 'K' times the area of the square, prove that the ratio of the area of the other triangle to the area of the first triangle is given by $1 : 4k^2$.

36. $ABCD$ is a line segment, trisected by the points B and C. P is any point on the circle where BC is its diameter. If the angles $\angle APB$ and $\angle CPD$ are respectively α and β, prove that, $4\tan\alpha \cdot \tan\beta = 1$.

37. Prove in any ΔABC, if one angle is equal to 120°, then the triangle formed by the feet of the angle bisectors, is right angled.

38. Let M be a point on the side of ΔABC. Let r_1, r_2, r be the radii of the inscribed circles of triangles AMC, BMC and ABC respectively. Let q_1, q_2, q be the radii of the inscribed circles of the same triangles that lie, in the angle $\angle ACB$. Prove the following result: $\dfrac{r_1}{q_1} \times \dfrac{r_2}{q_2} = \dfrac{r}{q}$.

39. There are exactly 100 lattice points on the circumference of a circle with origin as the centre. Prove that the radius of this circle will either be an integer or $\sqrt{2}$ times an integer.

40. ABC is a triangle with side lengths 13, 14, 15 units. If I be its incentre and R its circumradius, prove that the value of the expression, $\dfrac{AI \cdot BI \cdot CI}{R}$ is an integer. Is it a square?

41. Let BB' and CC' be altitudes of triangle ABC. Assume that $AB \neq AC$. Let M be the mid-point of BC, H the orthocentre of ABC and D the intersection of $B'C'$ and BC. Prove that $DH \perp AM$.

42. Prove that if the internal and external bisectors of $\angle C$ of ΔABC are congruent, then, $AC^2 + BC^2 = 4R^2$, where R is the circumradius of ΔABC.

43. Point P is inside ΔABC. Determine points D on side AB and E on side AC such that $BD = CE$ and $PD + PE$ is minimum.

44. Given a triangle ABC, let I be its incentre. The internal bisectors of the angles meet the sides in D, E, and F respectively. Prove that the area of ΔDEF is given by $\dfrac{2abcs}{(a+b)(b+c)(c+a)}$, in the usual notation, S being the area of ΔABC.

45. Diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that $\dfrac{AM}{AC} = \dfrac{CN}{CE} = r$

Determine r if B, M and N are collinear. **[IMO, 1982]**

46. $ABCDEF$ is a hexagon inscribed in a circle. Show that the diagonals AD, BE, CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

47. Let $A = \tan\alpha \tan\beta + 5$; $B = \tan\beta \tan\gamma + 5$; $C = \tan\gamma \tan\alpha + 5$; where α, β, γ are positive and $\alpha + \beta + \gamma = \pi/2$. Prove the inequality: $\sqrt{A} + \sqrt{B} + \sqrt{C} \le 4\sqrt{3}$.

48. Let ΔABC be a right triangle with $\angle A$ being the right angle. Prove the inequality: $\sin B \sin C \le \dfrac{1}{2}$. Find the condition for which the equality holds.

49. In ΔABC, prove that, in the usual notation,
$3(bc + ca + ab) \le (a + b + c)^2 < 4(ab + bc + ca)$.

50. If Δ is the area of ΔABC with sides a, b, c prove that,

(i) $\Delta \le \dfrac{1}{4}\sqrt{(abc)(a+b+c)}$.

(ii) When does the equality hold?

(iii) Also deduce the formula for the area of an equilateral triangle.

51. Let A, B, C be an equilateral triangle. Let K, L, M be arbitrary points, chosen on the sides BC, CA, AB respectively.

(i) Prove that the area of one of the triangles AML, BKM, CLK is less than or equal (ΔABC). (That is a quarter of the area of ΔABC)

(ii) When does the equality hold?

52. Let $ABCD$ be a convex quadrilateral with $\overline{AC} \cap \overline{BD} = \{E\}$. Let F_1, F_2, F be the area of ΔAED, ΔBEC, and quadrilateral $ABCD$. Prove the inequality: $\sqrt{F_1} + \sqrt{F_2} \le \sqrt{F}$. When does the equality occur?

53. In an acute angled triangle ABC, prove the inequalities;

(i) $\cot A + \cot B + \cot C \ge \sqrt{3}$

(ii) $\tan^2 A + \tan^2 B + \tan^2 C \ge 9$

(iii) $\sin^2 A + \sin^2 B + \sin^2 C \le 9/4$

54. Prove that, in an acute angled triangle ABC, the following inequalities hold:

(i) $\cos A \cos B \cos C \le 1/8$

(ii) $\dfrac{1+\cos A+\cos B+\cos C}{2\cos A\cos B\cos C} \ge 10$

55. Prove that, $a^2pq + b^2qr + c^2rp \le 0$, whenever a, b, c are the lengths of the sides of a triangle and $p + q + r = 0$. ($p, q, r \in R$)

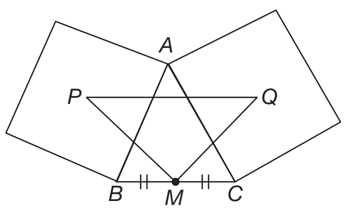
56. In ΔABC, show in the usual notation that, $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} \ge \dfrac{2s}{abc}$.

57. Which regular polygons can be obtained (and how) by cutting a cube with a plane?

58. The sides AB, BC and CA of a triangle are c, a, and b respectively.

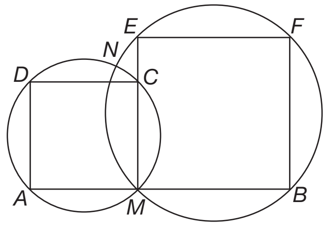
If $a^2 + b^2 - 1993c^2$, find the value of $\dfrac{\cot C}{\cot A+\cot B}$

59. Given a circle, a point P on it and a line intersecting the circle in two points, construct all chords of the circle through P which are divided by the line in the ratio 1 : 2.

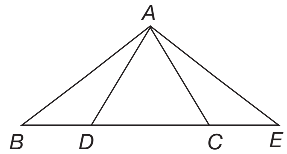


60. Given an arbitrary triangle ABC, let P and Q be the centres of squares on AB and AC, respectively, as shown in the figure. If M is the mid-point of BC, show that triangle PMQ is an isosceles right-angled triangle.

61. Let, M be any point on AB. Squares $AMCD$ and $BMEF$ are constructed and the circumscribed circles of $AMCD$ and $BM\ BF$ intersect at M and N. Show that the lines AE and BC pass through N.



62. The exterior and interior bisectors of the angle A of ΔABC meet the side BC at E and D as shown in the figure. If $AD = AE$, find $\angle BCA - \angle CBA$.



63. ABC is a triangle with $\angle B = 120°$ and BT is the bisector of $\angle B$ meeting AC at T. Prove that BT is the Harmonic Mean between BC and BA.

64. ABC is a triangle. The internal and external bisectors AP and AQ of $\angle A$ meets the line BC at P and Q, respectively. Prove that BC is the Harmonic Mean between BP and BQ.

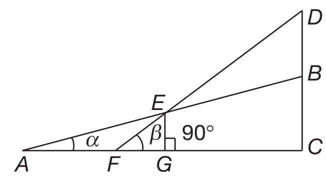
65. $ABCD$ is a cyclic quadrilateral. The chords AB and DC produced to meet at Q. AD and BC produced to meet at P. The bisectors of angles Q and P meet the circle at U, V, T and S, respectively. Show that PV and QS intersect at right angles.

66. $ABCD$ and $PQRS$ are two squares circumscribed and inscribed about a circle with centre O and radius 1 unit and the diagonals PR and QS of $PQRS$ lie along the diagonals AC and BD. If K, L, M and N are the mid-points of PA, QB, RC and SD, show that $KLMN$ is a square and compare the perimeter of this square to that of the circumference of the circle.

67. AB is a directed line segment and is divided at C, so that $BA \cdot BC = AC^2$. Prove that $AB^2 - AC^2 = AB \cdot AC$.

68. In an acute angled triangle ABC, $\angle A = 30°$, O is the ortho-centre and M is the mid-point of BC on the line OM; T is the point, such that $OM = MT$. Show that $AT = 2BC$. **[INMO, 1995]**

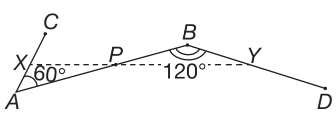
69. Two right-angled triangles ABC and FDC are such that their hypotenuses $AB = p$ and $FD = q$ intersect in E as shown in the figure. Find x (the distance of the point E from the side FC) in terms of $\alpha = \angle BAC$, $\beta = \angle DFC$ and the length of the two hypotenuses.



70. Equilateral ΔADC is drawn externally on side AC of ΔABC. Point P is taken on BD. Find $\angle APC$ if $BD = PA + PB + PC$.

Challenge Your Understanding

1. Prove that the bisector of an angle of a triangle is equal to or less than half the sum of the arms of the angle.
Apply this result to prove the following problem:
In the figure, P is the mid-point of the line segment AB, $\angle BAC = 60°$ and $\angle ABD = 120°$. X is any point on AC such that, XP extended meets BD at Y. Prove that the length of XY is greater than or equal to the length of AB.



2. A circle passing through vertices B and C of triangle ABC intersects sides AB and AC at C' and B', respectively. Prove that BB', CC' and HH' are concurrent, where H and H' are the orthocentres of triangles ABC and $AB'C'$, respectively.
[IMO shortlisted problem 1995]
3. Point C lies on the minor arc AB of the circle centreed at O. Suppose the tangent line at C cuts the perpendiculars to chord AB through A at E and through B at F. Let D be the intersection of chord AB and radius OC. Prove that $CE \cdot CF = AD \cdot BD$ and $CD^2 = AE \cdot BF$.
4. Two circles P_1 and P_2 intersect in two points P and Q. The common tangent of P_1 and P_2, nearer P than Q, touches P_1 and P_2 at A and B respectively. The tangent to P_1 at P intersects P_2 at E (distinct from P). The tangent to P_2 at P meets P_1 at F (distinct from P). Let H and K be two points on the rays AF and BE respectively, such that, $AH = AP$, $BK = BP$. Prove that the points A, H, Q, K, B are all concyclic.
[AMTI, 2008]
5. Suppose A is a point inside a given circle and is different from the centre. Consider all chords (excluding the diameter) passing through A. What is the locus of the intersection of the tangent lines at the endpoints of these chords?
6. The circumference of the circle is divided into 8 arcs by a convex quadrilateral $ABCD$, with four arcs lying inside the quadrilateral and the remaining four arcs lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counterclockwise direction, starting from some arc. Suppose $p + r = q + s$. Prove that the quadrilateral $ABCD$ is cyclic. **[RMO, 2002]**
7. If A, B, C, D are four distinct points such that every circle through A and B intersects or coincides with every circle through C and D, prove that the four points are either collinear or concyclic. **[Putnam MO, 1965]**
8. The cyclic octagon $ABCDEFGH$ has sides a, a, a, a, b, b, b, b respectively. Show that the radius of the circle circumscribing the octagon is given by, $\frac{1}{\sqrt{2}}(\sqrt{a^2+\sqrt{2}ab+b^2})$. **[RMO, 2002]**
9. A circle intersects a triangle ABC at six points A_1, A_2, B_1, B_2, C_1, C_2, where the order of appearance along the triangle is A, C_1, C_2, B, A_1, A_2, C, B_1, B_2, A. Suppose B_1C_1, B_2C_2 meets at X, C_1A_1, C_2A_2 meets at Y and A_1B_1, A_2B_2 meets at Z. Show that AX, BY, CZ are concurrent.
10. In ΔABC, let D be the mid-point of BC. If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$. **[RMO, 2005]**
11. Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the mid-points of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.
[APMO, 1998]
12. Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Prove that QO is perpendicular to BC. **[APMO, 2000]**
13. Assume ΔABC is isosceles with $\angle ABC = \angle ACB = 78°$. Let D and E be points on the sides AB and AC respectively, so that, $\angle BCD = 24°$ and $\angle CBE = 51°$. Find $\angle BED$.
14. Two circles with centres O_1 and O_2 intersect at points A and B. A line through A intersects the circles with centres O_1 and O_2 at points Y, Z, respectively. Let the

tangents at Y and Z intersect at X and lines YO_1 and ZO_2 intersect at P. Let the circumcircle of ΔO_1O_2B have centre at O and intersect line XB at B and Q. Prove that PQ is a diameter of the circumcircle of ΔO_1O_2B.

15. Let D, E, F be points on the sides BC, CA, AB respectively of ΔABC. Let R be the circumradius of the ΔABC. Prove that the geometrical inequality:

$$\left(\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}\right)(DE+EF+FD)\geq\frac{2s}{R}.$$

Where 's' is the semi perimeter of the ΔABC.

16. Let $ABCDEF$ be a convex hexagon such that

$\angle B+\angle D+\angle F=360°$ and $\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FA}=1$. Prove that $\frac{BC}{CA}\cdot\frac{AE}{EF}\cdot\frac{FD}{DB}=1$.

[IMO Shortlisted Problem, 1998]

17. ΔABC is scalene with $\angle A$ having a measure greater than 90°. Determine the set of points D which lie on the extended line BC for which $|AD|=\sqrt{|BD|\cdot|CD|}$ where $|BD|$ refers to the (positive) distance between B and D. **[INMO, 1989]**

18. Let $ABCD$ be a cyclic quadrilateral. Let E and F be variable points on the sides AB and CD, respectively, such that $AE : EB = CF : FD$. Let P be the point on the segment EF such that $PE : PF = AB : CD$. Prove that the ratio between the areas of triangles APD and BPC does not depend on the choice of E and F.

[IMO Shortlisted Problem, 1998]

19. For three points P, Q, R in the plane, we define $m(PQR)$ to be the minimum of the lengths of the altitudes of ΔPQR (Note that $m(PQR)=0$, where P, Q, R are collinear). Let A, B, C be the given points in the plane. Prove that for any point X in the plane, $m(ABC)\leq m(ABX)+m(ACX)+m(BCX)$. **[IMO, 1993]**

20. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

[IMO, 1998]

21. Circles G_1 and G_2 touch each other externally at a point W and are inscribed in a circle G. A, B, C are points on G such that A, G_1 and G_2 are on the same side of chord BC, which is also tangent to G_1 and G_2. Suppose AW is also tangent to G_1 and G_2. Prove that W is the incentre of triangle ABC. **[IMO Shortlisted Problem, 1992]**

22. Four points are given in space, in general position (*i.e.*, they are not coplanar and any three are not collinear). A plane π is called an *equalizing* plane if all four points have the same distance from π. Find the number of equalizing planes.

[Israeli MO, 1995]

23. Circles G_1 and G_2 touch each other externally at a point W and are inscribed in a circle G. A, B, C are points on G such that A, G_1 and G_2 are on the same side of chord BC, which is also tangent to G_1 and G_2. Suppose AW is also tangent to G_1 and G_2. Prove that W is the incentre of triangle ABC.

24. Hexagon $ABCDEF$ is inscribed in a circle so that $AB = CD = EF$. Let P, Q, R be the points of intersection of AC and BD, CE and DF, EA and FB respectively. Prove that triangles PQR and BDF are similar.

25. Given a non-equilateral triangle ABC and its circumcircle S; let A' denotes the point of intersection of the tangents to S at B and C; define likewise the points B' and C'.

(i) Show that the lines AA', BB', CC' concur.

(ii) Let the point of concurrence be K. Let G denotes the centroid of ΔABC. Prove that, $KG \,||\, BC$, iff $2a^2 = b^2 + c^2$ (where a, b, c are the lengths of the sides of ΔABC).

26. In a disk with centre O, there are four points such that the distance between every pair of them is greater than the radius of the disk. Prove that there is a pair of perpendicular diameters such that exactly one of the four points lies in side each of the four quarter disks formed by the diameters.

27. ABC is a triangle. On AB and AC as sides, two squares $ABDE$ and $ACFG$ are drawn outside the triangle. Prove that, CD, BF and the altitude through A of $\Delta\, ABC$ are concurrent.

28. Two intersecting circles Σ_1 and Σ_2 have a common tangent, which touches Σ_1 at P and Σ_2 at Q. The two circles meet at M, and N, where N is nearer to PQ than M. The line PN meets the circle Σ_2 again at R. Prove that MQ bisects $\angle PMR$.

29. In a non-equilateral triangle ABC, the sides a, b, c form an arithmetic progression. Let I and O denote the incentre and circumcentre of the triangle

(i) Prove that $IO \perp BI$.

(ii) Suppose BI extended meets AC in K and D, E are the mid-points of BC, BA respectively. Prove that I is the circumcentre of ΔDKE.

30. Let a, b, c denote the measures of the sides of ΔABC, while their respective opposite angles be denoted by α, β and γ. If $a + b = \tan\dfrac{\gamma}{2}\,(a \tan\alpha + b \tan\beta)$, prove that, the triangle is isosceles always.

31. In ΔABC, $\angle A$ is a right angle. Squares $ACDE$ and $ABGF$ are described on AC and AB respectively, externally to the triangle. BD cuts AC in M and CG cuts AB in N. prove that $AM = AN$.

32. ΔABC has a right angle at A. Among all points P, on the perimeter of the triangle find the position of P, such that $AP + BP + CP$ is minimized.

33. Let n be an integer ≥ 3. Prove that there is a set of 'n' points in the plane, such that, the distance between any two points is irrational and each set of three points determines, a non-degenerate triangle with rational area.

34. 2009 concentric circles are drawn with radii one unit to 2009 units. From a point on the outer most circles, tangents are drawn to the inner circles. Discover the number of tangents which will have integer measure in this problem. Also locate these tangents.

35. Given a triangle ABC let I be its incentre. The internal bisectors of the angles A, B and C meet the opposite sides in A', B', C' respectively. Prove the inequality

$$\frac{1}{4} < \frac{AI}{AA'}\cdot\frac{BI}{BB'}\cdot\frac{CI}{CC'} \leq \frac{8}{27}$$ **[IMO, 1991]**

36. Let P be a point inside ΔABC and D, E, F be the feet of the perpendiculars from P to the lines BC, CA, and AB respectively. Find all P, which will minimize the expression $\dfrac{BC}{PD} + \dfrac{CA}{PE} + \dfrac{AB}{PF}$.

37. In $\Delta\, ABC$, r is the inradius and r_A (similarly r_B, r_C) the radius of the circle, which touch the incircle and the sides emanating from the vertex A (similarly B and C). Prove the inequality: $r \leq r_A + r_B + r_C$.

38. Let ΔABC and a point P in its interior be given. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

39. Let a, b, c denote the measures of the sides of a triangle, prove the following inequality $a^2(a + b + c) + b^2(b + c + a) + c^2(c + a + b) \leq 3abc$.

40. In a triangle of base 'a', the ratio of the other two sides is 'r' where $r < 1$. Prove that the altitude to the triangle is less than or equal to $\left(\dfrac{ar}{1-r^2}\right)$.

41. Let A, B, C be a triangle with sides a, b, c. Consider a triangle $A_1B_1C_1$ with sides lengths as $a + b/2$, $b + c/2$, $c + a/2$. Prove the inequality: $[A_1B_1C_1] \geq 9/4\ [ABC]$ in the usual notation.

42. In an acute angled triangle ABC, the internal bisector of $\angle A$ intersects BC at L and intersects the circumcircle of ΔABC at N. From the point L, perpendiculars are drawn to AB and AC, the feet of the perpendiculars being K and M respectively. Prove that the quadrilateral $AKNM$ and triangle ABC have the same area.

43. A Pythagorean triangle is a right angled triangle, in which all the three sides are of integer lengths. Let a, b be the legs of a Pythagorean triangle and h be the altitude to the hypotenuse c. Determine all such triangles, for which the relation $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{h}=1$ is true.

44. If the perimeter of a triangle is given, prove the inequality,

(i) $\Delta \leq \dfrac{s^2}{3\sqrt{3}}$

(ii) $r \leq \dfrac{s}{3\sqrt{3}}$

(iii) $R \geq \dfrac{3\sqrt{3}}{4s^2}$

(iv) Hence deduce the inequality: $R \geq 2r$

(v) When does the equality hold (in iv)?

45. In a quadrilateral $ABCD$, it is given that $AB \parallel CD$. The diagonals AC and BD are perpendicular to each other. Prove the following inequalities:

(i) $AD \cdot BC \geq AB \cdot CD$

(ii) $AD + BC \geq AB + CD$.

46. Given two non-intersecting circles in a plane. They have two internal common tangents and two external common tangents. Show that the mid-points of these four tangents are collinear.

47. Let r_1, r_2, r_3, r_4 be the radii of four mutually externally tangent circles. Prove that

$$\sum_{k=1}^{4}\frac{2}{r_k^2}=\left(\sum_{k=1}^{4}\frac{1}{r_k}\right)^2.$$

Note: This result is known as Descartes's circle theorem.

48. In convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

[IMO, 1998]

49. Let $ABCD$ be a convex quadrilateral with perpendicular diagonals meeting at O. Prove that the reflections of O across AB, BC, CD, DA are concyclic.

[USA MO, 1993]

50. The incircle of triangle ABC touches BC, CA and AB at D, E and F respectively. X is a point inside triangle ABC such that the incircle of triangle XBC touches BC at D also, and touches CX and XB at Y and Z respectively. Prove that $EFZY$ is a cyclic quadrilateral. **[IMO Shortlisted Problem, 1995]**

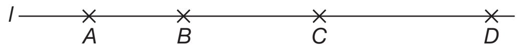
51. ABC is a right-angled triangle with $\angle C = 90°$. The centre and the radius of the inscribed circle is I and r. Show that

$AI \times BI = \sqrt{2} \times AB \times r.$

52. Let AB and CD be two perpendicular chords of a circle with its centre O and radius r. Let, X, Y, Z, W, in cyclical order, denote the four parts into which the disc is thus divided. Find the maximum and minimum of the quantity $\dfrac{E(X)+E(Z)}{E(Y)+E(W)}$, where $E(u)$ denotes the area of u.

53. Let, $ABCD$ be a rectangle and M, N and P, Q be the points of intersection of line l with sides AB, CD, and AD, BC, respectively (or their extensions). Given the points, M, N, P and Q and the length p of side AB. Construct the rectangle. Under what conditions can this problem be solved and how many solutions does it have?

54. Let A, B, C, D be the four given points on a line l. Construct a square, such that two of its parallel sides or their extensions go through A and B, and the other two sides (or their extensions) go through C and D.



55. The diagonals AC, BD of the quadrilateral $ABCD$ intersect at the interior point O. The areas of the ΔAOB and ΔCOD are s_1 and s_2, respectively, and the area of the quadrilateral is s. Prove that $\sqrt{s_1} + \sqrt{s_2} \le \sqrt{s}$. When does equality hold?

56. M is the mid-point of the hypotenuse AC of a right angled ΔABC. The perpendicular MP to AC meets AB produced at P and intersects BC at N. If $MN = 3$ cm and PN = 9 cm. Find the length of the hypotenuse. Also calculate the length of the sides AB and BC.

57. In ΔABC, $AB \ne AC$. The bisectors of $\angle B$ and $\angle C$ meet their opposite sides AC and AB *at* B' and C'. The two bisectors intersect at I. Prove that, if $IB' = IC'$ then $\angle BAC = 60°$.

58. Let $ABCD$ be a rectangle with $AB = a$ and $BC = b$. Suppose, r_1 is the radius of the circle passing through A and B and touching $CD \cdot r_2$ is the radius of the circle passing through B and C and touching AD.

Show that $r_1 + r_2 \ge \dfrac{5}{8}(a+b)$.

59. Let AC and BD be two chords of a circle with centre O and AC and BD intersect at right angle at the point M, in the interior of the circle. K and L are the mid-points of the chord AB and CD, respectively. Prove that $OKML$ is a parallelogram.

60. Given a circle of radius 1 unit and AB is a chord of the circle with length 1 unit. If C is any point on the major segment, show that

$AC^2 + BC^2 \le 2(2+\sqrt{3})$

61. From a point E on the median AD of ΔABC, the perpendicular EF is dropped to the side BC. From a point M on EF, perpendiculars MN and MP are drawn to the sides AC and AB, respectively. If N, E, P are collinear, show that M lies on the internal bisector of $\angle BAC$.

62. Prove that of all straight lines drawn through a point of intersection of two circles and terminated by them, the one which is parallel to the line of centres is the greatest.

63. $ABCD$ is a rectangle. Its diagonals AC and BD intersect at O. A straight line through B, intersects DC at E and DA at F. Here, $OE = OF$.

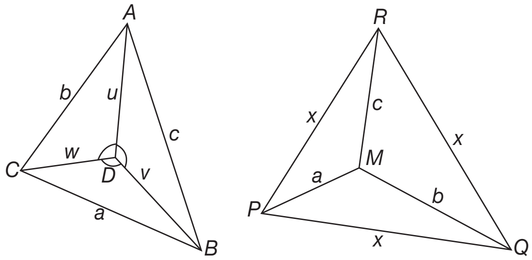
Show that $\dfrac{CD}{AF} = \dfrac{AF}{EC} = \dfrac{EC}{BC}$.

64. Let P be any point inside the parallelogram $ABCD$, and R be the radius of the circle through A, B and C. Show that the distance from P to the nearest vertex is not greater than R.

65. P is a variable point on the arc of a circle cut off by the chord AB. Prove that the sum of the lengths of the chords AP and PB is maximum when P is at the mid-point of the arc AB.

66. If A and B are two fixed points on a given circle and XY is a variable diameter of the same circle, then determine the locus of the point of intersection of lines AX and BY. You may assume that AB is not a diameter.

67. Consider the two triangles ABC and PQR shown in the figures. In ΔABC, $\angle ADB = \angle BDC = \angle CDA = 120°$. Prove that $x = u + v + w$.

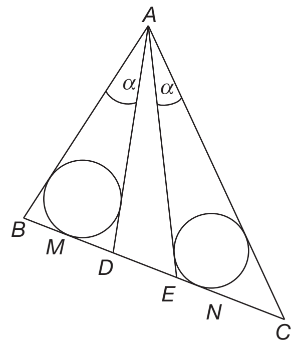


68. Let, OX and OY be two perpendicular lines meeting at O. A, C are points on OY such that $OA = 1$ unit and $OC = b$ units and B is a point on OX, such that $OB = a$ units. BD and CD are drawn perpendicular to OX and OY meeting at D. Circle on diameter AD, intersects OX at R_1 and R_2. Show that OR_1 and OR_2 are the roots of the quadratic equation $x^2 - ax + b = 0$.

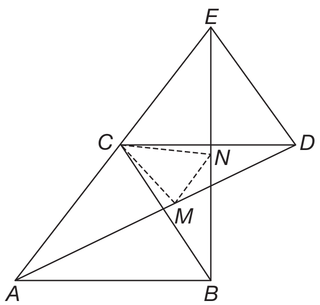
69. Let ABC be a right-angled triangle which is right angled at A. S be its circum-cirlce. Let, S_1 be the circle touching AB, AC and circle S internally. Let, S_2 be the circle touching AB, AC and S externally. If r_1 and r_2 are the radii of circles S_1 and S_2, show that $r_1 r_2 = 4$ area (ΔABC).

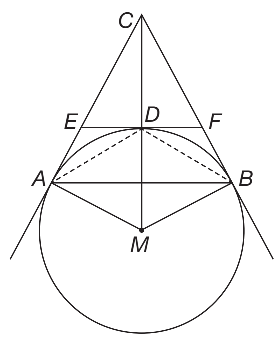
70. Let, D, E be points on the side BC of a ΔABC such that $\angle BAD = \angle CAE$. If the incircle of the ΔABD and ΔACE touch the side BC at M and N, show that

$$\frac{1}{MB} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE}.$$



71. ABC is an equilateral triangle and E is any point on AC produced and the equilateral ΔECD is drawn. If M and N are the mid-points of AD and EB, respectively, prove that ΔCMN is equilateral.

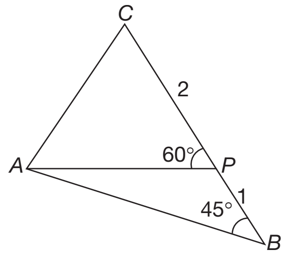




72. Let, M be the centre of a circle and A, B are two points on the circle, not diametrically opposite. The tangents at A and B intersect at C. Let, CM intersect the circle in D, and suppose that the tangent through D intersects AC and BC at E and F, respectively, as in the adjoining figure.

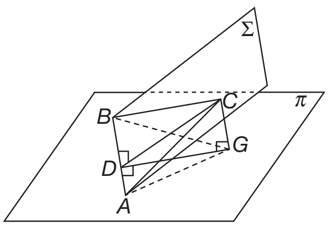
Show that (a) the area of the quadrilateral $ADBM$ is the geometric mean of the areas of triangle ABM and quadrilateral $ACBM$. (b) The area of pentagon $AEFBM$ is the harmonic mean of the areas of quadrilaterals $ADBM$ and $ACBM$.

73. The point P on the side BC of ΔABC divides BC in the ratio $1:2$. *i.e.*, $BP:PC = 1:2$. $\angle ABC = 45°$, $\angle APC = 60°$. Calculate $\angle ACB$. [Without using trigonometry.]

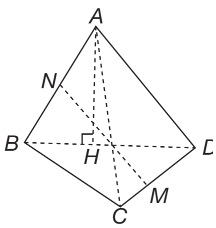


74. A line cuts a rectangular region into two regions of equal area. Show that it passes through the intersection of the diagonals of the rectangle.

75. Let A, B, C and D be non-coplanar point such that $ABCD$ is a three-dimensional pyramid like solid. Given $BA = BC = DB = AC = CD = AD = a$ unit, R and S are the mid-points of CD and AB, respectively. Prove that RS is perpendicular to both BA and CD.



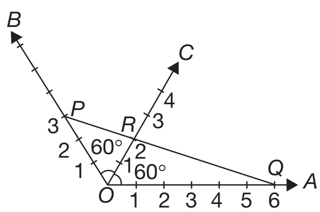
76. In the given figure, plane π and plane Σ intersect at the line AB. The angle between the planes π and Σ, *i.e.*, the dihedral angle $\angle \pi AB\Sigma$ is formed. CG is perpendicular to the plane π (c on Σ and G on π) and DG is perpendicular to AB and CD is perpendicular to AB. D is the mid-point of AB, $BC = AC$. If $AB = 4\sqrt{6}$, $AG = 6$, $\angle CBG = 45° = \angle CAG$, then find the length of CG and measurement of the dihedral angle $\angle \Sigma BA\pi$.



77. $ABCD$ is a regular tetrahedron, that is, it is a solid with four faces, each of which is an equilateral triangle. N and M are the mid-points of the sides AB and CD, respectively. Show that $NM = \dfrac{AB}{\sqrt{2}}$. If AH is drawn perpendicular to the plane of the base ΔBCD, show that $AH = \sqrt{2}\,AB$.

78. An equilateral triangle has one side in a given plane. The plane of the triangle is inclined to the given plane at an angle of 60°. What is the ratio of the area of the triangle to the area of its projection on the plane?

79. $ABCD$ is a square, and E is a point on AB extended. CE is joined and F is a point on AD, such that $\angle FCE = 90°$. If the ratio of the area of ΔFCE and the square $ABCD$ is p/q, find BE in terms of side AB of the square. For what values of p/q, BE is of rational length?



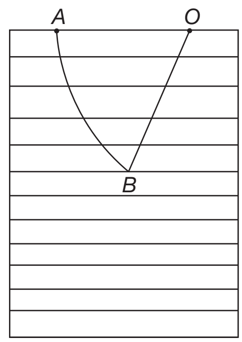
80. **Problem on electricity:** If we have an electrical circuit consisting of two wires in parallel with resistances R_1 and R_2, then the resistance R of the circuit is given by the equation $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$. The following diagram helps in finding the values of one of R, R_1, and R_2 given the value of the other two. $\angle BOA = 120°$ and $\overrightarrow{OC}$ bisects $\angle BOA$ and $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are marked with numbers (co-ordinated) at equal distances as shown in the figure.

The segment joining the point marked 3(P) on OB and the point marked 6(Q) on OA cuts OC at the point 2(R) showing that the sum of the reciprocals of 6 and 3 is the reciprocal of 2(R).

Prove that this method works for all points.

81. ABC is a triangle and a square $PQRS$ is inscribed in the ΔABC with the side PQ lying along BC. AD is the altitude from A to BC of the triangle. Prove that $2PQ$ is the Harmonic Mean between BC and AD.

82. The adjoining drawing shows how a sheet of ruled paper can be used to divide a line segment AO into n equal parts (here, into 5 equal parts). With O as centre, an arc of radius OA is drawn to intersect the $(n + 1)$th line from AO at B. Explain, how OA can be divided into n equal parts. Prove your construction. Assume that the lines of paper are evenly spaced.



83. ABC and $A'B'C'$ are two triangles in the same plane, such that the lines AA', BB' and CC' are mutually parallel. Let, $[ABC]$ denote the area of triangle ABC, with appropriate I sign, etc. Prove that
$3[ABC] + [A'B'C] = [AB'C] + [BC'A'] + [CA'B'] + [A'BC] + [B'CA] + [C'AB]$.

84. $ABCD$ and $A'B'C'D'$ are square maps of the same region, drawn to different scales and super-imposed as shown in the figure. Prove that there is only one point O on the small map which lies directly over point O' of the large map, such that O and O' each represent the same place of the country**.** Also, give an Euclidean construction (Straight edge and compasses) for O.

[USA MO, 1978]

85. In a triangle ABC, choose any points $K \in BC$, $L \in AC$, $M \in AB$, $N \in LM$, $R \in MK$ and $F \in KL$. If E_1 E_2, E_3, E_4, E_5, E_6 and E denote the areas of the triangles AMR, CKR, BKF, ALF, BNM, CLN and ABC respectively, show that $E \geq 8\,(E_1 \cdot E_2 \cdot E_3 \cdot E_4 \cdot E_5 \cdot E_6)^{1/6}$.

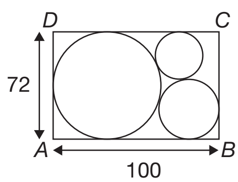
86. Let, ABC be an acute angled triangle. Three lines L_A, L_B and L_C are constructed through the vertices A, B, and C, respectively, according to the following prescription. Let, H be the foot of the altitude drawn from the vertex A to the side BC. Let, S_A be the circle with diameter AH; let SA meet the side AB and AC at M and N respectively, where M and N are distinct from A, then L_A is the line through perpendicular to MN. The lines L_B and L_C are constructed, similarly. Prove that L_A, L_B, and L_C are concurrent.

87. ABC is a right-angled triangle at A, and two circles with radii r_1 and r_2, respectively, touches both AB and AC. One of them touches the circumcircle of ABC internally, and the other externally. Show that $4\,\Delta ABC = r_1 r_2$. **[INMO, 1993]**

88. Given any acute-angled ΔABC, let points X, Y and Z be located as follows: X is the point where the altitude from A on BC meets the outward facing semi-circle drawn on BC as diameter. Points Y and Z are located similarly, prove that $[BCX]^2 + [CAY]^2 + [ABZ]^2 = [ABC]^2$. **[INMO, 1991]**

89. Let, C_1 and C_2 be two concentric circles in the plane with radii R and $3R$. Show that the orthocentre of any triangle inscribed in circle C_1 lies in the interior of circle C_2. Conversely, show that every point in the interior of C_2 is the orthocentre of some triangle inscribed in C_1.

90. In the figure $ABCD$ is a rectangle, find the radius of all circles.



Answer Keys

Chapter 1 POLYNOMIALS

Build-up Your Understanding 1

1. $x^4 - 20x^2 + 16 = 0$
2. $x^3 - 18x - 110 = 0$
3. $x^4 - 10x^3 + 32x^2 - 34x + 7 = 0$
5. $\frac{1}{2}$
8. 1984
9. k is a multiple of 3
10. $P(x) = a(x-2)(x-4)(x-8),\ a \in \mathbb{R}$

Build-up Your Understanding 2

1. $x = 1, 1, 1, 1$
2. $x = -1, -2, -3, -4$
3. $x = -\frac{1}{2}, \frac{4}{3}$
4. $x = -1$
5. $x = -\frac{1}{2}, \frac{1}{2}, b$; where $a = -4b,\ b \in \mathbb{Q}$
6. $a \in (-\infty, -6) \cup (2, \infty),\ b = -2a$
8. $8x^3 - 6x + 1 = 0$
9. $64x^6 - 96x^4 + 36x^2 - 3 = 0$
10. (a) $64x^6 - 96x^4 + 36x^2 - 3 = 0$

    (b) $8x^3 - 6x - 1 = 0$
11. (a) $3x^6 - 27x^4 + 33x^2 - 1 = 0$

    (b) $x^6 - 33x^4 + 27x^2 - 3 = 0$
12. $64x^6 - 96x^4 - 36x^2 - 1 = 0$
13. $-3, 24$
14. 86

Build-up Your Understanding 3

2. $a^{2n-1} + b^{2n-1} + c^{2n-1} = (a+b+c)^{2n-1},\ n \in \mathbb{N}$
3. -5
4. 135
5. 11182
6. $\frac{2}{3}, 4$
7. $x^3 - 9x^2 + 26x - 24 = 0$, 353
8. $\frac{209}{2}, 334$
9. $(x, y, z) = (a,0,0), (0,a,0), (0,0,a)$

11. $(x, y) = (3, 2), (2, 3),$
$\left(\frac{5-\sqrt{51}i}{2}, \frac{5+\sqrt{51}i}{2}\right), \left(\frac{5+\sqrt{51}i}{2}, \frac{5-\sqrt{51}i}{2}\right)$

12. 16, 81

Build-up Your Understanding 4

1. -2
2. $2, -2$
4. -1

Check Your Understanding

1. $\sqrt{2}$
2. 888883
3. $x \in [3, \infty)$
4. -5
6. $\pm\sqrt{2}$
7. $-\frac{3}{2}$
10. $\sqrt{p}$
13. $a = 3, k = 17$
15. 3
20. At most one positive root and at most three negative roots.
22. 899
28. $x = \frac{2abc}{ab+bc-ca}, y = \frac{2abc}{-ab+bc+ca},$
$z = \frac{2abc}{ab-bc+ca}$
29. $(x, y, z) = (a, b, -a-b)$ where $a, b \in \mathbb{R}$
30. $(x, y, z) = (-1, 3, 2), (1, -3, -2),$
$\left(-\frac{5}{\sqrt{13}}, \frac{11}{\sqrt{13}}, \frac{7}{\sqrt{13}}\right), \left(\frac{5}{\sqrt{13}}, -\frac{11}{\sqrt{13}}, -\frac{7}{\sqrt{13}}\right)$
31. $-\frac{1}{3}, 1, \frac{3}{2}, -4$
33. $(x, y, z) = \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$
36. $-1970, 500$
37. 41
40. $n = 1$
42. (i) $z^4 - 2(x^2 + y^2 - 2x^2y^2)z^2 + (x^2 - y^2)^2 = 0$
(ii) $(x, y) = (0, \pm 1), (\pm 1, 0)$ or $(\pm a, \pm a)$, where $a \in [-1, 1]$
44. 1996002
48. $x^2 - \frac{1}{2}$
49. $(a, b, c) = (-2, 0, 1), (2, 0, -1)$

Challenge Your Understanding

1. $P(x) = ax(x-1)(x-2)(x-3)\ldots(x-29), a \in \mathbb{R}$
2. $P(x) \equiv 0$
6. 5
7. $x^3 + 2x^2 + 2x + 2$
12. $\min(a^2 + b^2) = \frac{4}{5}; (a, b) = \left(-\frac{4}{5}, -\frac{2}{5}\right)$
13. $a_0 = 4, a_1 = -4^k, a_2 = \frac{4^{2k-1} - 4^{k-1}}{3}, a_{2k} = 1$
16. $a = b = c = d = e = 0, -\frac{1}{3}$ or $\frac{1}{3}$
17. $(x, y) = (0, 0), (19, 95)$
18. $(x, y) = (3, 4), (-3, -4)$
19. 8
20. 105336
21. 5, 5, −13, 5, 5, 5, −13, 5, 5, −13, 5, 5, 5, −13, 5, 5
22. $\frac{(C_1 - A_1)B_2 + (B_1 - C_1)A_2}{B_1 - A_1}$
26. $x = -2, 1 \pm \sqrt{5}$
27. $2008! - 1$
28. $\frac{1+(-1)^n}{n+2}$
29. $\frac{n+1+(-1)^{n+1}}{n+2}$
33. $\pm(x-1), \pm(x+1), \pm(x^2+x-1), \pm(x^2-x-1),$
$\pm(x^3+x^2-x-1), \pm(x^3-x^2-x+1)$
37. No
41. $f(x) \equiv 0;\ f(x) = -x^n(x-1)^n, n \in \mathbb{N}_0;$
$f(x) = -(x^2+x+1)^n, n \in \mathbb{N}_0$
42. $f(x) \equiv 0;\ f(x) = (x^2+1)^n, n \in \mathbb{N}_0$
43. $f(x) \equiv 0;\ f(x) = (x^2+1)^n, n \in \mathbb{N}_0$

44. $f(x) \equiv 0$; $f(x) = (-x)^m (1-x)^n (x^2+x+1)^p$ where $m, n, p \in \mathbb{N}_0$.

48. Only possible for $n = 2$ and 4
for $n = 2$, $(a_1, a_2) = (a, a+2)$, $a \in \mathbb{Z}$; for $n = 4$,
$(a_1, a_2, a_3, a_4) = (a, a-1, a+1, a+2)$, $a \in \mathbb{Z}$

Chapter 2 INEQUALITIES

Build-up Your Understanding 1

6. 2

Build-up Your Understanding 3

8. (a) 8 (b) $\frac{81}{4}$

Build-up Your Understanding 6

1. 1

Check Your Understanding

3. 2
19. 96 for $x = 4, y = 2, z = 4$.
24. Yes
33. $a = b = c = d$
34. $(x, y) = \left(-\frac{1}{2}, -\frac{1}{2}\right)$

Challenge Your Understanding

3. P becomes incentre
5. Equality never holds
6. $a = b = c$
18. Hypotenues $= \sqrt{30}$, Area $= 3\sqrt{6}$
26. a, b, c are negative
27. $a = b = c = d = 3$
31. $S \in (1, 2)$
32. 3

Chapter 4 RECURRENCE RELATION

Build-up Your Understanding 1

1. $a_n = 3 \cdot 2^{n-1} - 1,\ n \in \mathbb{N}$

2. $a_n = \sqrt{\frac{16}{5} + \frac{44}{5}\left(-\frac{1}{4}\right)^n},\ n \in \mathbb{N}$

3. $a_1 = 2,\ a_n = 2n - 1,\ \ n \geq 2$

4. $a_n = \frac{2}{n(n+1)},\ \sum_{k=1}^{n} a_k = \frac{2n}{n+1}$

5. $a_n = \frac{2}{n^2} - \frac{1}{2^{n-1}},\ n \in \mathbb{N}$

6. $a_n = (n-1!) \sum_{k=0}^{n-1} \frac{1}{k!}$

7. $a_n = n!\left(4 - \frac{n+3}{2^n}\right)$

8. $a_n = 1 - 2\left(\frac{1+(-1)^n}{n}\right)$

9. $a_n = 2\big((n-1)!\big) - 1,\ n \in \mathbb{N}$

10. $a_n = \frac{1}{n(n+1)},\ n \in \mathbb{N}$

11. $x_n = n^2(2^{n-1}+1),\ n \in \mathbb{N}$

12. $a_n = \frac{n^2-1}{3n^2},\ n \in \mathbb{N}$

Build-up Your Understanding 2

1. $a_n = \frac{3^{n-1}}{3^{n-1}+1}$

2. $a_n = \frac{1}{2 \cdot 3^{n-1} - 1},\ n \in \mathbb{N}$

3. $a_n = \frac{3^{n-1}}{1+3^{n-1}}$

4. $a_n = \frac{3 \cdot 2^{n-1}+1}{3 \cdot 2^{n-1}-1}$

6. $a_n = 3^{2^{-n}}$

7. $a_n = 2^{2^{n-1}-1},\ n \in \mathbb{N}$

8. $x_n = 1 - (1-x_1)^{2^{n-1}},\ n \in \mathbb{N}$

9. $a_n = 3n(n+1),\ n \in \mathbb{N};\ \sum_{k=1}^{n} \frac{1}{a_k} = \frac{n}{3(n+1)},\ n \in \mathbb{N}$

Build-up Your Understanding 3

1. $x_n = \dfrac{2^{n+1} + (-1)^n}{3}$
2. $a_n = 2 \cdot 3^n - (-1)^n$
3. $a_n = (n + 1)\, 2^{n-2}$
4. $a_n = (29n - 81)\,(-3)^{n-3},\ n \ge 0$
5. $a_n = 6^n + 1$
6. (a) $a_n = 3^{n+1} - 2^{n+1}$
   (b) $a_n = (3 - n)\, 3^{n-1}$
7. $a_n = 2 \cdot (-1)^{n+1} + 2(n - 1)2^n$.
8. $a_n = a + (b - a - 1)n + n^2$
9. $a_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{2n} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{2n}\right]$
10. $a_n = \dfrac{1}{n!}\left[2\left(\dfrac{2\sqrt{5}-2}{\sqrt{5}}\right)\left(\dfrac{-1-\sqrt{5}}{2}\right)^n + 2\left(\dfrac{\sqrt{5}+2}{\sqrt{5}}\right)\left(\dfrac{-1+\sqrt{5}}{2}\right)^n\right]$
11. $x_n = \cos(2^n \arccos x_0)$
14. $a_1 = 9,\ a_n = 6n+1,\ n \ge 2;$
    $\sum_{k=1}^{n} a_k^2 = 12n^3 + 24n^2 + 13n + 32, n \ge 1.$
15. $a_1 = 0,\ a_n = \dfrac{1}{n(n-1)},\ n \ge 2$
16. $c_1 = 9,\ c_n = \dfrac{3^n(n+1)(n+3)}{2n+5},\ n \ge 2;$
    $\sum_{k=1}^{n} a_k b_k c_k = \dfrac{29}{72} - \dfrac{1}{3^n(n+2)};\ \sum_{k=1}^{\infty} a_k b_k c_k = \dfrac{29}{72}$
17. $a_n = 2^{2^n - n}$
18. $a_n = 10^{\left(2 - \frac{1}{2^{n-2}}\right)}$
19. $a_n = 2^{\frac{n(n-1)}{2}} + \sum_{k=0}^{n-1} 2^{1+k\frac{(2n-1-k)}{2}}$
20. $a_n = \sum_{k=0}^{\infty}\left\lfloor \dfrac{n}{2^k} \right\rfloor$
    or $a_n = 2n - S_2(n)$, where $S_2(n)$ denote sum of the binary digits of n.

Build-up Your Understanding 4

1. $b_n = 2^{n+1} - n - 1,\ n \in \mathbb{N}$
2. $a_n = n2^{n-1},\ n \in \mathbb{N};\ \sum_{k=1}^{n} a_k = (n-1)2^n + 1,\ n \in \mathbb{N}$
3. $a_n = 3^n + 2^n$
4. $a_n = 2^n(2n^2 - 15n + 11),\ n \in \mathbb{N};$
   $\min a_n = a_5 = a_6 = -448$
5. $a_n = n^2 - 2^{n-1} + 1,\ n \in \mathbb{N}$
6. $a_n = 3^n + n^2$
7. $a_n = 5 \cdot 3^n + 2n + 3 - 6 \cdot 2^n$
8. $a_n = 2 \cdot 9^n + 7n$
9. $a_n = (n-2)2^n + 3,\ n \in \mathbb{N}$
10. $a_n = \dfrac{1}{n^2 - 2n + 3},\ n \in \mathbb{N}$
11. Sequence is periodic with period 5. As $2017 \equiv 2 \bmod 5$
    $\Rightarrow x_{2017} = x_2 = 2$

Check Your Understanding

1. (a) $a_n = 3 \cdot 2^n - 2 \cdot 4^n$
   (b) $a_n = 2 \cdot 3^n + n^2 + 3n - 1$
   (c) $a_n = (5n - 9)\, 3^{n-1} + 2^{n+2}$
2. $a_{2017} = 1$
4. $a_n = n \cdot n!$
5. $a_{2017} = 2^{2015} \cdot 2018 = 2^{2016} \cdot 1009$
6. $a_n = \dfrac{1}{n(n+1)}$
7. $a_1 = 1,\ a_n = (n-1)((n-1)!),\ n \ge 2$
8. $a_n = -\dfrac{n-1}{n}$

Challenge Your Understanding

2. $x_n = \dfrac{1}{5} - \dfrac{1}{5}(5x_0 - 1)^{2^n}$
3. $a_n = 2^{2^n - 1} - \dfrac{1}{2}$
4. (a) $n+2$ (b) $2n+3$ (c) $7 \cdot 2^n - 3$
5. $a_n = (1-(-1)^n) \cdot \dfrac{1}{2^n \cdot n}\dbinom{n-1}{\frac{n-1}{2}}\ \forall\ n \in \mathbb{N}$
6. (b) $\dfrac{\pi}{12}$
7. $T_n = n! + 2^n,\ n \ge 0$

8. $a_n = \dfrac{(a+1)^{2^{n-1}} + (a-1)^{2^{n-1}}}{(a+1)^{2^{n-1}} - (a-1)^{2^{n-1}}}$

9. $a_{2017} = 2017^2$

10. $a_n = \dfrac{4\sqrt{29}-15}{2\sqrt{29}}(5+\sqrt{29})^n + \dfrac{4\sqrt{29}+15}{2\sqrt{29}}(5-\sqrt{29})^n$

11. $p_n = 6 - 2^n,\ q_n = 2^{n-1} - 2$

12. $a_n = -(1+2n)2^n,\ b_n = (5+2n)2^n$

13. $2^{2^{2007}} + 1$

14. $x_n = \pm\sqrt{\dfrac{1}{n!}\sum_{k=0}^{n-1} k!}$, here for each term we can choose any sign.

15. $P_n(x) = \dfrac{x-2017}{x-2019}\,[(x-2018)^n - 1]$

Chapter 5 FUNCTIONAL EQUATIONS

Build-up Your Understanding 1

1. $f(x) = 0$
2. $f(x) = \dfrac{x^3 - x + 1}{2x(x-1)}$
3. 6044
4. $f(x) = 1$
5. $f(x) = a\cos x + b\sin x$
6. $f(x) = \dfrac{x+1}{x-1}$
7. $f(x) = \dfrac{2x^3 + x^2 + 5x - 2}{24x(x-1)}$

Build-up Your Understanding 2

3. $f(x) = x$
4. $f(x) = 0;\ f(x) = x$
5. No such function exist!

Build-up Your Understanding 3

1. 2^{1992}
4. $2^{1996} + 1$
6. $f(x) = x - \dfrac{3}{2}$
7. $f(x) = x + c$
8. $f(x) = \dfrac{1}{x+1}$

Build-up Your Understanding 4

1. $f(x) = cxa^x$
2. $f(x) = \sqrt{ax^2 + b}$
3. $f(x) = 1;\ f(x) = a^x - 1$
4. $f(x) = ax\ln|x|$
5. $f(x) = 0;\ f(x) = a^{\frac{x^2}{2}+bx}$, where $b \in \mathbb{R}^+$
6. $f(x) = \tan(ax)$

Build-up Your Understanding 5

1. $P(x) = x$
2. $f(x) = -\dfrac{x}{x+1}$
3. No such function exist!

Check Your Understanding

1. $f(x) = \dfrac{(x-1)^3}{1-c}$
2. $\dfrac{2011}{2012}$, 2013
4. 1173
5. $\cos((4k+1)x) = f(\cos x) \Leftrightarrow \sin((4k+1)x) = f(\sin x)$
6. (b) $\min(a) = 3$
7. $\dfrac{n-1}{2}$
8. $\dfrac{2}{n(n+1)}$
9. 9
12. $f(x) = x;\ f(x) = \dfrac{1}{x}$
13. $f(n) = n$
14. $f(x) = x;\ f(x) = x + 1$
15. $f(x) = x$

Challenge Your Understanding

1. $P(x) = (x-2)(x-4)(x-8)(x-16)$
2. $f(x) = ax + b + c;\ g(x) = ax + b;\ h(x) = ax + c$

3. $f(x) = -\dfrac{2}{(1+\sqrt{5})x}$

4. $f(n) = \dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)$

5. $(x, y, z) = (0, 0, 0), \left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$

6. $f(x) = \dfrac{3}{2}x + \dfrac{3}{2};\ g(x) = \dfrac{5}{2}x;\ h(x) = -x + \dfrac{1}{2}$

7. No

8. $f(n) = n$

9. $f(x) = \begin{cases} 0, & x = 0 \\ 1, & 0 < x < 2 \\ \dfrac{1}{2}(1 + 3^{n+2} - 2^{n+3}), & 2^n \le x < 2^{n+1}, n \ge 1 \end{cases}$

10. 127

11. $f(n) = \begin{cases} n + 3^m, & 3^m < n < 2 \times 3^m \\ 3n - 3^{m+1}, & 2 \times 3^m \le n \le 3^{m+1} \end{cases}$

$f(2016) = 3861$

12. $f(x) = 1 - x$

13. $f(x) = 1 - \dfrac{x^2}{2}$

14. $f(x) = 0;\ f(x) = \dfrac{1}{2};\ f(x) = x^2$

15. $f(n) = n$

Chapter 6 NUMBER THEORY

Build-up Your Understanding 1

4. n can be only 2, 5, 11, 29 and corresponding expression will have values: 1, 3, 6, 25

Build-up Your Understanding 2

8. 28 at $a = 23$, $b = 5$
16. 42
18. $(a, b) = (1, 5), (14, 5)$
24. 3

Build-up Your Understanding 3

7. $p = 3$, $q = 2$
8. 6
9. 5
11. $(p, q) = (3, 11), (11, 3), (r, r)$ where r is a prime number
12. $(p, q, r) = (2, 3, 5), (3, 2, 5)$

Build-up Your Understanding 4

1. 1999
2. 256
3. 661
4. $2^5 \times 3^2 \times 5 \times 7 \times 11 = 110880$

Build-up Your Understanding 5

1. (a) $x \equiv 14 \pmod{21}$
   (b) $x \equiv 1 \pmod{8}$
   (c) No solution
   (d) $x \equiv 99 \pmod{105}$
2. 69
3. 00
4. 4
8. Same as for 7
13. $(a, b, c) = (mn, n, n), (n, mn, n), (n, n, mn);\ m, n \in \mathbb{N}$

Build-up Your Understanding 6

4. 1024
5. 49
6. 143
7. 0. Also last five digits 03125
9. n = odd or multiple of 8
15. 29348, 29349, 29350, 29351; In general 44100 m + 29348, 44100 m + 29349, 44100 m + 29350, 44100 m + 29351, $m \in \mathbb{N}_0$
16. $x \equiv 653 \pmod{770}$
17. $x \equiv 25 \pmod{60}$, Minimum number of students = 25.
18. $x \equiv 3930 \pmod{4080}$, Minimum number of coins = 3930.
20. (a) 9 (b) 7
21. 1
24. (c) $n = 1, 2^a 3^b;\ a \in \mathbb{N}, b \in \mathbb{N}_0$

Build-up Your Understanding 7

1. 1
3. $9 \times 98765432 = 888888888$;
$V+E+X+A+T+I+O+N$
$= 9+8+7+6+5+4+3+2 = 44$
7. There exist no such b.
8. (a) 625×10^n, $n \in \mathbb{N}$

Build-up Your Understanding 8

6. $x = 100.15$, $y = 100.95$, $z = 99.05$
7. 2499
8. $x = 29/12$, $19/6$, $97/24$
12. 1500
13. 330
14. 250
15. 1210, 1211, 1212, 1213, 1214
16. 781
20. $2^k \,\Big\|\, \left\lfloor (1+\sqrt{3})^{2n+1} \right\rfloor \Rightarrow \max k = n+1$. But for first it depends on n even or odd.
21. (i) $n \in \mathbb{N}$ (ii) $n \neq 2^k$, $k \in \mathbb{N}_0$
22. 43

Build-up Your Understanding 9

10. $(a, b, c) = (0, 0, 0)$
12. 12, 16, 60, 144, 320, 588, 1936.
13. $n = 3, 41, 119$
14. $n = 0, 280$
15. 861, 168, 259 and 952
18. 3 pages
20. $(x, y, z) = (0, 0, 0)$
21. $(x, n) = (59, 12), (-59, 12)$
22. $(x, y, z) = (0, 1, 2), (3, 0, 3), (4, 2, 5)$,
23. $(x, y, z) = (1, 1, 18), (-1, -1, 18), (2, 2, 3), (-2, -2, 3)$
24. No solution
25. $(x, y, z) = (5, 8, 11), (5, 11, 8), (8, 5, 11), (8, 11, 5), (11, 5, 8), (11, 8, 5)$.
26. $(x, y) = (-11, 0), (0, 11)$
28. Primitive solution set
$(x, y, z) = (|a^2 - 2b^2|, 2ab, a^2 + 2b^2)$, $\gcd(a, b) = 1$

Check Your Understanding

2. (a) $28^2 - 11^2$, $32^2 - 19^2$, $112^2 - 109^2$, $774^2 - 773^2$
(b) $43^2 - 9^2$, $47^2 - 21^2$, $223^2 - 219^2$, $443^2 - 441^2$
4. $(x, y) = (9, 4)$
9. $n = 12$
10. 1972
12. $(x, y) = (1, 0), (1, -2), (-2, -3)$.
13. 600
16. 36
18. No such number
21. $N = 2 \times 3 \times 5 = 30$
27. 1995
28. $t_1 = 2012$, $t_2 = 5^2 = 25$, $t_3 = 7^2 = 49$, $t_4 = 13^2 = 169$, $t_5 = 16^2 = 256$, $t_6 = 13^2 = 169$ and so on sum $= 429211$
29. Number of medals is 30 and medals awarded on the successive days are 16, 8, 4, 2.
30. (12, 16) , (4, 48)
31. $(a, b, c, d) = (1, 1, 2, 6); (2, 2, 2, 5); (2, 2, 3, 3)$
32. No such number.
36. 729
38. 44 years
39. $x = y = z = 1$ or $x = y = z = -2$
40. $(a, b) = (9, 1), (8, 2)$
41. Smallest possible value of c is 675.
42. There are no consecutive integers of this type.
43. 86
51. $(x, y) = (20, 40); (8,44)$
52. 3
57. $(x, y) = (9, 11)$
58. Few such six digit numbers are 145690, 235780.
59. (i) Min $ab = 10$ at $(a, b) = (1, 10), (10, 1)$; (ii) Min $ab = 20$ at $(a, b) = (4, 5),(5, 4)$

Challenge Your Understanding

3. The Funny Numbers' are 2, 3, 5, 7, 23, 37, 53, 73, 373 (in all 9 numbers).
4. (ii) 120
6. $142857 \times 5 = 714285$
7. 2013
10. $(b, c) = (30, 60), (35, 140), (36, 180), (38, 380), (39, 780)$
12. 550, 803
15. $(a, b) = (18, 1)$
16. $T = 174, 175, 339, 505$
17. 1978, 1981, 1984, 2002
18. The sequence which is square free is
202, 291, 445, 581, 869, 949, 1207, 1273, 1403, 1711, 1643, 1739, 1763 (13 terms)

19. $1503^2 = 2259009$
20. $n + 1$
21. $n = 1, 3$
25. 1989
26. (i) For every n, there exits S_n. Define LCM $(1, 2, 3, \ldots, n) = l$.
Now $S_n = \{1^l, 2^l, 3^l, ..., n^l\}$
(ii) No
27. 337
30. $F_4 = 3$
32. $n \in \mathbb{N}$
37. 2, 3, 6
38. 5, 6, 7, 7, 8, 10, 12, 13, 14, 15
44. $(a, b, c) = (2, 4, 13), (2, 5, 8), (3, 3, 7)$
45. (2, 2, 3), (1, 3, 8), (1, 4, 5), and their permutations.
47. $B = \{2, 3, 4, 5, 6\}, \{2, 5, 8, 9\}, \{3, 4, 6, 10\}$
50. 1996002
51. $(m, n) \equiv (a, -a), \left(\frac{(a+1)(a+2)}{2}, \frac{a(a+1)}{2}\right),$
$\left(\frac{a(a+1)}{2}, \frac{(a+1)(a+2)}{2}\right), a \in \mathbb{Z}$
54. $n = 1, 2$
57. Write each of the terms of these sequences in (mod 8) and use the proof by induction to show the result.
59. 12

Chapter 7 COMBINATORICS

Build-up Your Understanding 1

1. (a) 1296, (b) 360
2. 9(9!)
3. 240
4. 376
5. (a) 60, (b) 107
6. 286
7. 15
8. Time required $= \frac{15 \times 15 \times 15 - 1}{2} \times \frac{10}{60 \times 60} = \frac{1687}{360}$ hrs.
$\approx$ 4 hrs. 41 min. 10 Seconds $> 4\frac{1}{2}$ hrs.
9. 720
10. 18
11. 64800
12. 505
13. 69760
14. 162
15. 3×4^4
16. 45×10^4
17. 216
18. 36
19. 108
20. 1620
21. 103
22. 154
23. 1020
24. $4 \cdot 7!$
25. $17 \cdot 8!$
26. 8!
27. 2^n
28. 91.
29. $n^m - 1$
30. $6^n - 3^n$
31. 300
32. 300
33. 31
34. 15
35. 134055
36. 1769580
37. 6399960
38. 2239986
39. Except $5k + 1$, for $k = 0, 1, 2, ..., 199$, all numbers will be unmarked.
40. 180

Build-up Your Understanding 2

1. (i) $n = 5$ (ii) $n = 7$
2. 8
3. ${}^{20}C_{10}$
4. (a) 20 (b) 21 (c) 10
5. ${}^{25}C_5, {}^{24}C_4$
6. 10
7. 226
8. 378
9. 16
10. 1512
11. 124
12. 292
13. 135
15. ${}^{20}C_{10} \cdot 2^{10}$
16. (i) 243 (ii)1, 10, 40, 80, 80, 32

17. $p = {}^5C_4 \cdot {}^2C_1 = 10$, $q = {}^5C_2\,({}^2C_1)^3 = 80$
$r = {}^5C_0({}^2C_1)^5 = 32$
$\Rightarrow 2q = 5r$, $8p = q$, and $2(p + r) > q$
18. 6
19. 1023
20. 126
21. $(p + 1)^n - 1$
22. $2^n - 2n - 2$
23. $(m + 1)\, 2^n - 1$
24. 3150
25. 25
26. ${}^nC_2$
27. 37
28. 20
29. 9
30. 16
31. 126
32. 6
33. 72
34. 5
35. 945
36. n^m
37. 91
38. ${}^{n-1}C_2$
39. mk
40. 2^{2n}
41. 2^{n-1}
42. ${}^nC_2 \cdot 3^{n-2}$
43. $\dfrac{(n+1)(n+2)(2n+3)}{6}$
44. 23
45. $63\times121\times31 = 3^2\cdot7^1\cdot11^2\cdot31$
46. 84
47. 276
48. ${}^{11}C_6$
49. $\dfrac{(m+n-2)!}{(m-1)!(n-1)!}$
50. 5
51. Total number of different tickets = 30 and number of selection = ${}^{30}C_{10}$
52. ${}^{10}C_3$
57. 15
58. $2^9 - 1$
59. 560
60. 140

Build-up Your Understanding 3

1. (a) 4
(b) 3
(c) 8
2. ${}^6P_3$
3. ${}^6P_3\times{}^5P_3\times{}^4P_3$
4. 50400
5. ${}^{10}C_6\times{}^4C_3\times9!$
6. 900
7. 40
8. 30
9. $\dfrac{11!}{(2!)^3}$, $\dfrac{8!}{(2!)^2}\times12$
10. 8!4!
11. (a) 7!, (b) 6!, (c) 5!, (d) 6!2!
12. ${}^8C_4\cdot 4!$
13. 719
14. 3600
15. 1800
16. Number of ways $= n + n^2 + \cdots + n^r$
17. $\dfrac{n^r(n^{n-r+1}-1)}{n-1}$ and $\dfrac{n^{r+1}-1}{n-1}$
18. 2
19. ${}^7C_2\, 2^5$
20. 20
21. $2(n!)^2$
22. $2\cdot6!\cdot6!$
23. 20
24. $\dfrac{10!}{2}$
25. $3n^2 - 2n$
26. $m(m - 1)(n - 5)^{m-2}$
27. 1440
28. 172800
29. 528
30. 1620
31. 43200
32. ${}^{10}C_3\times2\times7!$
33. 24
34. 185
35. 2454
36. 758
37. 917
38. 89
39. 236

40. (a) 213564, (b) 267^{th}
41. 24678
42. (i) 72^{nd}, (ii) 51342
43. 3840
44. 32
45. 8
46. $6(7! - 4!)$
47. $8!$
48. $\dfrac{9!}{3}$
49. 36×55^3
50. ${}^{14}C_5$

Build-up Your Understanding 4

1. $10!$
2. $20!$, $2 \cdot 18!$
3. (a) 240 (b) 480
4. (a) $2 \cdot 18!$ (b) $19! - 2 \cdot 18!$
5. (a) $2 \cdot 18!$ (b) $19! - 2 \cdot 18!$
(c) $18!\ (1/2)\{19! - 2 \cdot 18!\}$
6. (i) $(2n - 2)! \times 2$ (ii) $(2n - 2)!$
7. ${}^{10}C_2 \times 2! \times {}^{10}C_8 \times 8!$
8. 288
9. $\dfrac{24}{25}$
10. 18
12. 3
13. 225
15. 30
16. $\dfrac{(n-1)!}{r!}$

Build-up Your Understanding 5

1. $\dfrac{15!}{8!4!3!}$
2. $\dfrac{(8!)}{(3!)^2(2!)}$
3. $\dfrac{14!}{(2!)^5 \cdot (3!)^2 \cdot 4!}$
4. $\dfrac{k}{3!}$
5. $\dfrac{16!}{4!\,5!\,7!}$
6. 25200
7. 2940
8. $m^n - m$
9. 20^3, 19^2
10. $\dfrac{10\,!}{2\,!\ 3\,!\ 5\,!}$
11. 210
12. 125, 60
13. $n!\ {}^{n}C_2$
14. 5^7
15. $L = {}^{p+q}C_p \cdot {}^{q}C_q,\ M = {}^{p+q}C_p \cdot {}^{q}C_q \times 2!,\ N = {}^{p+q}C_p \cdot {}^{q}C_q \cdot \Rightarrow$
$L = M/2 = N$
$\Rightarrow 2L = M = 2N$

Build-up Your Understanding 6

1. 286
2. 4851
3. ${}^{17}C_2$
4. ${}^{27}C_3$
5. ${}^{28}C_4$
6. 13
7. $\dbinom{11}{3}$
8. 210
9. 100
10. 56
11. 330
12. ${}^{52}C_2$
13. ${}^{27}C_4$
14. 685
15. ${}^{5}C_2 \cdot {}^{10}C_3 + {}^{9}C_2 \cdot {}^{6}C_3 + {}^{23}C_2 \cdot {}^{4}C_1 + {}^{24}C_3 \cdot {}^{3}C_1$
16. The possibilities are (0, 10, 0), (2, 7, 1), (4, 4, 2) and (6, 1, 3), where (r, b, g) denotes the number of red, blue and green balls.
17. $\dfrac{(n+2)(n+1)}{2}$
18. (i) 35 (ii) 47 (iii) $\dbinom{8}{2}$
19. 110551
20. ${}^{9}C_4$
21. ${}^{93}C_3$
22. 10
23. 246
24. 15
25. 27
26. 1875
27. 64

28. $2^{10}\binom{15}{6}\binom{12}{3}\binom{13}{4}$

29. $2^{9}\binom{15}{6}\binom{12}{3}\binom{13}{4}$

30. $\sum_{r=0}^{9} 2^{10-r}\binom{10}{r}\binom{99}{9-r}$

Build-up Your Understanding 7

1. 28
2. 134
3. 33
4. $6^n - 5^n - 5^n + 4^n$
5. 738
6. 99989526
7. (a) $(1 + 26 + 26^2 + 26^3) \cdot (1 + 10 + 10^2 + 10^3 + 10^4) - 1$
(b) $(1 + 26 + 26^2 + 26^3 - 85) \cdot (1 + 10 + 10^2 + 10^3 + 10^4) - 1$
8. 5
9. 24×13^4
10. 2301
11. $7! - 5! - 5! + 3!$
12. 169194
13. 864
14. 10
15. 485
16. 540
17. ${}^9C_2 \cdot 360 + {}^9C_3 \cdot 540 = 58320$
20. 5400
22. 191
23. 101
24. 233
25. 144
26. 44

Build-up Your Understanding 8

1. $2^n - 2$
2. $2^{n-1} - 1$
3. 771
4. 540
5. 141
6. 462
7. $\dfrac{12!}{4!}$
8. ${}^9C_5$
9. 7000
10. 440
11. (i) ${}^{19}C_3$ (ii) ${}^{15}C_3$ (iii) ${}^{11}C_3$ (iv) ${}^7C_3$
13. (a) $3^{15} - 3 \cdot 2^{15} + 3$, (b) 2250
14. 11508
15. ${}^{12}C_5 \cdot 2^7$
16. (i) 150 (ii) 6 (iii) 25 (iv) 2
17. 1275
18. $\dfrac{25!}{5!}$
19. (a) 4^9 (b) 186480 (c) $4 \times ({}^9C_4)^2 = 63504$

Check Your Understanding

8. $2 \times 4^{n-1}$
10. $2^n - 1$
11. (i) 11111111111
(ii) 999999999999
15. $45 \times n \times 10^{n-1}$
16. $f(n) = \begin{cases} \dfrac{n+2}{2}, & n \equiv 0 \pmod 2 \\ \dfrac{n+1}{2}, & n \equiv 1 \pmod 2 \end{cases}$
18. For $a = 2^m + \alpha, 0 < \alpha \le 2^m, b = 2^n + \beta, 0 < \beta \le 2^n$, $c = 2^p + \gamma,\ 0 < \gamma \le 2^p$, minmum number of cuts $= m + n + p + 3$
19. 462
26. First player has the advantage if he start with 8.
28. 229
29. 35, 37, 40, 8, 0 for $m = 0, 1, 2, 3, 4$ respectively.
30. 2047
31. 37
32. 4351
33. 840
34. 715
35. 171700
36. The equal score can be either 4 or 8 according as the number of participants of std. XII is either 7 or 14.
37. Each of the smaller triangles can cover only one vertex of the larger triangle.
43. 1405
46. 450
48. m^2n^2
49. 12
50. 36
51. 32
52. 7
53. $(n + 1)^2$

54. 56
55. 147
56. 57
58. 473
59. 8
60. 12

Challenge Your Understanding

3. (i) 3 (ii) 3
5. Consider the following 18 subsets: {1}, {4, 100}, {7, 97}, …, {49, 55}, {52}. Now from each subset pick one element.
6. $N - 2$
7. ${}^{18}C_9 - 1, 511$
8. (i) m^n, (ii) ${}^mC_n \times n!$, (iii) ${}^mC_n$, (iv) ${}^{m+n-1}C_n$
9. 262
29. 60
30. $\sum_{r=0}^{n}\left((-1)^r \binom{n}{r}((n-r)!)^2\right)$
35. $a_n = \frac{1}{6}(2(-1)^n + (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1})$
38. *EDACB*
39. 315
41. 6
42. 6 days and 36 medals
46. $n \equiv \pm 1 \pmod 3$
47. 8
48. 33

Chapter 8 GEOMETRY

Build-up Your Understanding 1

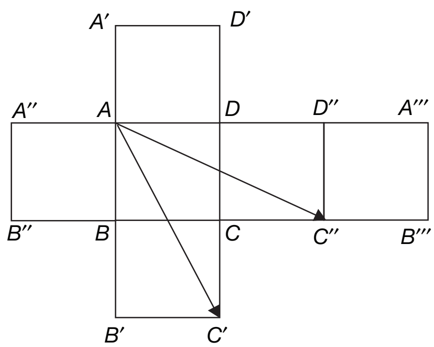
1. 30°, 60°, 90°
2. No
3. 50°, 70°, 60°
4. 45
5. 12°
7. 10
8. 30
9. 9
10. 15 and excluded angle 130°
11. 13
12. 3
13. 540°
18. 12, 12, 3; 5, 5, 10
19. $(a, b, c) \equiv (3, 7, 42), (3, 8, 24), (3, 9, 18), (3, 10, 15), (3, 12, 12), (4, 5, 20), (4, 6, 12), (4, 8, 8), (5, 4, 20), (5, 5, 10), (6, 4, 12), (6, 6, 6)$
20. $\angle BCA = 60°, \angle DBC = 10°$.

Build-up Your Understanding 2

4. 30°.
8. 45°
9. 1
10. 45°
12. 2
13. 60°
17. 45°
18. 30°

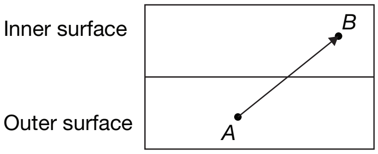
Build-up Your Understanding 3

1. 6
2. 7
3. 35
5. Point of intersection of diagonals *AC* and *BD*
8. *P* is the point of intersection of perpendicular bisector of *AB* with the line in both cases.
11. Take reflection of *A* in both arms of the angle and Join reflections. Let this line meets arms of the angle at *B* and *C* respectively. Now make the triangle *ABC*.
14. Open the cube as shown in the following figure.



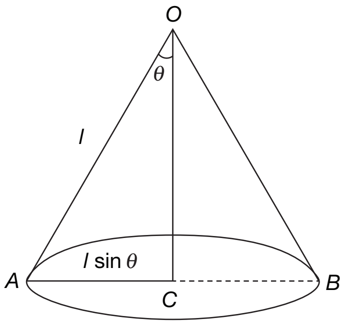
This is a flat diagram of a cube net, such that you could cut it out and fold it to make the cube. In the figure there are two acceptable routes, we can easily see that there are in total six such routes. Through each root we will travel $\sqrt{5}$ units assuming side of the cube 1 unit.

15. Open the surface so that glass become flat as shown in the figure.

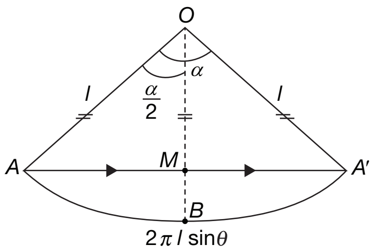


16. Cut the cone through a generatrix passing through the vertex and make it flat as shown:

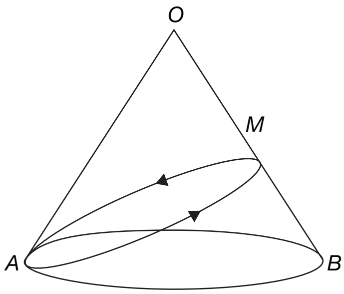
(i) For $\theta < 30°$



(ii)



(iii)



$$\alpha = \frac{\text{arc}}{\text{radius}} = \frac{2\pi l \sin\theta}{l} = 2\pi \sin\theta$$

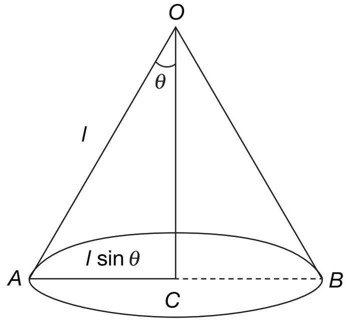
$$\Rightarrow AA' = 2AM = 2l\sin\frac{\alpha}{2} = 2l\sin(\pi\sin\theta)$$

Shortest path is $AA' = 2l\sin(\pi\sin\theta) \le 2l$, for $\theta < 30°$.

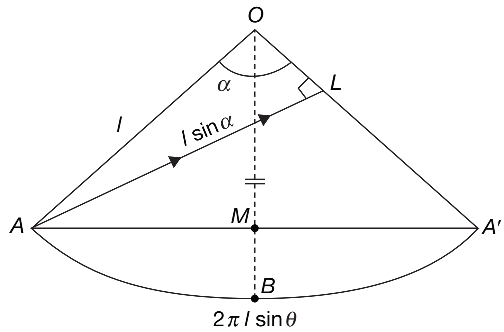
Shortest path is $AOA' = 2l > 2l\sin(\pi\sin\theta)$, for $\theta \ge 30°$.

17. Cut the cone through a generatrix passing through the vertex and make it flat as shown:

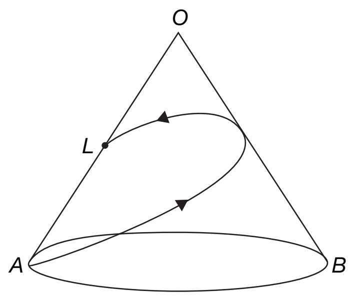
(i)



(ii)



(iii)



Let AL be perpendicular to OA' at L.

Then $AL = l\sin\alpha = l\sin(2\pi\sin\theta) < l$, for $\theta < \sin^{-1}\left(\frac{1}{4}\right)$

AL will be the shortest path.

19. P will be at the vertex of the triangle containing smallest angle.

Build-up Your Understanding 4

1. $\dfrac{\lambda}{\mu(1-\lambda)}$

2. $\dfrac{19}{96}$

3. 315

4. $\dfrac{4}{13}$

6. 4500.

8. (i) $\frac{AR}{RD}=\frac{2\mu-1}{2\lambda-1};\frac{BS}{CS}=\frac{2\lambda-1}{2\mu-1}$

9. $\frac{9}{5}$

11. $[ABCDE]=(\sqrt{5}+5)/2$

13. $\frac{27}{160}$

14. 441

15. $\frac{7}{5}$

17. Equality holds when $\frac{AP}{PD}=\frac{BP}{PE}=\frac{CP}{PF}$

Build-up Your Understanding 5

1. $\frac{9}{4}$

2. $\frac{3}{2}$

10. $PQ=4, XY=2$

11. 45°–45°–90° triangle and quadrisected angle is right.

20. 12 cm

25. 20°

Build-up Your Understanding 6

1. $5\sqrt{2}$

3. $a^2\sqrt{5}$

4. 108

5. 15

6. $7\sqrt{2}, 6\sqrt{2}$

7. $\frac{4\sqrt{13}}{5};\frac{126}{25}$

8. (ii) $OG=\sqrt{R^2-\frac{1}{9}(a^2+b^2+c^2)}$

Build-up Your Understanding 7

6. $\frac{1}{5}$

7. 1 unit square

8. $\frac{140}{3}$ unit square

11. 20 unit square

12. Equality holds if and only if L coincides with A, *i.e.*, $AB=AC$.

Build-up Your Understanding 9

2. $\frac{AX}{XD}=\frac{n(m+1)}{m}$

7. $\frac{\sqrt{3}}{3}$.

Build-up Your Understanding 10

5. $4+3\sqrt{3}$

6. 60°

11. $\frac{EG}{EF}=\frac{t}{1-t}$

Build-up Your Understanding 11

3. $\frac{5}{3}$

7. $5\sqrt{2}$

Build-up Your Understanding 12

9. 11 units

12. $\frac{2r_2r_3\left(\sqrt{r_1+r_2+r_3}+\sqrt{r_1}\right)^2}{(r_2+r_3)^2}$

Build-up Your Understanding 14

3. $\frac{25\sqrt{3}}{4+3\sqrt{3}}$

9. Equality for C being the mid-point of the major segment.

Build-up Your Understanding 15

5. $\frac{313}{338}$

8. (5, 5, 6), (5, 6, 5), (6, 5, 5)

Build-up Your Understanding 16

6. Shortest side is 10 units and area is 84 sq. units.

7. P is the centroid of ΔABC.

Build-up Your Understanding 17

6. $a=13, b=15, c=14$

18. (i) 42 (ii) 65/8

22. (i) $r=\sqrt{\frac{xyz}{x+y+z}};\ R=\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}$

Build-up Your Understanding 18

3. Let 'a' be the measure of a side, b the shortest diagonal and d the longest diagonal of a regular nonagon, then $a + b = d$.

9. 7

Check Your Understanding

4. The locus is the solid annulus centered at P with inner radius 1 and outer radius 2.

7. $\frac{3}{5}$

12. Locus of P is a circle passing through A, E, C.

13. $\frac{28}{3}$

21. Locus is a circle with OB as a diameter.

27. 45°.

43. The minimum is attained when $ADPE$ is a cyclic quadrilateral.

45. $\frac{1}{\sqrt{3}}$

48. Equality holds when $b = c$, *i.e.*, when the right ΔABC is isosceles also.

50. Equality holds when $a = b = c$, *i.e.*, when the Δ is equilateral.

51. Equality holds when K, L, M are the mid-points of the sides BC, CA, AB respectively.

52. $ABCD$ is a rhombus.

57. Regular polygon with 3, 4, or 6 sides are possible.

58. 996

62. 90°

66. Perimeter of $KLMN$ $= 2(2+\sqrt{2}) > 2\pi =$ Circumference of the circle.

69 $\dfrac{p\sin 2\alpha \sin\beta - q\sin\alpha\sin 2\beta}{2\sin(\alpha-\beta)}$

70. 120°

Challenge Your Understanding

5. Locus is a line perpendicular to OA, at A' where A' be the point on OA extended beyond A such that $OA \times OA' = r^2$, O be the center of the given circle and r be the radius.

10. 30°

13. 12°

22. There are exactly 7 equalizing planes.

32. $PA + PB + PC$ is minimized when P coincides A.

34. There are only two tangents with integer length, *i.e.*, 441, and 1960.

36. When P is the incentre of ΔABC.

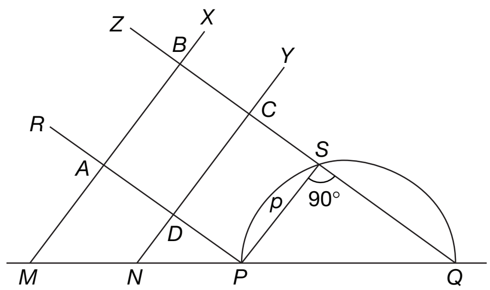
43. The only triplet forming a right triangle according to the given condition is the $3 - 4 - 5$ triangle.

52. If O be in Z part then maximum $\frac{\pi+2}{\pi-2}$ and minimum 1. If O be in Y part then maximum 1 and minimum $\frac{\pi-2}{\pi+2}$.

56. $AB = \frac{12}{\sqrt{5}}$; $BC = \frac{24}{\sqrt{5}}$; $CA = 12$

53. Construct a right ΔPSO (by constructing a semi-circle on PQ, we get $\angle PSQ = 90°$) with $\angle PSQ = 90°$ and $PS = p$.

Through M and N draw lines MX and NY parallel to PS and through P and Q draw lines PR and QZ perpendicular to MX and NY meeting them at $ABCD$. $ABCD$ is the required rectangle.



In the right angled ΔPQS, PQ is the hypotenuse hence $PQ > PS = p$. Thus, the construction of ΔPSQ is possible only if $PQ > p$.

By constructing the semicircle on the other half plane determined by l, we get a rectangle say $A'B'C'D$ which is the reflection of $ABCD$ about the line l. Thus there are two solutions.

73. 75°

76. 60°

78. 2 : 1

90. 36, 16, $\frac{81}{4}$

* *Comprehensive solutions to all exercises are available on the companion website.*

Appendix

Notations, Symbols and Definitions

A.1 Glossary of Notation

$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	The set of all non-negative integers.
P	The set of all prime numbers which are 2, 3, 5, 7, 11, 13, 17, . . . Note that $1 \notin P$. We call 1 a 'unit', it is neither prime nor composite.
$\mathbb{Z}_n$	The collection of all remainders of any integer divided by 'n' which are $0, 1, 2, 3, \ldots, n-1$
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\lvert A\rvert$	Cardinality of a set A or the number of elements in A
$[a, b]$	All x such that $a \le x \le b$ (closed interval)
$]a, b[$ or (a, b)	All x such that $a < x < b$ (open interval)
$a \mid b$	a divides b or b is a multiple of a
$a \nmid b$	a does not divide b
(a, b)	Greatest common divisor (gcd)
$\lfloor x \rfloor$	Integer part of x or the largest integer less than or equal to x
$\{x\}$	Fractional part of x
$a \equiv b \pmod{c}$	'c' divides $(a - b)$
$\Leftrightarrow$	If and only if (iff)
$\approx$	Approximately equal to
$\equiv$	Identically equal to
Σa_i	Sum $a_1 + a_2 + a_3 + \cdots + a_n$
$n!$	n factorial, *i.e.*, $1 \cdot 2 \cdot 3 \cdots n$

$\binom{m}{n}$ or ${}^mC_n$	The binomial coefficient; the number of combinations of m things taken 'n' at a time, *i.e.*, $\binom{m}{n} = \frac{m!}{n!(m-n)!}$
$f \circ g$	Composition of the functions f and g; $f \circ g(x) = f(g(x))$
$[ABC]$	Area of ΔABC
AB	The segment AB; also the length of segment AB
$\overrightarrow{AB}$	The vector AB

A.2 Glossary of Symbols

α	alpha	β	beta
γ	gamma	δ	delta
ε	epsilon	θ	theta
ι	iota	κ	kappa
λ	lamda	μ	mu
ν	nu	π	pi
ρ	rho	σ	sigma
τ	tau	ψ	psi
ω	omega	φ	phi

A.3 Glossary of Definitions

Trigonometry

1. **Trigonometric ratios of the sum and difference of two angles:**
   - $\sin(A+B) = \sin A \cos B + \cos A \sin B$
   - $\sin(A-B) = \sin A \cos B - \cos A \sin B$
   - $\cos(A+B) = \cos A \cos B - \sin A \sin B$
   - $\cos(A-B) = \cos A \cos B + \sin A \sin B$
   - $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
   - $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
2. **Product to sum formulae:**
   - $2\sin A \cos B = \sin(A+B) + \sin(A-B)$
   - $2\cos A \sin B = \sin(A+B) - \sin(A-B)$
   - $2\cos A \cos B = \cos(A+B) + \cos(A-B)$
   - $2\sin A \sin B = \cos(A-B) - \cos(A+B)$
3. **Product to sum formulae:**
   - $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
   - $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$
   - $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
   - $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$

4. Trigonometric ratios of multiple angles:

- $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A = 4\sin(60° - A)\sin A \sin(60° + A)$
- $\cos 3A = 4\cos^3 A - 3\cos A = 4\cos(60° - A)\cos A \cos(60° + A)$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A} = \tan(60° - A)\tan A \tan(60° + A)$

5. Maximum and minimum values of some trigonometric functions:

- Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$.
- Maximum and minimum value of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.
- If $\alpha, \beta \in \left(0, \dfrac{\pi}{2}\right)$ and $\alpha + \beta = c$ (constant) then the maximum values of the expression $\cos\alpha\cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha\sin\beta$ and $\sin\alpha + \sin\beta$ occurs when $\alpha = \beta = \dfrac{c}{2}$.
- If $\alpha, \beta \in \left(0, \dfrac{\pi}{2}\right)$ and $\alpha + \beta = c$ (constant) then the minimum values of the expression $\sec\alpha + \sec\beta$, $\operatorname{cosec}\alpha + \operatorname{cosec}\beta$, $\tan\alpha + \tan\beta$ occurs when $\alpha = \beta = \dfrac{c}{2}$.
- If A, B, C are the angles of a triangle then maximum value of $\sin A + \sin B + \sin C$ and $\sin A \sin B \sin C$ occurs when $A = B = C = 60°$.
- In case a quadratic in $\cos\theta$ or $\sin\theta$ is given then the maximum or minimum values can be interpreted by making a perfect square.

6 Trigonometric ratios of the sum of three angles:

- $\sin(A + B + C)$
  $= \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$
- $\cos(A + B + C)$
  $= \cos A\cos B\cos C - \sin A\sin B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C$
- $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

7. Sum of sines or cosines of *n* angles:

- $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \cdots + \sin(\alpha + (n-1)\beta)$
  $$= \frac{\sin\left(n\left(\frac{\beta}{2}\right)\right)}{\sin\left(\frac{\beta}{2}\right)}\sin\left(\alpha + (n-1)\frac{\beta}{2}\right).$$
- $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + (n-1)\beta)$
  $$= \frac{\sin\left(n\left(\frac{\beta}{2}\right)\right)}{\sin\left(\frac{\beta}{2}\right)}\cos\left(\alpha + (n-1)\frac{\beta}{2}\right).$$

8. Conditional identities:

In a triangle ABC we have following:

- $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
- $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\sin 2A \sin 2B \sin 2C = 4 \sin A \sin B \sin C$
- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
- $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

Geometry

1. Pythagoras's theorem and its converse:

Given any ΔABC, with sides a, b, c and angles $\angle A$, $\angle B$, $\angle C$, we have:
- $a^2 + b^2 > c^2 \Leftrightarrow \angle C$ is acute,
- $a^2 + b^2 = c^2 \Leftrightarrow \angle C$ is a right angle,
- $a^2 + b^2 < c^2 \Leftrightarrow \angle C$ is obtuse.

2. Apollonius theorem:

If D is the mid-point of the side BC of ΔABC, then, $AB^2 + AC^2 = 2(AD^2 + BD^2)$

An important consequence:

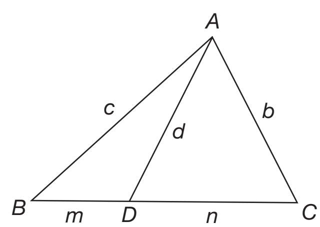
$4AD^2 = 2AB^2 + 2AC^2 - BC^2$ or $4AD^2 = 2c^2 + 2b^2 - a^2$ (where D is the mid-point of side BC)

3. For problem solving following are very useful facts:

If G is the centroid of ΔABC then

- $AG^2 = \frac{1}{9}(2AB^2 + 2AC^2 - BC^2)$
- $BG^2 = \frac{1}{9}(2BC^2 + 2AB^2 - AC^2)$
- $CG^2 = \frac{1}{9}(2BC^2 + 2AC^2 - AB^2)$
- $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$; where m_a, m_b, m_c are medians to sides a, b, c.
- $GA^2 + GB^2 + GC^2 = \frac{1}{3}(a^2 + b^2 + c^2)$
- $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3PG^2$; where P is any point in the plane of ΔABC

4. Stewart's theorem:



Let D be a point on side BC of ΔABC, and let $BD = m$, $DC = n$, $AD = d$. Then:

$a(d^2 + mn) = b^2m + c^2n$ (or $mb^2 + nc^2 = ad^2 + amn$)

If D is the mid-point of BC, this reduces to Apollonius theorem.

Another form of Stewart's theorem:

Let $BD : DC = p : q$. Then $(p + q)\, AD^2 + p\, DC^2 + q\, BD^2 = p\, AC^2 + q\, AB^2$

5. SAS inequality:

In ΔABC, let the lengths of sides AB, AC be fixed, but let $\angle A$ vary. Then the length a of the third side BC is an increasing function of $\angle A$. That is, the larger the angle A, the larger the side a, and conversely.

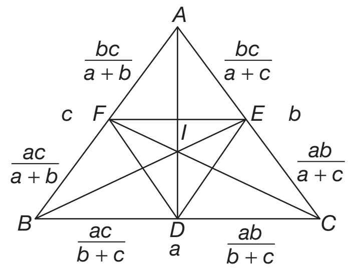
6. Angle bisector theorems:

- In ΔABC, let the internal bisector of $\angle A$ meet the opposite side BC at D. Then $BD : DC = AB : AC$.
- If D is a point on side BC of ΔABC such that $BD : DC = AB : AC$, then AD bisects $\angle A$.
- AD, BE, CF are the angle bisectors of $\angle A$, $\angle B$, $\angle C$ respectively meeting the opposite sides at D, E, F, then

$$BD = \frac{ac}{b+c};\ DC = \frac{ab}{b+c}$$

$$CE = \frac{ab}{a+c};\ EA = \frac{bc}{a+c}$$

$$AF = \frac{bc}{a+b};\ FB = \frac{ac}{a+b}$$



- The internal and external bisectors of $\angle A$ meets the circumcircle at X and Y, then XY is the circum-diameter and is perpendicular to BC.
- The internal and external bisectors of the vertical angles of a triangle divide the base in the ratio of the sides containing the angle. These points of division on the base are said to be conjugates of each other. The line (base) itself is said to be divided harmonically.

7. Cevian:

Any segment joining the vertex of a triangle to a point on the opposite side.

8. Ceva's theorem and its converse:

Let ABC be a triangle and X, Y, Z points on lines BC, CA, AB respectively, distinct from A, B, C.

Then the lines AX, BY, CZ are concurrent, iff

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = +1 \text{ or equivalently } \frac{\sin\angle BAX \cdot \sin\angle CBY \cdot \sin\angle ACZ}{\sin\angle XAC \cdot \sin\angle YBA \cdot \sin\angle ZCB} = +1$$

Second form of Ceva's theorem is known as the Trigonometric Form of the Ceva's theorem.

Sometimes it will be useful to know Ceva's theorem as

$BX \cdot CY \cdot AZ = CX \cdot AY \cdot BZ$.

Let X, Y, Z be points on the side lines BC, CA, AB of ΔABC. Suppose that the following equality holds: $\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ}{ZB} = 1$ (Condition for concurrency)

Then the lines AX, BY, CZ meet in a point.

Note: The following concurrences are true for any triangle:

- The perpendicular bisectors of the sides of a triangle concur (at the circumcentre).
- The internal angle bisectors of a triangle concur (at the incentre).
- The medians of a triangle concur (at the centroid).
- The altitudes of a triangle concur (at the orthocentre).

9. **Carnot's theorem:**
Let Points X, Y and Z be located on the sides BC, CA and AB respectively of ΔABC. The perpendiculars to the sides of the triangle at points X, Y and Z are concurrent if
$$BX^2 - XC^2 + CY^2 - YA^2 + AZ^2 - ZB^2 = 0 \qquad (\text{In } \Delta ABC)$$

10. **Menelaus theorem and its converse:**
Let a straight line l cut the sidelines BC, CA, AB of ΔABC in the points D, E, F, respectively. Then the following equality holds: $\frac{BD}{DC}.\frac{CE}{EA}.\frac{AF}{FB} = -1.$
(As earlier, the lengths are signed lengths.)
Let D, E, F be points on the sidelines BC, CA, AB of ΔABC. Suppose that the following equality holds: $\frac{BD}{DC}.\frac{CE}{EA}.\frac{AF}{FB} = -1.$
Then the points D, E, F lie in a straight line.

11. **Thales theorem:**
Let lines AA', BB' intersect at the point O, $A' \neq O$, $B' \neq O$, then
$AB \parallel A'B' \Leftrightarrow OA/OA' = OB/OB'$ (Here a/b denotes the ratio of two non-zero collinear vectors)

12. **Bramhagupta's Theorem:**
If AD is the altitude through A of ΔABC, and if R is the circumradius of ΔABC, then, $AB \cdot AC = (2R) \cdot AD$

13. **Napolean triangles:**
Construct equilateral triangles on sides of triangle ABC either all inwardly or all outwardly. Then the centres X, Y, Z of these triangles themselves form the vertices of an equilateral triangle called inner or outer Napoleon triangle.

14. **Medial triangle:**
A triangle having vertices at mid-point of sides of a given triangle is called medial triangle.
- Centroid of the triangle and its medial triangle is same.
- Circumcentre of the triangle is the orthocentre of the medial triangle.

15. **Pedal triangle and orthic triangle:**
Let ABC be a triangle, P a point and X, Y, Z respectively the feet of the perpendiculars from P to BC, CA, AB respectively. Now ΔXYZ is called a pedal triangle of ΔABC corresponding to the point P.
- The pedal triangle formed by the feet of the altitudes is called '**orthic triangle**'.
- Perimeter of orthic triangle = $2\Delta/R$ (where Δ is the area and R is the circumradius of ΔABC). It is least among all triangles inscribed in the triangle ABC.

16. **The nine point circle:**
The feet of the altitudes from A, B, C and the mid-points of AB, BC, CA as well as mid-points of AH, BH, CH lie on a circle called the nine point circle. Sometimes it is known as mid-point circle. Where H is the orthocentre of the ΔABC.

17. **Feuerbach's theorem:**
The nine point circle of a triangle is tangent to the in-circle and all three excircles of the triangle.

18. **Euler's formula:**
- If O and I are the circumcentre and in-centre of ΔABC, then, $OI^2 = R^2 - 2Rr$ where R and r respectively the circumradius and in-radius of ΔABC.
- Also $R \geq 2r$ or $R/r \geq 2$

19. Euler's line:
The orthocentre H, centroid G, the circumcentre S of an arbitrary triangle, lie on a line called Euler's line and satisfy $HG = 2GS$.

20. Simson–Wallace line (or pedal line):
If a point lies on the circumcircle, then the pedal triangle of P gets degenerated into a straight line, known as the Simson–Wallace line of P or the pedal line.
Converse is also true, *i.e.*, if the feet of the perpendiculars from a point to the sides of a triangle are collinear, then the point lies on the circumcircle of the triangle.

21. Fermat point (or Torricelli's point):
If no angle of ΔABC is greater than or equal to 120° and equilateral triangles $AC'B$, $BA'C$, $CB'A$ are constructed outwardly on the sides AB, BC, CA of ΔABC then, the lines AA', BB', CC' concur at a point, say P such that $AA' = BB' = CC'$; such a point P is called Fermat point or Torricelli's point of ABC.

22. Gergonne point:
Let the in-circle of ΔABC touch the sides BC, CA, AB at points P, Q, R, respectively. Then AP, BQ, CR meet in a point K called the Gergonne point.

23. Nagell point:
Let the excircles of ΔABC opposite to vertices A, B, C touch the sides BC, CA, AB at points U, V, W, respectively. Then AU, BV, CW meet in a point J called the Nagell point.

24. Symmedian point:
If the median of ΔABC through vertex A is reflected in the bisector of $\angle A$, the resulting line is called the symmedian through A. There are three such lines, one through each vertex of the triangle, and they meet in a point called the symmedian point.

25. Brocard point:
Given any triangle ABC, points W, W' may be found within it such that
$\angle WAB = \angle WBC = \angle WCA$, and $\angle W'BA = \angle W'CB = \angle W'AC$. These are the Brocard points of ΔABC. Let $\angle WAB = \theta$, $\angle W'BA = \theta'$. Then:
- $\theta = \theta'$
- $\cot\theta = \cot A + \cot B + \cot C$
- $\csc^2\theta = \csc^2 A + \csc^2 B + \csc^2 C$
- $\sin^3\theta = \sin(A - \theta)\cdot\sin(B - \theta)\cdot\sin(C - \theta)$

26. For arbitrary points A, B, C, D in space, AC perpendicular to BD iff
$AB^2 + CD^2 = BC^2 + AD^2$

27. Newton's theorem:
Let $ABCD$ be a quadrilateral; $\overline{AD} \cap \overline{BC} = \{E\}$; $\overline{AB} \cap \overline{CD} = \{F\}$. Such points A, B, C, D, E, F form a complete quadrilateral. Then, the mid-points of AC, BD and EF are collinear.
If $ABCD$ circumscribes a circle (called in-circle), then in-centre also lies on this line.

28. Brocard's theorem:
Let $ABCD$ be a quadrilateral, inscribed in a circle with centre 'O' and Let $\overline{AB} \cap \overline{CD} = \{P\}$, $\overline{AD} \cap \overline{BC} = \{Q\}$, $\overline{AC} \cap \overline{CD} = \{R\}$. Then '$O$' is the orthocentre of ΔPQR.
Here 'O' is also called Brocard point.

29. Cyclic quadrilateral:
A quadrilateral $ABCD$ is a cyclic (*i.e.*, there exists a circumcircle of $ABCD$) iff $\angle ACB = \angle ADB$ and $\angle ADC + \angle ABC = 180°$

30. Ptolemy's theorem:
A convex quadrilateral $ABCD$ is a cyclic iff $AC \cdot BD = AB \cdot CD + AD \cdot BC$

31. Condition for an in-circle of a quadrilateral *ABCD*:
A convex quadrilateral $ABCD$ is a tangent (*i.e.*, there exists an in-circle of $ABCD$) iff
$AB + CD = AD + BC$ (Pitot's theorem)

If this condition is satisfied, then it's in radius $r = \dfrac{\text{Area of } ABCD}{\text{Semi-perimeter of } ABCD}$

32. Alternate angle theorem:
In any circle, the angle between a tangent and a chord through the point of contact of the tangent is equal to the angle in the alternate segment (formed by the chord)

33. A common tangent to two circles divides a straight line segment joining the centres, externally or internally in the ratio of their radii.
The point S and S' dividing the line segment of the centres of two circles in the ratio of their radii are known as the centres of similitude of the two circles. The two common tangents from the external centre of similitude are the direct common tangents and two common tangents from internal centre of similitude are the transverse common tangents.

34. Power of a point:
Let $C(O, r)$ be a circle (the notation means that its centre is O, and its radius is r), and let P be a point. Consider any line l through P. Suppose it cuts $C(O, r)$ at A, B. Then the product $PA \cdot PB$ does not depend on l, and so is the same no matter which line is drawn (so it depends only on P, O, r). In fact: $PA \cdot PB = OP^2 - r^2$.
As in Ceva's and Menelaus's theorems, the lengths here are signed lengths.
Hence, if PA and PB are oriented in opposite directions, then $PA \cdot PB < 0$. (which will be the case if P lies within the circle)
The quantity $OP^2 - r^2$ is called the power of P with respect to circle C.

- If P lies on C, then its power wrt C is 0.
- If P lies outside C, then its power wrt C is the square of the length of the tangent from P to C.
- If P lies within C, then its power wrt C is negative.

Two very useful consequence of power of a point:

- If AB and CD are any two chords of a circle intersecting at P, then
  $PA \cdot PB = PC \cdot PD$ (secant property of a circle).
  Intersection point may be internal or external.
- If two straight line segments AB and CD (or both being produced) intersect at P such that $PA \cdot PB = PC \cdot CD$, then the four points A, B, C, D are concyclic (Condition for concyclicity).

35. Radical axis of two circles:
Let C_1 and C_2 be two given circles. Consider the locus of points P for which the power of P wrt C_1 is the same as the power of P wrt C_2. This locus is a straight line; it is called the radical axis of C_1, C_2.

- If C_1, C_2 intersect at points A, B, then the radical axis is line AB.
- If C_1, C_2 touch each other at a point P, then the radical axis is the line tangent to both circles at P.
- If C_1, C_2 are concentric, then the locus is empty.

36. Given the base and the ratio of the other two sides of triangle, locus of its vertex is a circle called Apollonius circle.

37. Area of a triangle:

There are several formulas for the area of a given ΔABC:

- $[ABC] = \frac{1}{2}(\text{Base})\times(\text{Height})$
- $[ABC] = \left(\frac{1}{2}\right) bc \sin A = \left(\frac{1}{2}\right) ca \sin B = \left(\frac{1}{2}\right) ab \sin C$
- $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter of the triangle;
- $[ABC] = rs$, where r is the radius of the in-circle of the triangle
- $[ABC] = \dfrac{abc}{4R}$

(where a, b, c are sides, R is the circumradius and r in radius and s the semi perimeter of ΔABC)

38. Area of a quadrilateral:

The area S of a quadrilateral $ABCD$ with semi perimeter p and angles α, γ at vertices A, C respectively is

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\left(\frac{\alpha+\gamma}{2}\right)}$$

- If $ABCD$ is a cyclic quadrilateral, the above formula reduces to
  $S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$
- **Area of a bicentric quadrilateral:** A bicentric quadrilateral is one which has both a circumcircle and an in-circle. If $ABCD$ is such a quadrilateral, then: Area (quadrilateral $ABCD$) $= \sqrt{abcd}$.

39. For ΔABC, in the usual notation ($O \equiv$ Circumcentre, $H \equiv$ Ortho-centre, $N \equiv$ Centre of nine point circle, $I \equiv$ In-centre, $I_a \equiv$ Ex-centre opposite to angle A, $I_b \equiv$ Ex-centre opposite to angle B, $I_c \equiv$ Ex-center opposite to angle C, $r_a \equiv$ Ex-radius opposite to angle A, $r_b \equiv$ Ex-radius opposite to angle B, $r_c \equiv$ Ex-radius opposite to angle C, etc.)

- $AI = r\operatorname{cosec}\dfrac{A}{2}$; $BI = r\operatorname{cosec}\dfrac{B}{2}$; $CI = r\operatorname{cosec}\dfrac{C}{2}$
- $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$
- $OI^2 = R^2 - 2rR, OI_a^2 = R^2 + 2Rr_a, OI_b^2 = R^2 + 2Rr_b, OI_c^2 = R^2 + 2Rr_c$
- $(HI)^2 = 4R^2 \cos A \cos B \cos C$
- $OH^2 = R^2(1 - 8\cos A \cos B \cos C) = 9R^2 - a^2 - b^2 - c^2$
- $NI = \dfrac{R}{2} - r; NI_a = \dfrac{R}{2} + r_a$, etc.
- $OG^2 = R^2 - \dfrac{a^2+b^2+c^2}{9}$
- $AH^2 + BH^2 + CH^2 = 3R^2$
- $OI^2 + OI_a^2 + OI_b^2 + OI_c^2 = 12R^2$
- For the orthic triangle, the sides are $a\cos A$ or $R\sin 2A$, $b\cos B$ or $R\sin 2B$ and $c\cos C$ or $R\sin 2C$. Its angles are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$.

- $IA \cdot IB \cdot IC = 4Rr^2$ or $\dfrac{AI \cdot BI \cdot CI}{R} = 4r^2$
- $r = \dfrac{\Delta}{s} = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2} = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$
- $r_a = \dfrac{\Delta}{s-a}, r_b = \dfrac{\Delta}{s-b}, r_c = \dfrac{\Delta}{s-c}$
- $r_a = s\tan\dfrac{A}{2}, r_b = s\tan\dfrac{B}{2}, r_c = s\tan\dfrac{C}{2}$
- $r_a = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}, r_b = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}, r_c = 4R\ \cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2},$
- $rr_ar_br_c = \Delta^2$
- $r_ar_b + r_br_c + r_cr_a = s^2$
- $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c} = \dfrac{1}{r}$
- If X, Y, Z are points of contact of the in-circle of ΔABC with its sides, then,
  (a) The sides of XYZ are $2r\cos\dfrac{A}{2}$, $2r\cos\dfrac{B}{2}$ and $2r\cos\dfrac{C}{2}$
  (b) Its angles are $\dfrac{\pi - A}{2}, \dfrac{\pi - B}{2}, \dfrac{\pi - C}{2}$
  (c) Its area is $\dfrac{\Delta r}{2R}$ or $Rr\sin A\sin B\sin C$
- **Cosine rule:**
  $a^2 = b^2 + c^2 - 2bc\cos A$
  $b^2 = c^2 + a^2 - 2ca\cos B$
  $c^2 = a^2 + b^2 - 2ab\cos C$
- **Sine rule:** Let the radius of the circumcircle of ΔABC be R. Then:
  $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$
- **Projection rule:**
  $a = b\cos C + c\cos B$
  $b = c\cos A + a\cos C$
  $c = a\cos B + b\cos A$
- **Napier's rule:**
  $$\tan\frac{B-C}{2} = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$$
  $$\tan\frac{C-A}{2} = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$$
  $$\tan\frac{A-B}{2} = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$$
- **Half angle ratios:**
  $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}},\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}},\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Inequalities

1. **Trivial inequality:**
   If x is any real number, we have: $x^2 \ge 0$.
   This seems 'trivial' but is the basis for every other inequality!
2. **Mean inequality:**
   Let $a_1, a_2,\ldots, a_n$ be n positive numbers. Then $A \ge G \ge H$

   Where $A=\dfrac{a_1+a_2+a_3\ldots+a_n}{n}$ (AM); $G=\sqrt[n]{a_1\cdot a_2\cdot a_3\ldots a_n}$ (GM);

   $$H=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}} \text{ (HM)}$$

   (Also equality holds if all numbers are equal)

   - $\text{Min}(a,b)\le\dfrac{2ab}{a+b}\le\sqrt{ab}\le\dfrac{a+b}{2}\le\sqrt{\dfrac{a^2+b^2}{2}}\le\text{Max}(a,b)$
   - More generally, let $a_1, a_2,\ldots, a_n$ be n positive numbers; then

   $$\min\{a_1,a_2,...,a_n\}\le\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\le\sqrt[n]{a_1a_2\ldots a_n}\le\frac{a_1+a_2+\cdots+a_n}{n}$$
   $$\le\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}\le\max\{a_1,a_2,...,a_n\}$$

   with equality if and only if $a_1=a_2=\cdots=a_n$.

   The following inequalities derived from AM ≥ GM ≥ HM, will be very useful for problem solving.

   - $x^2+y^2+xy\ge(x+y)^2$ **(Sophie inequality)**
   - $x^2+y^2-xy\ge xy$
   - $x^3+y^3\ge xy(x+y)$
   - $\dfrac{ab}{a+b}\le\dfrac{a+b}{4}$
   - $\dfrac{a^2+b^2}{a+b}\ge\dfrac{a+b}{2};\dfrac{a^2+b^2+c^2}{a+b+c}\ge\dfrac{a+b+c}{3}$
   - $xy\le\left(\dfrac{x+y}{2}\right)^2$
3. **Quadratic inequality:**
   If $x\in R$, and $Ax^2+Bx+C=0$, then $D\ge0$ or $B^2-4AC\ge0$
   If $A>0$, $D<0$ or $4AC-B^2>0$ and x is real, then $Ax^2+Bx+C\ge0$

4. **Triangle inequality:**
   - If a, b, c are the measures of the sides of triangle, then, $b - c < a < b + c$; $c - a < b < c + a$; $a - b < c < a + b$
   - The lengths a, b, c can represent the sides of a triangle iff, $a + b > c$, $b + c > a$, $c + a > b$.
   - If a, b are real numbers, then $|a + b| \le |a| + |b|$, $|a - b| \ge ||a| - |b||$.
5. **Weirstras's inequality:**
   For positive numbers $a_1, a_2, a_3, \ldots, a_n$

   $$(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n) > 1 + a_1 + a_2 + a_3 + \cdots + a_n$$

   If a_1 are fractions (*i.e.*, less than unity), then,

   $$(1-a_1)(1-a_2)(1-a_3)\cdots(1-a_n) > 1 - (a_1 + a_2 + a_3 + \cdots + a_n)$$

6. **Cauchy–Schwarz inequality: (C–S Inequality)**
   If a, b, c, x, y, z are real numbers (positive, zero, or negative)
   Then, $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \ge (ax + by + cz)^2$; With equality iff $a : b : c :: x : y : z$
   In general, let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any $2n$ real numbers; then

   $$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \ge \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

   with equality precisely when there exist constants μ, λ, not both zero, such that $\mu a_i = \lambda b_i$ for all i.
7. **Tchebycheff's inequality:**
   If $x_1 \le x_2 \le x_3 \le \cdots \le x_n$ and $y_1 \le y_2 \le y_3 \le \cdots \le y_n$ then

   $$\frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{n} \ge \left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right)\left(\frac{y_1 + y_2 + y_3 + \cdots + y_n}{n}\right)$$

   If one of the sequences is increasing and the other decreasing, then, the direction of the inequality changes.
8. **Holders inequality:**

   $$\left(a_1^p + a_2^p + \cdots + a_n^p\right)^{\frac{1}{p}} \left(b_1^q + b_2^q + \cdots + b_n^q\right)^{\frac{1}{q}} \ge \left(a_1b_1 + a_2b_2 + \cdots + a_nb_n\right)$$

   where $\frac{1}{p} + \frac{1}{q} = 1$ and a, b are non-negative real numbers.

9. **Ptolemy's inequality:**
   For any four points A, B, C, D; $AB \cdot CD + AD \cdot BC \ge AC \cdot BD$
   Equality occurs if and only if $ABCD$ is cyclic.
10. **The parallelogram inequality:**
    For any four points A, B, C, D we have $AB^2 + BC^2 + CD^2 + DA^2 \ge AC^2 + BD^2$.
    Equality occurs if and only if $ABCD$ is a parallelogram.
11. **Toricelli's (or Fermat's) point for maxima/minima:**
    For a given triangle ABC, the point X for which $AX + BX + CX$ is minimal is Toricelli's point, when all angles of ΔABC are less than 120° and is the vertex of the obtuse angle otherwise.
12. Let P be a point in the plane of the triangle. Then point P for which $AP^2 + BP^2 + CP^2$ is minimal is the centroid of the triangle. (Leibniz's theorem)

13. The Erodos–Mordell inequality:
Let P be a point in the interior of ΔABC and X, Y, Z projections of P onto BC, CA, AB respectively. Then $PA + PB + PC \geq 2(PX + PY + PZ)$
Equality holds iff ΔABC is equilateral and P is its centroid.

14. Jensen's inequality:
If $f(x)$ is open down (or concave) for all $x \in [a, b]$ then we have following inequality:

$$\frac{w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + \cdots + w_n f(x_n)}{w_1 + w_2 + w_3 + \cdots + w_n}$$
$$\leq f\left(\frac{w_1x_1 + w_2x_2 + w_3x_3 + \cdots + w_nx_n}{w_1 + w_2 + w_3 + \cdots + w_n}\right)$$

for all $x_1, x_2, x_3, \ldots, x_n \in [a, b]$ and where $w_1, w_2, w_3, \ldots, w_n \in \mathbb{R}^+$ called weights.
Equality will holds when $x_1 = x_2 = x_3 = \cdots = x_n$
In case of function is open up (or convex) inequality will be reverse.

Algebra

1. $\sqrt[n]{a^n} = |a|$, if n is even and $\sqrt[n]{a^n} = a$, if n is odd.
2. **Difference of two squares:**
This is of use more often than one would expect:
$a^2 - b^2 = (a - b) \cdot (a + b)$.
3. **Two simple and useful factorizations:**
$xy + x + y + 1 = (x + 1)(y + 1)$, $xy - x - y + 1 = (x - 1)(y - 1)$.
4. **Sophie Germain identity:**
$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab) = ((a + b)^2 + b^2)((a - b)^2 + b^2)$.
5. **Important identities and concepts (Useful for problem solving):**
   - $a^3 + b^3 + c^3 - 3abc \equiv (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
   - $(a^2 + b^2)(x^2 + y^2) \equiv (ax + by)^2 + (ay - bx)^2$
   - $(x^n - y^n)$ is always divisible by $(x - y)$.
   - $(x^n + y^n)$ is divisible by $(x + y)$ when 'n' is odd.
   - $a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \equiv (a + b)(b + c)(c + a) - 2abc$
   - $(a + b)(b + c)(c + a) + abc \equiv (a + b + c)(ab + bc + ca)$
   - $(x + y + z)(xy + yz + zx) \equiv (x + y)(y + z)(z + x) + xyz$
   - $(x + y + z)^3 \equiv x^3 + y^3 + z^3 + 3(x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2) + 6xyz$
   - $(x + y + z)^3 - (x^3 + y^3 + z^3) \equiv 3(x + y)(y + z)(z + x)$
   - $(x + y)(y + z)(z + x) \equiv \sum x^2y + 2xyz$
   - $x^2(y - z) + y^2(z - x) + z^2(x - y) \equiv x^2y - xy^2 + y^2z - z^2y + z^2x - zx^2$
   $\equiv -(x - y)(y - z)(z - x)$
   - $a^4 + b^4 + a^2b^2 \equiv (a^2 + ab + b^2)(a^2 - ab + b^2)$
   - $x^2 + y^2 + xy \equiv \dfrac{3}{4}(x + y)^2 + \dfrac{1}{4}(x - y)^2$
6. If $a + b + c = 0$, $a^3 + b^3 + c^3 = 3abc$
7. If u, v are given numbers, then the quadratic equation whose roots are u, v is $(x - u)(x - v) = 0$.
8. Let a, b, c be real numbers, $a \neq 0$. Then the roots of the quadratic equation $ax^2 + bx + c = 0$ are real if and only if $D \geq 0$ or $b^2 - 4ac \geq 0$.

9. Relations between the roots and coefficients:
   - If α, β are the roots of $ax^2 + bx + c = 0$, then $\alpha + \beta = -\frac{b}{a}; \alpha\beta = \frac{c}{a}$
   - If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then
     $\alpha + \beta + \gamma = -\frac{b}{a}; \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}; \alpha\beta\gamma = -\frac{d}{a}$
   - If α, β, γ and δ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then
     $\alpha + \beta + \gamma + \delta = -\frac{b}{a}; \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a};$
     $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}; \alpha\beta\gamma\delta = \frac{e}{a}.$

10. **Polynomials:**
    - Every polynomial equation of degree $n \geq 1$, has exactly 'n' roots.
    - If a polynomial equation with real coefficients has a complex root ($p + iq$). Where p, q are real numbers, $q \neq 0$, then, it also has a complex root ($p - iq$).
    - If a polynomial equation with rational coefficients has an irrational root $(p + \sqrt{q})$, (p, q rational, $q > 0$, q not the square of any rational number), then, it also has an irrational root $(p - \sqrt{q})$.

11. **Remainder/factor theorem:**
    If $f(x)$ is a polynomial in x, and c is any real number, then the remainder in the division of $f(x)$ by $(x - c)$ is $f(c)$.
    If $f(c) = 0$ then $x - c$ is called a factor of $f(x)$.
12. A number α is a common root of the polynomial equations $f(x) = 0$ and $g(x) = 0$ iff, it is a root of $h(x) = 0$, where $h(x)$ is the GCD of $f(x)$ and $g(x)$.
13. A number α is repeated root of a polynomial equation $f(x) = 0$ iff it is a common root of $f(x) = 0$ and $f'(x) = 0$.
14. **Rational root theorem:**
    If the rational number p/q (where p, q are integers $q \neq 0$, $(p, q) = 1$) is a root of the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ where $a_0, a_1, a_2, \ldots, a_n$ are integers and $a_0 \neq 0$, then p is a divisor of a_n and q ia a divisor of a_0.
15. **Integral root theorem:**
    Let $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$, represent a polynomial equation that has leading coefficient of 1, all coefficients and constant integer. Any rational root of this equation must be an integer and divisor of a_n.
16. **Descarte's rule of signs:**
    Suppose $P(x)$ be a polynomial whose terms are arranged in descending powers of x of the variable. Thus, the number of positive real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms or less than this by an even number.
    The number of negative real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$ or is less than this number by an even number.
17. The sum of a n-term arithmetic progression $a, a + d, a + 2d, \ldots, a + (n - 1)d$ is
    $$\sum_{k=0}^{n-1}(a + kd) = \frac{n(2a + (n-1)d)}{2} = n \times \frac{(\text{First term} + \text{Last term})}{2}$$
    Examples: $1 + 2 + 3 + \cdots + n = \left(\frac{1}{2}\right)n(n + 1)$, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

18. The sum of a n-term geometric progression a, $ar, ar^2, \ldots, ar^{n-1} (r \neq 1)$ is $a\left(\frac{r^n - 1}{r - 1}\right)$;

for $r = 1$, sum is na.

Examples: $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$, $1 + 3 + 3^2 + \cdots + 3^{n-1} = \left(\frac{1}{2}\right)(3^n - 1)$.

19. We have:

- $1 + 2 + 3 + \cdots + n = \left(\frac{1}{2}\right) n(n + 1)$.
- $1^2 + 2^2 + 3^2 + \cdots + n^2 = \left(\frac{1}{6}\right) n(n + 1)(2n + 1)$.
- $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{1}{4}\right) n^2(n + 1)^2$.

20. The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ does not converge to a finite number. There is no simple formula for the sum $1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Rather: $1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n + \gamma$, where $\ln n$ is the 'natural logarithm' of n, and $\gamma \approx 0.577216$ is the 'Euler–Mascheroni constant'.

Number Theory

1. Notation: $a|b$ means: 'a is a divisor of b'. We read it as: 'a divides b'.

Example: $4|12$, but $4 \nmid 13$.

2. If $a|b$ and $a|c$ then $a|(pb + qc)$

3. **Greatest common divisor (GCD):**

Let a and b two non zero integers. Then the gcd of a and b exists and is written as (a, b) and it is unique also. **Examples:** GCD$(10, 15) = 5$, GCD $(8, 9) = 1$.

- The gcd of a, b can be represented as a linear function of a, b, *i.e.*, there exists integers m, n for (a, b) such that $(a, b) = am + bn$. (Linearity property)
- If $(a, b) = 1$, then a and b are said to be relatively primes or co-primes of each other.

  Example: 15 and 22 are co-prime.
- Two consecutive integers are always co-prime.

4. **Congruencies:**

$a \equiv b \pmod{c}$ means: '$a - b$ is divisible by c'. We read it as: 'a is congruent to b modulo c'.

Example: $19 \equiv 4 \pmod{5}$.

(a) The congruence relation modulo n for a fixed non-zero integer n is reflexive, symmetric, and transitive. Thus: if $a \equiv b \pmod{n}$, and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

(b) Let $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$; then

- $a + c \equiv (b + d) \pmod{m}$
- $a - c \equiv (b - d) \pmod{m}$
- $ac \equiv bd \pmod{m}$
- $pa + qc \equiv pb + qd \pmod{m}$ $\forall$ integers p, q
- $a^n \equiv b^n$ (mod m) $\forall$ integers $n \in \mathbb{N}$
- $f(a) \equiv f(b) \pmod{m}$ for every polynomial with integer coefficients

5. An integer x_0, satisfying the linear congruence $ax \equiv b \pmod{m}$ has a solution. Further, if x_0 is a solution, then the set of solutions is precisely $(x_0 + km)$ where k is an integer.
6. Some extremely useful and far reaching results: For any $n \in \mathbb{Z}$, we have:
   - Either $n^2 \equiv 0 \pmod 3$ or $n^2 \equiv 1 \pmod 3$. That is, all squares are of the form $3k$ or $3k + 1$; a square cannot be of the form $3k + 2$.
   - Either $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$. That is, all squares are of the form $4k$ or $4k + 1$; a square cannot be of the form $4k + 2$ or $4k + 3$.
   - If p is a prime number, and $p \mid ab$, then $p \mid a$ or $p \mid b$.

     Note: this claim is not true for composite number. That is, if n is composite, and $n \mid ab$, we cannot conclude that $n \mid a$ or $n \mid b$.
   - If a, b are co-prime positive integers, and ab is a square, then both a and b are squares.
   - If a, b are co-prime integers, and ab is a cube, then both a and b are cubes.
   - Suppose that a, b, c, d are positive integers, and $ab = cd$. Further, suppose that a, b are co-prime, and c, d are co-prime. Then either $a = c$ and $b = d$, or $a = d$ and $b = c$. In any case, $\{a, b\} = \{c, d\}$.
7. **Multiplicative inverse:**
   If n is a number, and a is co-prime to n, then an integer b can be found such that $ab \equiv 1$ (mod n). We call 'b' the multiplicative inverse of 'a' modulo n.
   Example: Let $n = 11$. The multiplicative inverses of 2, 3, 4, 5 are 6, 4, 3, 9, respectively.
8. **Fermat's little theorem:**
   If p is a prime number, and a is co-prime to p, then $a^{p-1} \equiv \pmod{p}$.
   Example: $2^6 \equiv 1 \pmod 7$, and $3^4 \equiv 1 \pmod 5$.
9. **Another form of the Fermat little theorem:**
   If p is a prime number, and a is any integer, then $a^p \equiv a \pmod{p}$.
10. **Wilson's theorem:**
    If p is a prime number, then $(p-1)!+1 \equiv \pmod{p}$
    Example: $6! + 1 = 721 \equiv 0 \pmod 7$.
11. **Euler's totient function:**
    Let n be any positive integer. The number of all positive integers less than or equal to n and prime to it is denoted by $\phi(n)$; the function ϕ is called Euler's totient function.
    Example: $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(10) = 4$. Note that:
    - n is a prime number $\Leftrightarrow \phi(n) = n - 1$.
    - n is a power of 2 $\Leftrightarrow \phi(n) = \left(\frac{1}{2}\right)n$.
    - The Euler phi function is multiplicative. This means that if m, n are co-prime, then $\phi(mn) = \phi(m) \cdot \phi(n)$.

      Example: $\phi(12) = \phi(3) \cdot (4)$.
    - Here is a quick way of computing $\phi(n)$: List the distinct primes p which divide n, then multiply n by the product of $\dfrac{p-1}{p}$ for all such p.

      That is, if $n = a^p \cdot b^q \cdot c^r \ldots$where a, b, c are distinct primes and p, q, r are positive integers, then $\phi(n) = n\left(1-\frac{1}{a}\right)\cdot\left(1-\frac{1}{b}\right)\cdot\left(1-\frac{1}{c}\right)\ldots$

Example: Take $n = 20$. The distinct primes dividing 20 are 2 and 5, so

$$\phi(20) = 20 \times \frac{1}{2} \times \frac{4}{5} = 8.$$

Example: Take $n = 350$. The distinct primes dividing 350 are 2, 5, 7, so

$$\phi(350) = 350 \times \frac{1}{2} \times \frac{4}{5} \times \frac{6}{7} = 120.$$

12. Euler's theorem:
If x be any positive integer prime to n, then $x^{\phi(n)} = 1 \pmod{n}$
eg: $3^4 \equiv 1 \pmod{10}$, $15^{10} \equiv 1 \pmod{22}$.

13. Infinitude of primes:
- There are infinitely many prime numbers.
- There are infinitely many prime numbers of each of the types 1 (mod 4) and 3 (mod 4):
  1 (mod 4) : {5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113, . . .},
  3 (mod 4) : {3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83, 103, 107, . . .}.
- There are infinitely many prime numbers of each of the types 1 (mod 3) and 2 (mod 3):
  1 (mod 3) : {7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, 127, . . .},
  2 (mod 3) : {2, 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, 107, . . .}.

14. If $n \equiv 3 \pmod 4$, then n has at least one prime factor of the form 3 (mod 4).

15. If p is a prime number of the type 3 (mod 4), then it cannot be expressed as $x^2 + y^2$, where x, y are integers.

16. If p is a prime number of the type 1 (mod 4), then it can be expressed as $x^2 + y^2$, where x, y are integers. Moreover, this representation is unique.
Example: $13 = 2^2 + 3^2$, $89 = 5^2 + 8^2$.

17. If a positive integer n can be expressed as $x^2 + y^2$ where x, y are integers, then:
- n has at least one prime factor p of the form 1 (mod 4).
- the number of primes p which divide n and which are of the form 3 (mod 4) is even.

  Example: Take $n = 2205$. It can be expressed as $21^2 + 42^2$, and its prime factorization is $2205 = 3^2 \times 5 \times 7^2$. Note that it has a prime factor of the type 1 (mod 4), and the number of primes p which divide n and which are of the form 3 (mod 4) is 4 (two 3's and two 7's).

18. Pythagorean triples:
The equation $x^2 + y^2 = z^2$ has infinitely many 'primitive solutions' (*i.e.*, with x, y, z co-prime). They may be found as follows: Choose any two positive integers u, v of opposite parity, with $u > v$. Put $x = u^2 - v^2$, $y = 2uv$, $z = u^2 + v^2$.
(We can switch the roles of x and y: put $x = 2uv$, $y = u^2 - v^2$.) This generates the entire set of primitive solutions.
Example: Put $u = 5$, $v = 2$; we get $(x, y, z) = (21, 20, 29)$.

19. Let N be a positive integer, greater than 1, say $N = a^p \cdot b^q \cdot c^r \ldots$; where a, b, c are distinct primes and p, q, r are positive integers. The number of ways in which N can be resolved into two positive factors is $\frac{1}{2}(p+1)(q+1)(r+1)\ldots$

20. Number of ways in which a composite number can be resolved into two positive factors which are prime to each other is given by 2^{n-1}, where n is the number of distinct prime factors of n.

21. Let N be a positive integer greater than 1 and let $N = a^p \cdot b^q \cdot c^r \ldots$ where $a, b, c, \ldots$ are distinct primes and $p, q, r \ldots$ are integers (positive), then the sum of all the positive divisors in the product is equal to $\left(\frac{a^{p+1}-1}{a-1}\right)\left(\frac{b^{q+1}-1}{b-1}\right)\left(\frac{c^{r+1}-1}{c-1}\right)\ldots$

22. The greatest integer function:
The greatest integer written symbolically as $\lfloor\ \rfloor$, is defined by setting $\lfloor x \rfloor$ = the greatest integer not exceeding x for every real x, *i.e.*, $\lfloor x \rfloor \le x$.

23. The highest power of prime p which is contained in $n!$ is

$$v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

Combinatorics

1. Two laws of enumeration:
- Law of addition. If A, B are two sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.
- Law of multiplication. If A, B are two sets, then $|A \times B| = |A|\,|B|$. Here, $A \times B$ is the Cartesian product of the sets A, B.

2. One-to-one correspondence:
If the elements of two finite sets A, B can be placed into one-to-one mapping, then $|A| = |B|$.

3. Properties of binomial coefficient ^nC_r:
- $^nC_0 = {}^nC_n = 1$
- $^nC_r = {}^nC_{n-r}$
- If $^nC_r = {}^nC_k$, then $r = k$ or $n - r = k$
- $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- $r\,.\,{}^nC_r = n\,{}^{n-1}C_{r-1}$
- If n is even, ^nC_r is greatest for $r = \frac{n}{2}$ and if n is odd, ^nC_r is greatest for $r = \frac{n-1}{2}, \frac{n+1}{2}$.

4. Combinations:
From a set containing n distinct elements, a subset with k elements can be chosen in $\binom{n}{k}$ distinct ways.
- Number of points of intersection between n non-concurrent and non-parallel lines is $^nC_2$.
- Number of lines, joining any two points out of n points (no three are collinear), is $^nC_2$.
- Number of triangles formed using n points in which no three of them are collinear is $^nC_3$.
- Number of diagonals that can be drawn in a 'n' sided polygon is $^nC_2 - n$.
- The number of ways of selecting one or more items from n distinct items is $2^n - 1$.
- The number of subsets of n elements is 2^n; the number of non-empty subsets is $2^n - 1$.

- The number of ways to select r objects from n distinct objects where p particular objects should always be included in the selection $= {}^{n-p}C_{r-p}$.
- The number of ways to select r objects from n distinct objects where p particular objects should never be included in the selection $= {}^{n-p}C_r$.
- Number of ways to select r objects from n distinct objects where each object can be selected any number of times is ${}^{n+r-1}C_r$.
- The number of ways to select at least one object from n identical objects $= n$.
- The number of ways to select one or more objects from $(p + q + r + \cdots + n)$ objects where p objects are alike of one kind, q are alike of second kind, r are alike of third kind, ... and remaining n are distinct from each other $= [(p + 1)(q + 1)(r + 1)...2^n] - 1$.

5. Permutations:

Number of permutations of n distinct objects taken r objects at a time is ${}^nP_r = \binom{n}{r} r!$

- The number of 1–1 function from a set of m elements to a set of n elements $(m \le n)$ is ${}^nP_m = \dfrac{n!}{(n-m)!} = n(n-1)(n-2)\ldots(n-m+1)$.
- Total number of ways to permutate (arrange, order) n distinct objects in a row $= n!$.
- The number of bijections from a n-set on to itself is $n!$.
- Number of ways to permuate (arrange) n objects out of which p are identical of one kind, q are identical of another kind, r are identical of third kind and rest all are distinct is $\dfrac{n!}{p!q!r!}$.
- Total number of ways to permutate n distinct things taken r at a time when objects can be repeated any number of times is n^r.
- The number of functions from an r-set to an n-set is n^r.
- The number of ways to select and arrange (permutate) r objects from n distinct objects such that arrangement should always included p particular objects $= {}^{n-p}C_{r-p} \cdot r!$.
- The number of ways to select and arrange r objects from n distinct objects such that p particular objects are always excluded in the selection $= {}^{n-p}C_r \cdot r!$.
- The number of ways to arrange n distinct objects such that p particular objects remain together in the arrangement $= (n + 1 - p)!p!$.
- The number of ways to arrange n distinct objects such that out of p particular objects no two are together $= (n - p)!\ {}^{n-p+1}C_p\, p!$.

6. Circular permutations:

Number of ways to arrange n distinct objects in a circle $= (n - 1)!$.

Number of circular permutations of n distinct objects such that clockwise and anticlockwise arrangements of objects are same $= \dfrac{(n-1)!}{2}$, $n \ge 3$.

7. Derangement formulae (or no fix point formulae):

If n distinct objects are to be arranged in a line such that no object occupies its original place, then it is called derangement. Number of ways to derange is

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right].$$

8. Distribution Problems:

Number of objects is predefined in each group or box:

- Number of ways in which $a + b + c$ distinct objects (out of a, b, c no two numbers are equal) can be divided into 3 **unnumbered groups** containing a, b, c objects respectively
$$=\binom{a+b+c}{a}\binom{b+c}{b}\binom{c}{c}=\frac{(a+b+c)!}{a!b!c!}$$
- Number of ways in which $a + b + c$ distinct objects (out of a, b, c no two numbers are equal) can be divided into 3 **numbered groups** containing a, b, c objects
= Number of ways to divide $a + b + c$ objects (out of a, b, c no two numbers are equal) in 3 unnumbered groups × (Number of groups)! $=\dfrac{(a+b+c)!}{a!b!c!}\times 3!$
- Number of ways to divide mn distinct objects equally in m **unnumbered groups** (each group gets n objects) $=\dfrac{(mn)!}{n!^m\, m!}$.
- Number of ways to divide and distribute mn distinct objects equally in m **numbered groups** (each group gets n objects) $=\dfrac{(mn)!}{n!^m\, m!}\times m!=\dfrac{(mn)!}{n!^m}$.
- Number of ways to divide $ma + nb + pc$ distinct objects (out of a, b, c no two numbers are equal) in $m + n + p$ **unnumbered groups** such that m groups contains a objects each, n groups contains b objects each, p groups contains c objects each $=\dfrac{(ma+nb+pc)!}{(a!)^m(b!)^n(c!)^p\, m!n!p!}$
- Number of ways to divide and distribute $ma + nb + pc$ distinct objects (out of a, b, c no two numbers are equal) in $m + n + p$ **numbered groups** such that m groups contains a objects each, n groups contains b objects each, p groups contains c objects each $=\dfrac{(ma+nb+pc)!}{(a!)^m(b!)^n(c!)^p\, m!n!p!}\times(m+n+p)!$.

Number of objects is not predefined in each group or box:

- The number of ways to divide n identical objects into r **numbered groups** such that each group gets 0 or more objects (empty groups are allowed) $= {}^{n+r-1}C_{r-1}$.
- The number of ways to divide n identical objects into r **numbered groups** such that each group receives at least one object (empty groups are not allowed) $= {}^{n-1}C_{r-1}$.
- The number of ways to divide n identical objects in r **numbered groups** such that each groups gets minimum m objects and maximum k objects = Coefficient of x^n in $(x^m + x^{m+1} + \cdots + x^k)^r$.
- Number of ways to divide n non-identical objects in r **numbered groups** such that each groups gets 0 or more number of objects (empty groups are allowed) $= r^n$.
- Number of ways to divide n non-identical objects in r **numbered groups** such that each group gets at least one object (empty groups are not allowed) $= r^n - {}^rC_1\,(r-1)^n + {}^rC_2\,(r-2)^n - {}^rC_3\,(r-3)^n + \cdots + (-1)^{r-1}\,{}^rC_{r-1}\,1^n$.

9. Principle of inclusion-exclusion (PIE):

This is a far reaching generalization of the law of addition.

If A, B, C are three finite sets, then

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$
- Let $A_1, A_2, A_3, \ldots, A_n$ be n sets, then in general:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \ldots \cap A_n|$$

10. Pigeon hole principle: (PHP or Dirichlets's principle)

If more than 'n' objects are distributed in 'n' boxes, then, at least, one box has more than one object in it.

11. Recursion:

Sometimes a sequence is defined recursively. This means that we compute each element in terms of the elements preceding it, using some fixed rule. This applies to all elements except for a few initial terms which are fixed independently.

- **Powers of 2:** Let $a_n = 2^n$ for $n \in N$. Then: $a_1 = 2$, $a_n = 2a_{n-1}$ for $n > 1$.
- **Squares:** Let $a_n = n^2$ for $n \in N$. Then: $a_1 = 1$, $a_n = a_{n-1} + 2n - 1$ for $n > 1$.
- **Factorials:** Let $a_n = n!$ for $n \in N$. Then: $a_1 = 1$, $a_n = na_{n-1}$ for $n > 1$.

12. Compositions:

For $n \in \mathbb{N}$, let a_n be the number of ways of writing n as a sum of one or more positive integers, with order being taken into account (so, 1 + 2 is counted separately from 2 + 1). These expressions are called the compositions of n. So the compositions of 2 are 2, 1 + 1, and the compositions of 3 are 3, 2 + 1, 1 + 2, 1 + 1 + 1.

We may show that: $a_n = 2^{n-1}$.

13. Fibonacci numbers:

The Fibonacci numbers F_n for $n \in \mathbb{N}$ are defined thus: $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 3$.

Here are the first few Fibonacci numbers:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	...

These numbers are ubiquitous. For example:

- The number of compositions of n in which all the summands exceed 1 is a Fibonacci number; in fact it is F_{n-1}.
  Example: For $n = 6$ we get the compositions 6, 4 + 2, 3 + 3, 2 + 4, 2 + 2 + 2.
- The number of compositions of n in which the summands are only 1's and 2's is a Fibonacci number; in fact it is F_{n+1}.
  Example: For $n = 4$ we get the compositions 1 + 1 + 1 + 1, 2 + 1 + 1, 1 + 2 +1, 1 +1 +2, 2 + 2.
- The number of subsets of the n-element set $\{1, 2, \ldots, n\}$ in which no two consecutive numbers occur is a Fibonacci number; in fact it is F_{n+2}.
  Example: For $n = 3$ we get the 5 subsets: $\{\}, \{1\}, \{2\}, \{3\}, \{1, 3\}$.
  For $n = 4$ we get the 8 subsets: $\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}$.

14. Taxicab paths:

If we have to walk on the coordinate plane from the initial point $O(0, 0)$ to the terminal point $P(m, n)$ where $m, n \in \mathbb{N}_0$, so that our path consists of steps one unit 'North' or one unit 'East', then the number of possible paths is $\binom{m+n}{n}$.

15. Catalan numbers:

If we have to walk on the coordinate plane from the initial point $O(0, 0)$ to the terminal point $P(n, n)$ where $n \in \mathbb{N}$, so that our path consists of steps one unit

'North' or one unit 'East' and never goes above the line $y = x$, then the number of possible paths is defined to be the nth Catalan number, C_n. We may show that

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

They may be recursively defined:

$$C_0 = 1, \text{ and } C_{n+1} = C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \cdots = \sum_{k=0}^{n} C_kC_{n-k}$$

Here are the first few Catalan numbers:

n	1	2	3	4	5	6	7	8	9	10	11	...
C_n	1	2	5	14	42	132	429	1430	4862	16796	58786	...

Like the Fibonacci numbers, the Catalan numbers too are ubiquitous:

- The number of ways a convex n-sided polygon can be triangulated is a Catalan number; in fact it is C_{n-2}.
- The number of correctly matched strings of n pairs of parenthesis is C_n.
  Example: $n = 3$: ((( ))), ( )(( )), ( )( )( ), (( ))( ), (( )( )).

A.4 Glossary of Recommended Books

1. Gems Primary, Junior and Inter Levels (Published by the Association of Mathematics Teachers of India, Chennai)
2. Mathematical Circles by Dimtri Fomin, Sergey Genkin and Ilia Itenberg (University Press).
3. Problem Primer for the Olympiads by C. R. Pranesachar, B. J. Venkatachala and C. S. Yogananda (Prism Books Pvt. Ltd., Bangalore)
4. An Excursion in Mathematics Editors: M. R. Modak, S. A. Katre and V. V. Acharya and V. M. Sholapurkar (Bhaskaracharya Pratishthana, Pune).
5. Challenge and Thrill of Pre-College Mathematics by V. Krishnamurthy, C. R. Pranesachar, K. N. Ranganathan, and B. J. Venkatachala (New Age International Publications, New Delhi).
6. Functional Equations by B. J. Venkatachala (Prism Books Pvt. Ltd., Bangalore)
7. Inequalities an approach through problems (texts and readings in mathematics) by B. J. Venkatachala (Hindustan Book Agency).
8. Problems in Plane Geometry by I. F. Sharygin (MIR Publishers, Moscow)
9. Elementary Number Theory by David M. Burton (UBS)
10. Introduction to the Theory of Numbers by Niven and Zuckerman (Wiley).
11. Higher Algebra by Hall and Knight (Macmillan).
12. Higher Algebra by Barnard and Child (Macmillan).
13. Applied Combinatorics by A. Tucker (Wiley).
14. Introduction to Graph Theory by R. J. Wilson (Pearson Education India)
15. Problem Solving Strategies by Arthur Engel; Edited by K. Bencsath, P. R. Halmos. (Springer).
16. Mathematical Olympiad Challenges by Titu Andreesu, Razvan Gelca (Springer)
17. Mathematical Olympiad Treasures by Titu Andreesu, Bogdan Enescu (Springer)
18. The IMO Compendium by Dusan Djukic, Vladimir Jankovic, Ivan Matic, Nikola Petrovic.

Logarithms Table

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6471	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

(Continued)

Logarithms Table

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7768	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9997	9996	0	1	1	2	2	3	3	3	4

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